SLOPES OF MULTIFUNCTIONS AND EXTENSIONS OF METRIC REGULARITY

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Dedicated to Professor Phan Quoc Khanh on his 65th birthday

Abstract. This article aims to demonstrate how the definitions of slopes can be extended to multi-valued mappings between metric spaces and applied for characterizing metric regularity. Several kinds of local and nonlocal slopes are defined and several metric regularity properties for set-valued mappings between metric spaces are investigated.

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1. Introduction

This article aims to demonstrate how the definitions of slopes which have proved to be very useful tools for analyzing local properties of real-valued functions [1–3, 6, 10–13, 15–17] can be extended to multi-valued mappings between metric spaces and applied for characterizing metric regularity.

Several kinds of local and nonlocal slopes are defined in Section 2 following the scheme developed in [10] for real-valued functions and extended in [4, 5] to vector-valued functions. The idea is not quite new. Some elements of the definitions introduced in the current article are present implicitly in many publications [2, 3, 12, 13, 15, 16]. It seems the definitions can be useful and the time has come to formulate them explicitly.

In this article we investigate several metric regularity properties for set-valued mappings between metric spaces:

- conventional local *metric regularity* and *uniform metric regularity* for mappings depending on a parameter (Section 3);
- *metric regularity along a subspace* (Section 4);
- *metric multi-regularity* for mappings into product spaces (Section 5)

and formulate the corresponding necessary and sufficient regularity criteria in terms of slopes. For the definitions and characterizations of the mentioned above extensions of metric regularity we refer the readers to [8, 9].

Our basic notation is standard, see [14, 18]. Depending on the context, X and Y are either metric or normed spaces. Metrics in all spaces are denoted by the same symbol $d(\cdot,\cdot)$. $d(x, A) = \inf_{a\in A} \|x - a\|$ is the point-to-set distance

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from x to A. When dealing with product spaces we always assume that the product topology is given by the maximum type norm/distance. We also use the denotation $\alpha_+ = \max(\alpha, 0)$, where $\alpha \in \mathbb{R}$.

Recall that a set-valued mapping (multifunction) $F : X \rightrightarrows Y$ is a mapping which assigns to every $x \in X$ a subset (possibly empty) $F(x)$ of Y. As usual, we use the notation $\operatorname{gph} F := \{(x, y) \in X \times Y |\, y \in F(x)\}$ for the graph of F and $F^{-1} : Y \rightrightarrows X$ for the inverse of F. This inverse (which always exists) is defined by $F^{-1}(y) := \{x \in X |\, y \in F(x)\}$, $y \in Y$, and satisfies

$$(x, y) \in \operatorname{gph} F \quad \Leftrightarrow \quad (y, x) \in \operatorname{gph} F^{-1}.$$

2. Slopes

We start with considering an extended-real-valued function f on a metric space X. Recall that the local (strong) *slope* [7] of f at x ($|f(x)| < \infty$) is defined as

$$|\nabla f|(x) := \limsup_{u \to x,\, u \neq x} \frac{[f(x) - f(u)]_+}{d(u, x)}. \tag{1}$$

This quantity provides a convenient characterization of the local behaviour of f near x.

Given a $y \in \mathbb{R}$, we set

$$f_y(x) := \max\{f(x), y\}, \quad x \in X \tag{2}$$

and define the *nonlocal slope* of f at x relative to y:

$$|\nabla f|^{\diamond}_y(x) := \sup_{u \neq x} \frac{[f_y(x) - f_y(u)]_+}{d(u, x)}. \tag{3}$$

If $f(x) < y$, then $f(x) < f_y(u)$ and $f_y(x) < f_y(u)$, and consequently $[f_y(x) - f_y(u)]_+ = [f(x) - f_y(u)]_+ = 0$. Hence, $[f_y(x) - f_y(u)]_+ = [f(x) - f_y(u)]_+$ for all x and u, and subscript y in $f_y(x)$ in the last formula can be removed:

$$|\nabla f|^{\diamond}_y(x) = \sup_{u \neq x} \frac{[f(x) - f_y(u)]_+}{d(u, x)}. \tag{4}$$

As mentioned above, $|\nabla f|^{\diamond}_y(x) = 0$ if $f(x) \le y$. So only the case $f(x) > y$ can be of interest. Note that the supremum in the right-hand side of (3) (or (4)) can be restricted to a certain neighbourhood of x since $[f_y(x) - f_y(u)]_+/d(u, x) \to 0$ as $d(u, x) \to \infty$.

