Running title: Teaching linear functions in context with technology

Title: Teaching linear functions in context with graphics calculators: students' responses and the impact of the approach on their use of algebraic symbols.

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Abstract:

This study analyses some of the consequences of adopting a functional/modelling approach to the teaching of algebra. The teaching of one class of 17 students was observed over five weeks, with 15 students undertaking both pre- and post-tests and 6 students and the teacher being interviewed individually. Use of graphics calculators made the predominantly graphical approach feasible. Students made considerable progress in describing linear relationships algebraically. They commented favourably on several aspects of learning concepts through problems in real contexts and were able to set up equations to solve contextualised problems. Three features of the program exerted a 'triple influence' on students' use and understanding of algebraic symbols. Students' concern to express features of the context was evident in some responses, as was the influence of particular contexts selected. Use of graphics calculators affected some students' choice of letters. The functional approach was evident in the meanings ascribed to letters and rules. Students were very positively disposed to the calculators, and interesting differences were observed between the confidence with which they asked questions about the technology and the diffidence with which they asked mathematical questions.

Keywords:

Algebra, algebraic expressions, functional approach, graphics calculators, linear functions, modelling approach, real world problems, secondary school mathematics, teaching in context.

Teaching linear functions in context with graphics calculators: students' responses and the impact of the approach on their use of algebraic symbols.

Introduction

A brief overview of Australian school mathematics textbooks shows that linear functions are a key topic. Typically students are introduced to standard explicit and implicit forms of function rules such as $y = mx + c$ and $ax + by = c$. Following the textbook sequence, they are taught to graph functions, taking consideration of both the *x* and *y* intercepts and the gradient of the function. Next students solve simple equations for *x* graphically and perhaps find the intersection of two functions. Symbolic equation solving may precede or follow graphical work. Finally there is usually a section of context based 'word problems' designed to illustrate applications of the theory studied. Class teachers commonly report two things: first, that during the unit of work students repeatedly ask "What are we doing this for?" and second, that when the students start the context based problems (usually word problems) they are unable to apply what they have learned.

In the study reported in this paper, a group of students were taught algebra through an approach that developed concepts through a series of problems based in a real world context. To assist with this alternative approach, students were given access to graphics calculators in all lessons. From the pre-test, post-test, interviews and observation we see that students engaged with the context of problems, gained skills in using technology and made important progress in both their understanding of, and facility with, algebra. This paper reports on two different aspects of the impact of the teaching program: first the students' response to the emphasis on context-based learning and the use of technology, and second the influence of both the content and teaching approach on students' understanding of, and facility with, algebraic symbols.

This Study

The seventeen year 8 students (about 13 years old) who participated in this study were classified by their school as being of above average ability. In the experience of the class teacher, the range of abilities in this group was narrower than that of a standard class but there was still diversity in their mathematical backgrounds. The class teacher, a respected and experienced mathematics teacher, was thoroughly familiar with this class of students, whom she had been teaching for more than 12 months, and was also familiar with the usual textbook approach to teaching linear functions as well as the basic features of the graphics calculator. Teaching this topic with heavy use of graphing technology was, however, new to her.

For the purpose of this study, the students did not use the normal textbook, but followed the linear functions chapter from Asp, Dowsey, Stacey and Tynan's (1995, 1998) *Graphic Algebra: Explorations with a Function Grapher*, a book that arose from research conducted during the 1991-1994 Technology-Enriched Algebra Project conducted at the University of Melbourne. The class teacher followed the recommendation of the authors of this text that students work through this material at their own pace punctuated by teacher intervention and whole class discussion. These interventions were sometimes used to teach calculator skills and at other times used to draw the students' attention to important features of the algebra. For example at different points, the teacher emphasized the meaning of letters as variables, writing algebraic rules, function notation, and transformation of linear function graphs.

TI-83^+ calculators were made available to students during each class. The graphic facility of these calculators allows flexibility of scaling and in particular allows students to move easily between different views of a graph by zooming in and out. These features were used extensively in the program, for example as students read information from graphs to solve contextualised problems. The teaching program gives generic technical instructions, referring to features shared by many of the graphing technologies, including the TI83+, and not explicitly giving button-press sequences, which the teacher explained. The students had no prior experience of using a graphics calculator. To expedite the teaching with the graphics calculator there was a large poster, pinned to a classroom wall, which clearly showed the various calculator buttons. The teacher also had access to an overhead projector and calculator view screen if she wished to use these. A major goal of the teacher for this unit of work was to begin to teach students how to use graphics calculators, a skill she regarded as essential for learning mathematics to senior levels in the school, in parallel with teaching algebra.

The teaching and assessment for this study took place over five weeks. Pre- and post-tests on graphing and symbolic algebra were administered. A selection of items from each test can be seen in the sections that follow. The post-test items aimed both to parallel those of the pre-test where students had not shown mastery, and reflect the focus of the learning tasks undertaken by the students. Sixteen of the seventeen students sat each test. A different student was absent each time so only fifteen results could be paired. Further data was collected immediately following the post-test from interviews with six students selected by the class teacher. In her opinion, these students provided a representative sample of students. After the teaching, the class teacher was interviewed using items that parallel the student interview questions as a stimulus. Finally data was also available from the notes made by the researchers who observed 20 of the 25 class lessons.

The class teacher explained that in the previous year students had worked a little with algebraic symbols in an abstract way. They had used notation for the four processes (e.g. interpreting $4m$, a/b, etc.), collected like terms, simplified simple expressions with exponents such as x^2/x, but they had not done any applications of algebra to real world contexts and they had not used letters as standing for variables. They had solved simple linear equations with one unknown on one side of the equation, although not in a context. The teacher said that most students essentially solved the linear equations by arithmetic 'guess and check' methods rather than using an algebraic method. Some word problems that could have been solved by algebra had been encountered in a general problem-solving unit. Referring to this, she said that:

> Most of it [the algebra in the linear functions unit] is new except they might have seen this word type of question before where they have to work out the number of tickets or number of hours…but certainly the graphing and putting it into an equation - they haven't done that before.

Pre-test responses confirmed this. Most students could give a verbal explanation for a specific problem solved numerically, but not correctly use algebra. Evidence of this can be seen in the many examples included below. This suggests that the students were at a significant point of transition in their mathematical development.

The teaching emphasised a graphic approach to linear functions. Students solved some equations both graphically and symbolically but graphical solution methods predominated, assisted by zooming on the calculator to reach a desired accuracy. The teaching was almost all set in the context of problems, from which mathematical ideas were drawn. The unit begins with a story about a girl selling homemade lemonade. The profit she makes is set up as a function of how much lemonade she sells, first in a table and then on a graph. Students then graph the function and read various information related to the problem setting from the graph, changing the range and using zooming as required. Later, the story introduces other drink sellers with different prices and ingredient costs. These corresponding functions are graphed and the graphs and functions are compared. Points of intersection, slopes, intercepts and intervals are interpreted in context. The program then introduces another real world problem, a comparison of mobile phone charges, which leads to other aspects of scale and the intersection patterns for three linear functions. Problems drew out the significance of slope and intercepts in terms of both the original problem and the related algebraic equations. The next situation investigates the relationship between height and arm span, leading students to draw a line of best fit by eye through data, and to interpret the intercepts and slope of the line of best fit in terms of the real situation once again. The class omitted the next context (a series of problems based around comparison of school sizes in different countries) and replaced it with the section on word problems from the textbook. Based on past experience, the teacher expected students would find these difficult and, in her words, "not know where to start". The unit ended with several lessons where students created target designs from straight lines and line segments on the graphing screen. The first and second contexts involved price as the dependent variable, as did several of the problems in the pre-test and post-test.

Post-test responses and interviews showed that all students made progress in using symbolic and graphic algebra. However, even in this select class, there was a diversity of responses to the learning tasks. The focus of this paper is not to put a numerical measure on the change in the quality of students' responses between the pre- and post-tests (Although it was very good) but to look for evidence of the effect of learning in context and use of graphics calculators on students' responses and then, specifically, to analyse the impact of these features of the teaching on the students' use of algebraic symbols.

In each of the two main sections below we begin by briefly referring to background literature relevant to the aspect under consideration. In the first section we discuss the reasons for adopting this alternative teaching approach, and in the second, different approaches to algebra and where, based on both content and approach, this teaching fits. Following these preambles we consider students' affective and cognitive responses in to the teaching as evidenced by their comments and written work. The first section focuses on the students' and teacher's responses to the introduction of linear functions through well-chosen real world problems and the use of graphics calculators.

Learning in context with graphics calculators available

Rationale for this approach

It is the shared experience of mathematics teachers that early algebra is the point at which many students 'switch off' mathematics and start to define themselves as 'not good at maths', so an approach that revealed the relevance of mathematics is of interest.

The students participating in the present study had previously done some work on the four processes of addition, subtraction, multiplication and division with pronumerals but had no experience of using algebra to describe relationships or to express generality. Appropriate teaching of these concepts is clearly important in students' mathematical development. Freudenthal (1991) speaks of mathematics starting within common sense. Students' confidence can be maintained and their understandings developed by starting with problems based on situations that are experientially real for them. Mathematical sense may grow from and within common sense. Therefore the choice of introductory problems will be crucial in directing their thinking. Gravemeijer (2002) writes about the importance of developing models from which students may establish mathematical understandings. His work suggests that focusing on carefully selected problems set in a context, which the students already understand, may lead to deep and transferable conceptual understanding.

While these researchers' comments provide sound reasons for imbedding the teaching of fundamental algebra concepts, such as linear functions, in the solution of real world problems there is a risk that students will be faced with time consuming and error prone tasks such as sketching graphs and working with non-integer values which may distract their attention from the algebraic concepts. Technology can provide the necessary support for students working to solve real world problems graphically, because graphs become manipulable objects through changing domain and changing scale (zooming). Kissane (1999, 2001) also reminds us that graphics calculators enable students to enter a function rule and swap quickly and easily between graphical and numeric representations of the function. This facility, he says, enables students to explore a range of solution strategies using different representations. In addition graphics calculators support students' ability to create mathematical models based on real data. Even beginning algebra students, learning about linear functions, can enter data into lists, create a scatterplot, then make and test conjectures about the function which would best fit the data. Technically the task of changing the values of the coefficient and constant, then moving to the graph view to test each new conjecture, is quite simple thus allowing the student to concentrate on the mathematics and observing the effect each change to a parameter has on the slope and position of the graph.

It seems then that while the 'traditional' approach to teaching algebra achieved success for a minority of students the ready availability of technology such as graphics calculators should make it possible to reorient the curriculum so that more students can develop an understanding of and facility with algebra within familiar contexts and then progress to the solution of generalised but de-contextualised problems. As described above, this was the approach adopted for the teaching in the study reported in this paper. First we will consider the students' response to the use of graphics calculators.

Response to using a graphics calculator for working with linear functions

These students were younger than most students being introduced to graphical calculators. At the school involved in this study, graphics calculators were not usually introduced until year 10. However their teacher said:

> They were good initially with the calculators when they were just learning how the calculators worked and being able to do a graphs …They coped pretty well. They pick up things pretty quickly…They know how to put a list in now and graph lines and that there are idiosyncrasies. You have to put expressions in the right way or the calculator won't [accept them].