It is easy to see from definitions (1) and (3) that, when $y < f(x)$, the two slopes are related by the inequality:

$$|\nabla f|(x) \le |\nabla f|^{\diamond}_y(x).$$

At the same time, the nonlocal slope (3) is an important ingredient in the definition (1) of the local one: for any $y < f(x)$, it holds

$$|\nabla f|(x) = \lim_{\varepsilon \downarrow 0} |\nabla f_{B_\varepsilon(x)}|^{\diamond}_y(x),$$

where $f_{B_\varepsilon(x)}$ is the restriction of f to $B_\varepsilon(x)$.

The following relations hold true:

$$|\nabla f|(x) = \limsup_{u\to x,\, u\neq x} \frac{[f(x) - \operatorname{cl} f(u)]_+}{d(u,x)}, \quad |\nabla f|^\diamond_y(x) = \sup_{u\neq x} \frac{[f(x) - \operatorname{cl} f_y(u)]_+}{d(u,x)}, \tag{5}$$

where $\operatorname{cl} f$ is the lower semicontinuous envelope of f (defined by $\operatorname{cl} f(x) = \liminf_{u\to x} f(u)$).

In the special case $y = 0$, we will omit y in the denotation of the nonlocal slope. Thus

$$|\nabla f|^\diamond(x) := \sup_{u\neq x} \frac{[f(x) - f_+(u)]_+}{d(u,x)}, \tag{6}$$

where function f_+ is defined by $f_+(x) = [f(x)]_+$. We will refer to (6) simply as the *nonlocal slope* of f at x.

If f takes only nonnegative values, then (6) takes a simpler form:

$$|\nabla f|^\diamond(x) := \sup_{u\neq x} \frac{[f(x) - f(u)]_+}{d(u,x)} \tag{7}$$

and coincides with the global slope defined in [16].

Let $\bar{x} \in X$ and $\bar{y} = f(\bar{x})$, $|\bar{y}| < \infty$. Using (1) and (3), we define respectively the *strict outer* and *uniform strict slopes* [10, 11] of f at $\bar{x}$:

$$\overline{|\nabla f|}^{>}(\bar{x}) := \liminf_{x\to\bar{x},\ f(x)\downarrow f(\bar{x})} |\nabla f|(x), \tag{8}$$

$$\overline{|\nabla f|}^{\diamond}(\bar{x}) := \liminf_{x\to\bar{x},\ f(x)\downarrow f(\bar{x})} |\nabla f|^\diamond_{\bar{y}}(x). \tag{9}$$

The word "strict" reflects the fact that slopes at nearby points contribute to definitions (8) and (9) making them analogues of the strict derivative. The word "outer" is used to emphasize that only points outside the set $S_{\bar{y}}(f) := \{x \in X | f(x) \le \bar{y}\}$ are taken into account. The word "uniform" emphasizes the nonlocal character of $|\nabla f|^\diamond_{\bar{y}}(x)$ involved in definition (9).

Taking into account (5), we have the relations:

$$\overline{|\nabla f|}^{>}(\bar{x}) := \liminf_{x\to\bar{x},\ \operatorname{cl} f(x)\downarrow f(\bar{x})} |\nabla(\operatorname{cl} f)|(x),$$

$$\overline{|\nabla f|}^{\diamond}(\bar{x}) := \liminf_{x\to\bar{x},\ \operatorname{cl} f(x)\downarrow f(\bar{x})} |\nabla(\operatorname{cl} f)|^\diamond_{\bar{y}}(x).$$

Consider now a multifunction $F : X \rightrightarrows Y$ between metric spaces. We are going to define slopes of F using basically the same scheme as described above. To this end, an appropriate scalarization function is needed to replace (2). Given a $y \in Y$, we set

$$f_y(x) := d(y, F(x)), \quad x \in X. \tag{10}$$

Next we apply (1) and (7) to function (10) to define respectively the local and nonlocal slopes of F at x relative to y:

$$|\nabla F|_y(x) := |\nabla f_y|(x) = \limsup_{u\to x,\, u\neq x} \frac{[f_y(x) - f_y(u)]_+}{d(u,x)}, \tag{11}$$

$$|\nabla F|^\diamond_y(x) := |\nabla f_y|^\diamond(x) = \sup_{u\neq x} \frac{[f_y(x) - f_y(u)]_+}{d(u,x)}. \tag{12}$$