The students interviewed were all positive in their comments on learning to use the calculator:

> *Student 4*: The only problem I really had was learning to stop the line half way [restrict the domain of a graph].
>
> *Student 6:* It was good to learn how to use the calculator.
>
> *Student 7:* I enjoyed it. Not really difficult.
>
> *Student 15:* The cool calculators. [*Interviewer*: Did you find them easy to use?] No they are very detailed. So [its use] need to be explained first before I could use it, but once I learnt how to use it, it was very easy.

Asked to comment on their perceptions of using the graphics calculator for learning mathematics these students were again positive in their responses:

> *Student 1:* It [learning mathematics] was better with the calculator. Easier. I've never really got algebra before but with the calculator it helped [me] to understand everything. [It] made a lot more sense with the calculator. … [Using the calculator] makes it a lot easier. You don't have to do something and write it down; do something and write it down; just do it and write down an answer. It 's a bit more interesting
>
> *Student 4*: When we were doing other things the teacher talked to us when we needed help but when we were doing this subject she taught us to use the calculators. That helped us get a head start, before we even started it [the algebra], as we knew how to use it [the calculator], and how to get the answers, instead of having to ask for help. … [The calculator] helped [me] learn algebra... Very easy to use and instead of writing down all the pages of equations you just write it into the calculator and it is easier; easier to learn.

The researchers who observed the classes were interested that the teacher and students were so overwhelming positive in their interview responses. The classroom observers each recorded comments on the time and effort involved in the students' learning to use the technology. When students requested assistance, it was most commonly because they were not sure how to operate the calculator correctly or efficiently not because they needed help with the mathematics. However the students obviously did not perceive the time or effort involved as excessive. When a student did need to ask for some help with the mathematics they would commonly preface their question with a comment such as "I'm not good at algebra". They did not make similar personal comments when they met technology obstacles. Observation showed that learning to use the graphics calculator required a considerable investment of both teacher and student time. It is therefore important that these skills, once gained, be consolidated and applied in future topics.

Students' understanding of the graphical representation of linear functions

We move our focus now from the graphics calculators to the impact of taking a graphical approach to the topic of linear functions. For the first problem on the pre-test students were provided with a graphical representation of the costs of hiring either Jack or Jill's truck. The post-test question paralleled this with a graph showing the alternative costs of hiring a plumber, either Bob or Chris.

The pre-test results showed us that most of the students were already able to interpret key features of linear graphs. All of the students but one were able to read appropriate values from the graph in response to questions such as *"If you have $400, for how many days can you hire Jill's truck?"* or "*What is the fixed amount at the beginning of the hire of*

Jill's truck?" A large majority could interpret the point of intersection of two graphs as the cut-off point where hiring one truck became more expensive than hiring the other. Most students could also give a sensible verbal explanation of how to work out truck hire but only one quarter of the students could write an acceptable algebraic rule to connect the cost in dollars with the number of days of truck hire. The post-test results showed that all but one of the students could both correctly interpret the graph and also read off values. The biggest change was in writing an algebraic rule to describe the function that was portrayed graphically. At the post-test, three quarters wrote an acceptable rule.

A further question was added to the post-test to try to ascertain students' understanding of links between the slope/initial values of a linear graph with the parameters of its algebraic equation. They were given the following problem following the interpretation of the graphs of the costs of the plumbers Bob and Chris. "*At Christmas Bob gives all his customers a $10 discount but Chris gives all his customers a discount of $5 for every hour he works. Add new graphs on the axes below to show the new hiring cost for Bob and Chris."* One quarter of the students sketched two correct new graphs, almost half had one or the other (equally split) correct while the rest either did not answer or were incorrect. The class teacher was not surprised that the class still had difficulty with this item. She was already concerned that the role of the coefficient and constant in $y=mx+c$ had not been made sufficiently explicit. She felt that when using this material again she would give this greater emphasis. Teaching in context had not brought out the graphical roles of the parameters in the general abstract linear function for all students.

Response to using real world contexts for a first experience of linear functions

We will now consider the teacher's and the students' responses to learning new concepts and skills by working on context based problems. The teaching in this unit of work was based on working through a series of questions related to four contexts that were experientially real for the students (e.g. profits made by selling homemade lemon squash, comparing mobile phone charges, looking for a relationship between arm span and height and one context omitted by this class on comparing shoe sizes used in different countries). Only the fifth series of exercises were abstract: to create pictures using straight line graphs and segments. Rather than completing a large number of similar examples the students worked quite slowly and carefully through a few strategic problems in the real setting. When discussing the change in teaching approach the class teacher said:

> It was around the other way to the way it's often taught which is to get them to do a linear graph of some sort, generally the $y=x$ graph, then move it, then put a number in front of the x. [We] generally teach them that maths skill first then apply this to real problems, which they [the students] always have difficulty with. This was the other way around: started with a real problem then tried to work out the answers then to also generalize it and put it into some sort of function. So its interesting to see if there will be a greater connection with word problems than there normally would be.

Students responded positively to the revised teaching approach. Student 1 articulated this sentiment when she said that:

> It is a lot easier if you have word problems straight off. It's kind of giving you a different perspective; an image that helps you remember it.

The students related personally to these context problems. Student 1 said that she found the problems:

> More interesting, yes, I liked the lemonade one because I used to make lemonade squash when I was little.

Student 15 also found the work more meaningful, commenting that:

> Yes, then you can relate things - its not just a lot of numbers and things on the paper so when you have it [the answer] it actually makes sense.

While it is pleasing to see that the students were more interested and engaged, we wanted to know whether the focus on real situations had cognitive gains. Comments from both the class teacher and the students included below indicate increases in confidence, understanding and successful application of this algebra to new situations, including word problems from the textbook which the teacher expected them to find difficult.

> *Student 1:* Yes because sometimes with algebra you know what you are doing but you don't know why. With this, it kind of told you why as well.

> *Student 6:* It got sort of related to real things, instead of just being numbers.

> *Class teacher:* They understood it and … they can relate to it fairly well. Yes definitely better than *x*'s and *y*'s. 'What do they represent?' always comes up. At least they have some understanding of where all this [algebra] comes from… To me it makes more sense to do that. It does help them, although I think they need more skills practice [i.e. non-contextual graded series of manipulation exercises]. ... About three-quarters of the way through the unit of work, we did the word problems from their textbook. They coped with that. Generally year 9's find them really difficult and you [the teacher] land up going through question after question to show them a lot [of examples] before they start to pickup what's going on, whereas these kids seemed to manage it pretty well. They seemed to understand from the start.

This last comment was especially important for the evaluation. The researcher who observed the 'textbook' word problems lesson recorded that one student asked about one of the word problems: "At zero years, there are less than zero books. How can that be?" This question led to an informal discussion of restricted domains. The students' were applying common sense to their mathematics, which in turn led to further growth in their understanding of algebra. By learning algebra in context, they had less difficulty applying it to contexts.

Impact of learning in context evident in students' written solutions

Analysis of students' pre-test and post-test responses indicates differing levels of abstraction and differing levels of engagement with the context of the problem. A selection of these examples is discussed in the following paragraphs. Recall that on the post-test, students were given a pair of graphs representing the call-out fee and cost per hour of hiring two plumbers, Bob and Chris. The students' responses illustrate the degree to which they focused on the practical features of the real problem and the degree to which they considered the mathematical aspects that would be common to any such problem. A typical response from the quarter of the class deeply engaged with the context was given by Student 4, who wrote:

> It is cheaper to hire Chris for long jobs like plumbing the kitchen but vice versa for Bob for small jobs like a leaking tap.

A second group, comprising a further quarter of the class, took a more mathematical approach but still focused on taking an overview of the context of the problem. Student 1 provides an example of the category:

> Depending on how long they are there if they are there for 4 hours or less Bob is cheaper but more and Chris is cheaper. Bob goes up $50 an hour whereas Chris goes up $25.

The rest of the students reported the bare facts of the context. Student 7 provided one such response:

> If the job takes 4 hours or less Bob is cheaper. If the job takes more than 4 hours Chris is cheaper.

The change in students' thinking before and after the teaching is evidenced in responses to parallel pre- and post-test items. Consider the following examples displayed in figure 1, again taken from responses to question 1 in which students were provided with a graphical representation for two functions. The pre-test item referred to the cost of hiring Jill's truck for 14 days, at $50 per day with an initial fee of $100. The post-test item asked for the cost of hiring Bob, the plumber for 14 hours, at $50 per hour, with an initial callout fee of $25. All students provided answers to this item on both pre- and post-tests that indicated they had paid attention to the context but most provided more contextual detail in their post-test answers. No students showed evidence of generalized algebraic thinking in their responses to this item on the pre-test. In contrast half of the students did so on the post-test.

In figure 1, student 5 demonstrates the new attention to context details and new mastery of algebra. (The pre-test response is to read the value from the graph for x=9 and to successively add $50 for every hour of work). Student 8 shows detailed attention to both the context and algebra, and a clear explanation of the role of the rate and the initial fee, whereas the pre-test verbally describes an incorrect model.

Student	*Pre-test response*	*Post-test response*
5	Add $50 onto every number from #9 until you get to #14	Bob's prices go up by fifty dollars an hour starting at $25. 50 x 14=$700, plus the starting 25, and it would cost $725.
8	Well you find out how much it cost per day and x 4	Bob: Well the initial cost is $25, so you would add that to they're hourly, which for Bob is, $50 per hour. So the equation would be. $\underbrace{50}_{\text{Hourly rate}}\ \underbrace{x}_{\text{Number of hours}} + \underbrace{25}_{\text{Initial cost}}$

Figure 1. Examples showing development of understanding of contexts and algebra

In this first section we have considered the students' understanding of the graphical representation of linear functions and their affective responses to both the use of technology and working on problems based on familiar contexts. Their comments along with those of the observers suggest that the students engaged with these learning tasks. No student asked "What is this about?" or "What do we have to do this for?" On the contrary their questions in class and their written responses provide evidence that they were learning algebra by making sensible generalizations in order to solve problems which had meaning for them. In the next section we will consider the evidence, provided by students' written solutions and interview comments, of their cognitive progress in using symbols. We will discuss the influence of both the content and teaching approach on students' understanding of and facility with algebra.

Learning in context and students' use and understanding of algebraic symbols

Learning in context and the 'four approaches to algebra'

Bednarz, Kieran and Lee's (1996) book contains wide ranging discussions of the different options for introducing algebra in school and the impact of each option on the development of students' conceptual understanding of algebra; in particular on the meaning students assign to algebraic symbols and notations. More precisely, the authors promote didactical discussions in regard to four approaches to algebra that have stimulated interest among international researchers namely: generalization, problem-solving, modelling and functional perspectives. Scaffolded by epistemological and historical foundations, each of these approaches are carefully analysed theoretically. The studies included in this comprehensive book provide in-depth reflections on students' various understandings of algebra depending on whether the introduction which they experienced focused on generalizing patterns, solving specific (classes of) problems, modelling a physical phenomena, or focusing on the concepts of variable and function. The different chapters of Bednarz et al's book raise important questions regarding the teaching and learning of some of the fundamental concepts of algebra, suggesting that each perspective highlights these fundamental concepts in significantly different ways.