The following representations are straightforward:

$$|\nabla F|_y(x) = \limsup_{\substack{u\to x,\, u\neq x\\ v\in F(u)}} \frac{[f_y(x) - d(y,v)]_+}{d(u,x)},$$

$$|\nabla F|^{\diamond}_y(x) = \sup_{\substack{u\neq x\\ v\in F(u)}} \frac{[f_y(x) - d(y,v)]_+}{d(u,x)},$$

as well as the inequality:

$$|\nabla F|_y(x) \le |\nabla F|^{\diamond}_y(x).$$

Given a point $(\bar{x},\bar{y}) \in \operatorname{gph} F$, we now define the *strict outer* and *uniform strict slopes* of F at $(\bar{x},\bar{y})$:

$$\overline{|\nabla F|}^{>}(\bar{x},\bar{y}) := \liminf_{(x,y)\to(\bar{x},\bar{y}),\ f_y(x)\downarrow 0} |\nabla F|_y(x), \tag{13}$$

$$\overline{|\nabla F|}^{\diamond}(\bar{x},\bar{y}) := \liminf_{(x,y)\to(\bar{x},\bar{y}),\ f_y(x)\downarrow 0} |\nabla F|^{\diamond}_y(x). \tag{14}$$

It is easy to check that quantities (13) and (14) do not change if function (10) is replaced in definitions (11), (12), (13), and (14) by its lower semicontinuous envelope. Note also the obvious inequality:

$$\overline{|\nabla F|}^{>}(\bar{x},\bar{y}) \le \overline{|\nabla F|}^{\diamond}(\bar{x},\bar{y}).$$

Example 1. Consider a mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x) = (x_1+x_2, x_1-x_2)$ where $x = (x_1, x_2)$. If $y = (y_1, y_2)$, then

$$f_y(x) = \|y_1 - (x_1 + x_2), y_2 - (x_1 - x_2)\|.$$

Let $x \in \mathbb{R}^2$ and $y \in \mathbb{R}^2$ be such that $f_y(x) > 0$. Denote

$$z_1 := \frac{y_1 + y_2}{2} - x_1 \quad \text{ and } \quad z_2 := \frac{y_1 - y_2}{2} - x_2.$$

Then

$$z_1 + z_2 = y_1 - (x_1 + x_2), \quad z_1 - z_2 = y_2 - (x_1 - x_2),$$

and $\|z_1, z_2\| \neq 0$. Indeed, if we assume that $z_1 = z_2 = 0$, then $x_1 + x_2 = y_1$ and $x_1 - x_2 = y_2$ which contradicts the assumption that $f_y(x) > 0$. Take $u_1 = x_1 + tz_1$, $u_2 = x_2 + tz_2$ for $t > 0$, and $u = (u_1, u_2)$. Then

$$\begin{aligned} f_y(u) &= \|y_1 - (x_1 + x_2) - t(z_1 + z_2), y_2 - (x_1 - x_2) - t(z_1 - z_2)\| \\ &= (1-t)\|z_1 + z_2, z_1 - z_2\| \end{aligned}$$

and

$$\frac{f(x) - f(u)}{d(u,x)} = \frac{\|z_1 + z_2, z_1 - z_2\|}{\|z_1, z_2\|} \ge \gamma > 0,$$

where the positive constant γ depends only on the norm on $\mathbb{R}^2$. For instance, if $\mathbb{R}^2$ is equipped with the maximum type norm, then denoting $\alpha := |z_1|/|z_2|$ if $|z_1| \le |z_2|$ or $\alpha := |z_2|/|z_1|$ otherwise, one has

$$\frac{f(x) - f(u)}{d(u,x)} = \max\{1+\alpha, 1-\alpha\} \ge 1$$

64 and we can take $\gamma = 1$.

By (11) and (12), it follows that $|\nabla F|_y(x) \ge \gamma$ and $|\nabla F|^{\diamond}_y(x) \ge \gamma$. Since x and y are arbitrary, it also follows from (13) and (14) that $\overline{|\nabla F|}^{>}(0,0) \ge \gamma$ and $\overline{|\nabla F|}^{\diamond}(0,0) \ge \gamma$.