The question that we now raise is how does the teaching approach adopted here highlight the fundamental concepts, as seen from the point of view of the students' responses? Taking into account the teaching method adopted in this case, what can be said about students' ability to understand and utilise algebra?

When analysing the way students work with algebra, it seems that it is important to take into account two aspects: one related to the *content of teaching* and the other one related to the *way this content is taught*. In the study reported in this paper the content of the teaching was the key factor determining where this work is positioned among the four approaches to algebra listed above. The content of teaching focused on linear functions so we might say that this study is closely related to the functional approach. In particular, in this teaching sequence, the functions are viewed from the perspective of the relationship between two quantities such as number of litres of lemonade sold and profit. Such a view of functions as a process for computing one value by means of another (rather than as a relation between two sets; another possible way of emphasizing functions in a structural approach) can clearly be seen in the real situations studied. The *Lemonade* problem required students to work out the profit made per litre of lemonade sold, given the cost of the ingredients and the charge per litre). In the *Mobile phone charges* problem, in order to determine the most and least expensive charging schemes among different operators, students had to work out the different mobile schemes rates, depending on the call time, given the monthly cost and the charge per minute for each scheme. It should be noted that this unit of work was not an introduction to the formal study of functions as such. Function notation was used but there was no discussion of formal definitions, sets or mappings. As Kieran, Boileau, Garançon observe, "a functional approach to algebra does not necessarily mean the study of functions" (1996, p. 257).

The problems in this unit of work were not restricted to exploring deterministic linear relationships. It also included fitting linear functions to data, as illustrated by the *Arm span versus height* or the *International shoe size* problems, in which students were encouraged to find rules for functions, from a description or a table of data, and test whether the rules fitted the data reasonably, etc. These problems do not rigorously follow the modelling perspective. As Heid (1996) reminds us “a true modelling approach might be more open-ended in its choice of problems and models and fuzzier in its fit of data to particular families” (p.254). However, we do find some evidence of this approach in this part of the unit of work, since the teaching focused on the creation, use and evaluation of mathematical models for relationships derived from realistic contexts. In summary, after considering these two distinct types of problems that underpin the teaching, it can be seen that in this study the approach included some characteristics of both the functional and the modelling approaches.

From the examples of the teaching content given above, another feature of the unit of work emerges, this time related to the *way of approaching* linear functions. The two parts of the unit that led us to classify the study as both a functional and a modelling approach, namely exploring linear relationships and fitting linear functions to data, are both built on the same foundation *i.e*. working from real-word problems. Since learning in context is one of the main characteristics of this approach, it may sometimes be difficult to clearly distinguish between the teaching approach and the content itself. Yet this distinction has to be made, because the influence of *learning in context* on students’ relationship to algebra may be of a different nature than that which results from this particular dual functional-modelling approach. This leads us back to our earlier question, now reformulated and extended as follows: what understanding and facility with algebra have students developed during the five weeks of this study? More precisely, taking into account the functional-modelling framework, what relationship do students have with some of the algebraic concepts and/or algebraic notation? What are the imprints of learning in context? Finally, do we see any influence of the use of the graphics calculator in their relationship to algebra? The next paragraphs will be devoted to tackling these questions.

Students’ pre-test ability to understand and utilise algebraic rules

As mentioned earlier in this paper, pre-test responses showed that even though most students could give a correct verbal explanation for a specific problem solution, providing some evidence of a good understanding of the problem, very few of them showed confidence or facility when asked to use algebra. This is consistent with other research (see, for example, MacGregor & Stacey, 1993). About half of the students did not answer these latter parts of Qu.1 from the pre-test and most of those who made an attempt produced ‘inconsistent’ algebraic rules, as illustrated by the responses in figure 2:

Student	**Qu.1g***: Explain in words how to work out the cost of hiring Jill’s truck for 14 days WITHOUT USING THE GRAPH.*	**Qu.1h***: Use algebra to write a rule connecting the cost in dollars with the number of days of hire of Jill’s truck.*
1	Well the hire price goes up $50 a day so just add $50 for every extra day plus $100 dollars starting fee	$5c + 4x = 9y$
15	Well each day costs $50 so multiply $50 x 14 days then add the fixed amount	$50a = d$, a = dollars, d = days
17	You have the fixed rate of $100 the it cost $50 a day so you times $50x14 then add $100	$z + (y \times x)$

Figure 2. Examples showing students initial facility with verbal description but difficulty with algebra

Although the tasks underpinning questions *Qu.1g* and *Qu.1h* are clearly of different kinds (in *Qu.1g* students are working with specific values of the problem, and are asked to produce the answer in words, whereas in *Qu.1h* they have to deal with the generality of the problem and provide their answers using algebra), comparing students' answers in *Qu.1g* and *Qu.1h* illustrates their difficulty in 'translating' the general pattern of the problem, that they seem to have grasped, into algebraic expressions. It is also important to note that all the questions in the pre-test similar to *Qu.1h*, that is where students were asked to provide algebraic rules, presented the highest rates of non response (about half of the students).

After considering their writing of algebraic rules, we can say that most of the students showed limited facility with algebra and, as stated earlier, appeared to be at a significant point of transition in their mathematical development. Based on their answers on the post-test, let us now take a look at students' facility with, and understanding of, algebraic rules after the five weeks of teaching. We will do this by highlighting the precise detail of the students' use of the different components of algebraic expressions and their link to the features of the teaching. We will also discuss the thinking that may have underpinned these choices.

Perceiving and using letters - a triple influence

Kuchemann (1981) and MacGregor & Stacey (1997) outline some of the ways in which beginning algebra students interpret letters and Janvier (1996) recalls the various interpretations a student can give to letters even when they are firmly established as numbers:

> To summarize, for a student, the letter a can be interpreted in at least four different ways (...) First, it can be used as an indeterminate value in a formula such in P(erimeter) = $4a$; secondly, it can be considered as an unknown, such as in $2a+3 = 7$; thirdly, as a variable in $A = \pi r^2$ (if r varies); and finally, it can be interpreted as a polyvalent name as in the identity [such as $(a+1)^2 = a^2+2a+1$, where a may be any of the above]. (Janvier,1996, p. 227).

The way students interpret letters in given algebraic rules can be seen in a number of examples taken from their post-test answers. In particular consider their responses to part of the *Fun Park problem*, where students had to compare two ways of paying: the 'cheap entry' and the 'cheap rides' tickets, given different fixed prices of entry and different costs per ride in each case. One question near the end of that problem, was as follows:

> The manager of the park has been told to increase takings. At the finance meeting he presents the following new entry plans:
> $P(x)=20+5x$
> $Q(x)=3x+30$.
> What do you think the x in these rules stand for?

The letter x had not been used earlier in the problem, although probably the majority of students had selected it to name the variable when writing their own algebraic rules to describe the two costs.

In explaining what the manager may have meant by x, one student from the class attributed a specific value answering "7 (seven)". When interviewed and asked to comment on his answer, the student said "I thought it was a number so I just put seven". Apart from this case, all other students who answered this question did not feel the need to give a specific value to x, and either answered "number of rides", "rides", or "how many rides". Their responses suggest that, they seemed to accept that a single symbol such as x could stand for more than one value and, at the same time, seemed to accept that this unknown value could vary: x as both a variable and an unknown depending on context.

One might argue that the answer that *x* stands for "rides", given by some students, may be evidence of them interpreting the letter *x* as an object (see Kuchemann, 1981), rather than a symbolic representation of a variable number. But this seems unlikely because there is no evident link between *x* and 'rides', as might be the case if the letter used was *r*, for example. Moreover, the use of a letter as an object was not consistent with their responses to other questions. (Figure 3, Student 15, shows a clear case of the letter *d* as object or unit in the pre-test.)

Evidence of the students interpreting the letters as variables, that is as a symbolic representation of a number that can take on a range of values, can be found elsewhere in their post-test, and sometimes in a very explicit way. Note the use of the word 'insert' in the following answers; this will be discussed in more detail below.

> *Student 14*: Bob's linear equation is $50x+25$ so you insert 14 as *x*.
>
> *Student 15*: Bob, for each hour he works he costs an extra \$50. So we form the rule $50x +25$. (...) Using the rule we can insert 14 into *x*. By doing this we get an answer of \$725.

When we take into account the teaching and more specifically the approach that was adopted, the predominance of interpreting letters as variables in students' post-test responses is not surprising. In fact, as mentioned before, some characteristics of both the functional and the modelling approach can be found in this study, entailing, as pointed out in Bednarz et al (1996), the use of letters as variables.

Just as the imprint of the *content of teaching* is clearly present in students' *interpretation of letters*, the post-test responses also provide some evidence of the influence of the *way of teaching* in students' *use of letters* when writing algebraic rules. Features of the two main aspects that characterized the way of teaching this unit of work, that is, working from real-world problems and using graphics calculators, were both found in students' responses.

The imprint of the context in which a problem was set is obvious when we consider the letters chosen by the students for the purpose of creating an appropriate algebraic model. This can be illustrated by the following responses to the question: "*Use algebra to write a rule to work out the cost in dollars from the number of hours of hiring Bob.*"

> *Student 4*: $P=25+50\times x$, P=Profit, x=hours worked
>
> *Student 5*: $B=50h+25$
>
> *Student 15*: $50x+25=c$, x=hours, c=cost

These answers were typical of students' responses to this post-test question, and indicate their tendency to 'recall' the context in which the problem was set, as well as their ease in the post-test of writing algebraic rules. When writing an algebraic rule they commonly chose letters such as *h* for hours, *c* for cost, or *B* for Bob. Note also that some problems in this unit of work involved the concept of profit (such as the Lemonade problem). This might have influenced student 4's response.

When comparing these responses to the few algebraic rules students provided in the pre-test, shown in figure 3, we observe a clear shift in their use of algebra, and more specifically in their use of letters.

Student	**Qu. 2f**: *Use algebra to write the rule that shows how to work out the cost in dollars of a WIZARD vacuum cleaner from the number of dust bags that you buy with it.*
8	$a\text{-}b \div c = d$
15	$n = 3d$, dollars = d, number = n
16	$a + (b \times c) = d$
17	$w\text{-}b = db + w$

Figure 3. Examples of students' inappropriate use of symbols in pre-test responses

In fact, not only did the rules provided in the pre-test answers not model the problem set, but the use of letters, at that stage, was far from being 'consistent'. The previous answers illustrate the two main kinds of use of letters found in students' pre-tests. In the few cases where letters were used, either the students tended to choose random letters (usually the first and consecutive letters of the alphabet) with no link whatsoever with the context of the problem (*cf*. students 8 or 16), or they provided algebraic rules where the context tends to take over the 'algebraic consistency' of the rule. This can be found in student 17's response (above), where we might presume that *w* stands for 'Wizard', and *b* or *db* for dust bags.