3. Metric regularity

Recall (see e.g. [14, 18]) that a multifunction $F : X \rightrightarrows Y$ between metric spaces is said to be *metrically regular* near $(\bar{x},\bar{y}) \in \operatorname{gph} F$ if there exists a $\tau > 0$ and neighbourhoods U and V of $\bar{x}$ and $\bar{y}$ respectively such that

$$d(x, F^{-1}(y)) \le \tau d(y, F(x)), \quad \forall x \in U,\ y \in V. \tag{15}$$

The following (possibly infinite) constant is convenient for characterizing the metric regularity property:

$$r[F](\bar{x},\bar{y}) := \liminf_{\substack{(x,y)\to(\bar{x},\bar{y})\\ (x,y)\notin \operatorname{gph} F}} \frac{d(y,F(x))}{d(x,F^{-1}(y))}. \tag{16}$$

It is easy to check that F is metrically regular near $(\bar{x},\bar{y})$ if and only if $r[F](\bar{x},\bar{y}) > 0$. Moreover, when positive, constant (16) provides a quantitative characterization of this property. It coincides with the reciprocal of the infimum of all positive τ such that (15) holds for some U and V (metric regularity modulus). Constant (16) is also known as the *rate* or *modulus of surjection* or *covering* (see [12, 14]).

The next theorem provides an equivalent characterization of the metric regularity property in terms of slopes (13) and (14). It follows from [16, Theorem 5] where a slightly more general statement is established and formulated without the explicit use of constants (13), (14) and (16).

Theorem 2. *Let X and Y be a complete metric space and a metric space respectively, $F : X \rightrightarrows Y$ be a closed multifunction and $(\bar{x},\bar{y}) \in \operatorname{gph} F$. Then*

$$r[F](\bar{x},\bar{y}) = \overline{|\nabla F|}^{\diamond}(\bar{x},\bar{y}) \ge \overline{|\nabla F|}^{>}(\bar{x},\bar{y}).$$

If, additionally, Y is a normed linear space, then the last inequality holds as equality.

Corollary 3. *Let X and Y be a complete metric space and a metric space respectively, $F : X \rightrightarrows Y$ be a closed multifunction and $(\bar{x},\bar{y}) \in \operatorname{gph} F$. Consider the following conditions:*

(i) *F is metrically regular near $(\bar{x},\bar{y})$;*
(ii) $\overline{|\nabla F|}^{\diamond}(\bar{x},\bar{y}) > 0$*;*
(iii) $\overline{|\nabla F|}^{>}(\bar{x},\bar{y}) > 0$.

Then (iii) $\Rightarrow$ (ii) $\Leftrightarrow$ (i).

Moreover, the following assertions are true:

(a) *if* (15) *holds with some $\tau > 0$, U and V, then $\tau^{-1} \le \overline{|\nabla F|}^{\diamond}(\bar{x},\bar{y})$;*
(b) *if $0 < \tau^{-1} < \overline{|\nabla F|}^{\diamond}(\bar{x},\bar{y})$, then* (15) *holds with some U and V.*

If, additionally, Y is a normed linear space, then $\overline{|\nabla F|}^{\diamond}(\bar{x},\bar{y})$ in (a) *and* (b) *above can be replaced by $\overline{|\nabla F|}^{>}(\bar{x},\bar{y})$.*

Example 4. Considering the linear continuous mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ from Example 1 given by $F(x) = (x_1 + x_2,\, x_1 - x_2)$ where $x = (x_1, x_2)$, we see that it is surjective and consequently metrically regular near $(0, 0)$. This conclusion also follows from Corollary 3 thanks to the estimates for the strict slopes of F established in Example 1.

The statement of Theorem 2 can be extended to the case of set-valued mappings depending on a parameter.

Consider a multifunction $F : P \times X \rightrightarrows Y$, where X and Y are metric spaces and P is a topological space. Denote $F_p = F(p, \cdot) : X \rightrightarrows Y$. Let $(\bar{p}, \bar{x}, \bar{y}) \in \operatorname{gph} F$.

We say that F is *uniformly metrically regular* (see e.g. [8]) near $(\bar{p}, \bar{x}, \bar{y})$ with respect to (x, y) if there exists a $\tau > 0$ and neighbourhoods U, V and W of $\bar{x}$, $\bar{y}$ and $\bar{p}$ respectively such that

$$d(x, F_p^{-1}(y)) \le \tau d(y, F(p, x)), \quad \forall x \in U,\ y \in V,\ p \in W. \tag{17}$$

This property can be equivalently characterized using the following analogue of (16):

$$r_{\bar{p}}[F](\bar{x}, \bar{y}) := \liminf_{\substack{(p,x,y)\to(\bar{p},\bar{x},\bar{y}) \\ (p,x,y)\notin \operatorname{gph} F}} \frac{d(y, F(p, x))}{d(x, F_p^{-1}(y))}. \tag{18}$$

F is uniformly metrically regular near $(\bar{p}, \bar{x}, \bar{y})$ with respect to (x, y) if and only if $r_{\bar{p}}[F](\bar{x}, \bar{y}) > 0$.

To formulate uniform metric regularity criteria in terms of slopes, some modifications of definitions (10) – (14) are required:

$$f_{y,p}(x) := d(y, F(p, x)), \quad x \in X, \tag{19}$$

$$|\nabla F|_{y,p}(x) := |\nabla f_{y,p}|(x) = \limsup_{u\to x,\, u\ne x} \frac{[f_{y,p}(x) - f_{y,p}(u)]_+}{d(u, x)}, \tag{20}$$

$$|\nabla F|^{\diamond}_{y,p}(x) := |\nabla f_{y,p}|^{\diamond}(x) = \sup_{u\ne x} \frac{[f_{y,p}(x) - f_{y,p}(u)]_+}{d(u, x)}, \tag{21}$$

$$\overline{|\nabla F|}^{>}_{\bar{p}}(\bar{x}, \bar{y}) := \liminf_{(p,x,y)\to(\bar{p},\bar{x},\bar{y}),\ f_{y,p}(x)\downarrow 0} |\nabla F|_{y,p}(x), \tag{22}$$

$$\overline{|\nabla F|}^{\diamond}_{\bar{p}}(\bar{x}, \bar{y}) := \liminf_{(p,x,y)\to(\bar{p},\bar{x},\bar{y}),\ f_{y,p}(x)\downarrow 0} |\nabla F|^{\diamond}_{y,p}(x). \tag{23}$$

The required characterization of the uniform metric regularity property in terms of slopes (22) and (23) is similar to the one provided by Theorem 2 and follows from [16, Theorem 8], the latter one being formulated without slopes (22) and (23) and regularity constant (18).

Theorem 5. *Let X, Y and P be a complete metric space, a metric space and a topological space respectively, $F : P \times X \rightrightarrows Y$ be a closed multifunction and $(\bar{p}, \bar{x}, \bar{y}) \in \operatorname{gph} F$. Then*

$$r_{\bar{p}}[F](\bar{x}, \bar{y}) = \overline{|\nabla F|}^{\diamond}_{\bar{p}}(\bar{x}, \bar{y}) \ge \overline{|\nabla F|}^{>}_{\bar{p}}(\bar{x}, \bar{y}).$$

If, additionally, Y is a normed linear space, then the last inequality holds as equality.

Corollary 6. *Let X, Y and P be a complete metric space, a metric space and a topological space respectively, $F : P \times X \rightrightarrows Y$ be a closed multifunction and $(\bar{p}, \bar{x}, \bar{y}) \in \operatorname{gph} F$. Consider the following conditions:*

(i) *F is uniformly metrically regular near $(\bar{p}, \bar{x}, \bar{y})$;*

(ii) *$\overline{|\nabla F|}^{\diamond}_{\bar{p}}(\bar{x}, \bar{y}) > 0$;*

(iii) *$\overline{|\nabla F|}^{>}_{\bar{p}}(\bar{x}, \bar{y}) > 0$.*

Then (iii) $\Rightarrow$ (ii) $\Leftrightarrow$ (i).

Moreover, the following assertions are true:

(a) *if* (17) *holds with some $\tau > 0$, U, V and W, then $\tau^{-1} \le \overline{|\nabla F|}^{\diamond}_{\bar{p}}(\bar{x}, \bar{y})$;*

(b) *if $0 < \tau^{-1} < \overline{|\nabla F|}^{\diamond}_{\bar{p}}(\bar{x}, \bar{y})$, then* (17) *holds with some U, V and W.*

If, additionally, Y is a normed linear space, then $\overline{|\nabla F|}^{\diamond}_{\bar{p}}(\bar{x}, \bar{y})$ in (a) *and* (b) *above can be replaced by $\overline{|\nabla F|}^{>}_{\bar{p}}(\bar{x}, \bar{y})$.*