In summary, contrary to their pre-test, the way students alluded to the context, by the means of the use of letters, in the algebraic rules they wrote in the post-test, presented some evidence of what we might call a 'symbolic mastery of the context' and, more generally, since they expressed correct relationships, what we might call 'algebraic mastery of the context'.

Finally, some of the evidence examined in the previous paragraphs, suggesting that students perceived letters as variables, also alerted us to the fact that the second characteristic of the teaching that is the use of graphics calculators, also had some influence on students' use of letters. In particular, when students' expressed their substituting in terms of *inserting* a value into an 'expression' (*cf*. students 14 and 15 who talked about "insert 14 into/ as *x*"), we see a term usually related to the use of a calculator. This led us ask: what role, if any, did the graphics calculator play in the student's perception or use of the letters? The transcript of an interview carried out with one of the students, an excerpt of which is given below, provides us with some interesting information. It shows quite clearly the influence of the use of graphics calculators in the students' perception and use of letters:

> [Referring to question 3c of the post-test: "*Use algebra to write a rule connecting the cost of going to the park and the number of rides that you have for the cheap rides ticket and the cheap entry ticket*"]
>
> *Student 6*: The cheap rides ticket is $2x+32$ and the cheap entry ticket is $3x+10$.
>
> *Interviewer:* Why did you choose *x*?
>
> *Student 6:* Hmm…Just because in the calculators you can't use put anything else but *x*.
>
> [Referring to question 3e: "*The manager of the park has been told to increase takings. At the finance meetings he presents the following new entry plans: $P(x)= 20 + 5x$, $Q(x) = 3x+30$ (...)*"]
>
> *Interviewer*: The manager chose *x*, just like you did. Do you think it's a good choice?

Student 6: [He] might have been using the graphics [calculator] and so he used x.

Learning in context - *The use of units in algebraic rules*

Janvier (1996) points out a particular phenomena related to the modelling approach and student's use of symbols that is widely discussed in the literature (see Clement, Lochhead, & Monk, 1981): the frequent misunderstanding of some students in distinguishing between the letters that are the symbolic representation of magnitude units and the ones that stand for variables, especially when students tend to include magnitudes within their algebraic rules:

> For instance, the same g may represent the constant used in $E = mgh$, an abbreviated form for gram, or a variable such as a particular gain (that stands for a varying magnitude expressed in dollars). (Janvier, 1996, p. 231)

A similar question to the one pointed out by Janvier can be found in our study. But unlike Janvier's, the issue does not arise from the approach chosen in this unit of work (both functional and modelling), it comes from the way of teaching the content, specifically, from the teaching in context. In the previous paragraphs, the imprint of the context has been established through students' use of specific letters to represent the variables of the problem (h for hours, c for cost). Now we consider the influence of the context with regard to the use of symbols standing for the unit of measurement in students' writing algebraic rules. In particular, in this unit of work, students have often dealt with real-world problems where the idea of cost was involved (the *Lemonade* problem, or *Mobile phone charges* problem, for example). The questions we ask ourselves are: How 'abstract' and how 'concrete' are the formulas they provide? To what extent are they recalling the context? Do they tend to use the dollar symbol when writing the algebraic rule?

The inexpert use of algebraic rules in students' pre-test responses has already been discussed. Now, including an examination of their use of units in the formulas they provide seems to be another important characteristic to take into account when looking at students' use and understanding of algebraic expressions. The weakness of their pre-test ability to use algebraic symbols is illustrated by the following answer:

Student 6: $196 + ($x$×$3)

Among other interesting features we will examine, this student's rule is quite representative of students' difficulty in discerning 'unnecessary accessories' from the 'indispensable components' of an algebraic rule, in this case including the unit symbols in a formula.

The way students employed the units (and more specifically the dollar sign) in their answers at different stages of the post-test, shows clear progress in their use and understanding of algebraic rules. Even though the recalling of the context, by the means of the use of unit symbols, was also found in the post-test responses, this occurred at a different level. The following comparison (in figure 4) between some of the post-test students' wording and writing of algebraic rules is quite illustrative of this idea:

Student	**Qu. 1d***: Explain in words how to work out the cost of hiring Bob if you knew the number of hours he would be working*	**Qu. 1e***: Use algebra to write a rule to work out the cost in dollars with the number of hours of hiring Bob*
4	First times $50 by the number and add $25 at	$P = 25{+}50 \times x$

	the end	
6	Times the hour he works by \$50 and add \$25	Cost = $50x+25$

Figure 4. Examples showing a comparison of students' writing in words and symbols

In fact, even though the units were still present in students' description of the rule in words, these symbols faded out in the algebraic formula. In this sense, one might say that students seemed to move towards a more 'abstract' algebra - in the sense that the 'concrete' part coming from the context was not present in their rules. This distinction between the 'algebra' side of the problem and the context (here related to the unit symbols) that it has emerged from is also sometimes clearly expressed in students' answers, such as the following:

> *Student 1*: Well, you times the hours worked by 50 then plus 25 because he charges \$50 an hour and \$25 initial fee.

Other signs - Moving towards a conventional way of writing rules?

In the previous paragraphs, the influence of the three components underpinning this study (the dual functional/modelling approach, the learning from context and the use of graphics calculators) has been examined through students' writing and interpreting algebraic rules. More specifically, this has been done by pointing out the different components of algebraic expressions (such as letters) and their link to the features of the teaching. Examples have been given providing some evidence of what we may call a 'triple influence' in their writing of formulas. Nevertheless, even though the imprints of the different features appeared quite clearly in students' responses, their post-test responses also showed that the different characteristics of the teaching did not take over the algebraic consistency of the formula. This has been illustrated by their 'symbolic mastery of the context' (a referenced context not overwhelming the rule) when choosing letters to represent the variables of the problem, and also by their progressive abandoning of the use of units when writing a rule. When considering the influence of the context, we have mentioned an 'abstract' way of writing the rule, as opposed to the 'concrete' setting of the problem. Not only was the context progressively becoming 'algebraically discrete' in students' answers: in a more general way, the post-test indeed showed that students progressed towards writing algebraic rules in a conventional manner. Evidence of this significant advance is shown in the two examples in figure 5, below which were typical of the change we observed.

Student	*Pre-test response*	*Post-test response*
6	\$196 + (x × \$3)	$2x + 32$
12	100 + (50 × x) , x = Days	$50x$ +25 , x = hours

Figure 5. Examples showing students' desist from imbedding units in algebraic rules

Indeed, the students' post-test answers showed that not only the use of superfluous parenthesis tended to dim, but also that the use of the multiplicative sign faded out. These changes suggest that, in their thinking, the students' progressively detached the algebraic rules from the process they came from. Finally, some other students' answers to the post-test revealed their conscious decision to write rules in more conventional way. This was highlighted by one particular students' response: he crossed out his first attempt to write the rule "x×50", writing instead beneath it: "$50x$+25".

Conclusions

In this paper we set out to determine the impact of two different aspects within an alternative approach to the teaching of linear functions: an emphasis on context based learning and the use of technology. We have been particularly concerned to provide a detailed analysis of the influence of both the content and teaching approach on students' understanding of, and facility with, algebraic symbols.

We have shown that, while the use of graphics calculators required an initial overhead of time and effort the students and teacher felt this was worthwhile. This technology afforded the students support for their exploration of real world problems but at the same time the calculator syntax constraint of allowing only y and x to represent variables influenced some students' by-hand rule writing.

As a result of approaching the topic of linear functions through the solution of real world context problems, students came to see that mathematics is useful. They learnt to write algebraic rules in a conventional manner, and as a result of working in context, they quickly came to make sensible use of symbols and understand functions as giving one variable quantity in terms of another. Their comments suggested that they felt some sense of satisfaction in writing generalizations that allowed them to solve these problems. The students certainly became adept at identifying both initial values and the 'rate of change' pertinent to each example, and transferred this knowledge to the textbook word problems. The value of this teaching in context was perhaps best summed up by the student who said, "It gives you a different perspective, an image that helps you remember it."

The detailed analysis of students' verbal and symbolic responses to problems highlighted the influence the functional-modeling approach to teaching, the emphasis on context and the use of graphics calculators. The range of students' writings included in this paper illustrates very clearly the individual nature of students' responses to a teaching sequence. While all students made good progress towards writing conventional algebraic expressions and developing symbol sense.

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References

Asp, G., Dowsey, J., Stacey, K. & Tynan, D. (1995). *Graphic Algebra: Explorations with a function grapher.* Carlton, VIC: Curriculum Corporation.

Asp, G., Dowsey, J., Stacey, K. & Tynan, D. (1998). *Graphic Algebra: Explorations with a graphing calculator.* Berkeley, Ca: Key Curriculum Press.

Bednarz, N., Kieran, C. & Lee, L. (Eds.) (1996). *Approaches to Algebra: Perspectives for Research and Teaching.* Dordrecht, Netherlands: Kluwer Academic Publishers.

Clement , J., Lochhead, J., & Monk, G. (1981). Translation difficulties in learning mathematics. *American Mathematical Monthly*, *88*, 289-290.

Freudenthal, H. (1991). Mathematics as a New Educational Task. Dordrecht, Netherlands: Kluwer Academic Publishers.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning.1*(2), 155-177.

Heid, M. K. (1996). A technology-intensive functional approach to the emergence of algebraic thinking. In N. Bednarz, C. Kieran & L. Lee (Eds.) *Approaches to Algebra: Perspectives for Research and Teaching*, pp.239-255. Dordrecht, Netherlands: Kluwer Academic Publishers.

Janvier, C. (1996). Modeling and the initiation into algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.) *Approaches to Algebra: Perspectives for Research and Teaching*, pp.222-238. Dordrecht, Netherlands: Kluwer Academic Publishers.

Kieran, C., Boileau A., Garançon, M. (1996). Introducing algebra by means of a technology-supported, functional approach. In Bednarz, N., Kieran, C. & Lee, L. (Eds.) (1996). *Approaches to Algebra: Perspectives for Research and Teaching*, pp.257-293. Dordrecht, Netherlands: Kluwer Academic Publishers.

Kissane, B. (1999). The graphics calculator as an investigative tool. Paper presented at the *Australian Association of Mathematics Teachers Virtual Conference 1999 CD_ROM*.

Kissane, B. (2001). Technology and the curriculum; the case of the graphics calculator. In M.O.J.Thomas (Ed.) *Proceedings of TIME 2000:An International Conference on Technology in Mathematics Education* (pp.60-71). Auckland, New Zealand.

Kuchemann, D.E. (1981), Algebra. In K.M. (Ed.), *Children's understanding of mathematics, 11-16*, pp.102-119. London: John Murray.

MacGregor, M. & Stacey, K. (1993) Seeing a Pattern and Writing A Rule. In I. Hirabayashi, N. Nohda, K. Shigematsu & F.-L. Lin (Eds.) *Proceedings of Seventeenth International Conference for the Psychology of Mathematics Education*_, University of Tsukuba, Japan, July, Vol I, 181-188.

MacGregor, M. & Stacey, K. (1997) Students' understanding of algebraic notation: 11 – 16. *Educational Studies in Mathematics.* 33(1), 1 – 19.