4. Metric regularity along a subspace

Consider a multifunction $F : X \rightrightarrows Y$ from a normed linear space to a metric space. Let H be a (closed) subspace of X. F is called *metrically regular along H* [9] near $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if there exists a $\tau > 0$ and neighbourhoods U and V of $\bar{x}$ and $\bar{y}$ respectively such that

$$\inf_{h \in H} \{\|h\| \,|\, x + h \in F^{-1}(y)\} \le \tau d(y, F(x)), \quad \forall x \in U,\ y \in V. \tag{24}$$

Obviously, if $H = X$, then this property coincides with the conventional metric regularity of F near $(\bar{x}, \bar{y})$.

In the definition of metric regularity along H, it is convenient to use the point-to-set *distance along H* defined for $x \in X$ and $M \subset X$ as

$$d_H(x, M) := \inf_{h \in H} \{\|h\| \,|\, x + h \in M\} = d(0, (M - x) \cap H).$$

Of course, it is not a real distance on X. For instance, $d_H(x_1, x_2) = \infty$ if $x_1 - x_2 \notin H$. In general, $d_H(x, M) \ge d(x, M)$, and the equality holds when $H = X$.

The above property can be equivalently characterized using the following constant:

$$r_H[F](\bar{x}, \bar{y}) := \liminf_{\substack{(x,y) \to (\bar{x}, \bar{y}) \\ (x,y) \notin \operatorname{gph} F}} \frac{d(y, F(x))}{d_H(x, F^{-1}(y))}. \tag{25}$$

F is metrically regular along H near $(\bar{x}, \bar{y})$ if and only if $r_H[F](\bar{x}, \bar{y}) > 0$.

Evidently, $r_H[F](\bar{x}, \bar{y}) \le r[F](\bar{x}, \bar{y})$, and metric regularity of F along some subspace implies its conventional metric regularity.

The metric regularity along a subspace can be treated in the framework of the aforementioned property of parametric metric regularity.

For multifunction $F : X \rightrightarrows Y$, define another multifunction $\Phi : X \times H \rightrightarrows Y$ by the formula

$$\Phi(x, h) := F(x + h), \quad x \in X,\ h \in H. \tag{26}$$

Then, for this multifunction, X can be viewed as a parameter space and the above parametric definitions can be reformulated for this particular case, the point $\bar{h} = 0$ being of special interest. The next proposition (cf. [9, Proposition 4.1 (iii)]) shows that the uniform metric regularity of Φ near $(\bar{x}, 0, \bar{y})$ is exactly the metric regularity of F near $(\bar{x}, \bar{y})$ along H.

Proposition 7. *Let the mapping $\Phi : X \times H \rightrightarrows Y$ be defined by (26). Then $r_H[F](\bar{x}, \bar{y}) = r_{\bar{x}}[\Phi](0, \bar{y})$.*

Proof. Taking into account (26) and the obvious relations

$$F(x) = \Phi(x, 0), \quad \Phi_x^{-1}(y) = (F^{-1}(y) - x) \cap H, \quad d(h, \Phi_x^{-1}(y)) = d_H(x+h, F^{-1}(y)),$$

we have:

$$r_{\bar{x}}[\Phi](0, \bar{y}) = \liminf_{\substack{(x,h,y)\to(\bar{x},0,\bar{y}) \\ (x+h,y)\notin \operatorname{gph} F}} \frac{d(y, F(x+h))}{d_h(x+h, F^{-1}(y))}$$

$$\leq \liminf_{\substack{(x,y)\to(\bar{x},\bar{y}) \\ (x,y)\notin \operatorname{gph} F}} \frac{d(y, F(x))}{d_h(x, F^{-1}(y))} = r_H[F](\bar{x}, \bar{y}).$$

On the other hand,

$$r_{\bar{x}}[\Phi](0, \bar{y}) = \lim_{\delta\downarrow 0} \inf_{\substack{(x,h,y)\in B_\delta(\bar{x},0,\bar{y}) \\ (x+h,y)\notin \operatorname{gph} F}} \frac{d(y, F(x+h))}{d_h(x+h, F^{-1}(y))}$$

$$= \lim_{\delta\downarrow 0} \inf_{\substack{x\in B_{2\delta}(\bar{x}),\, y\in B_\delta(\bar{y}) \\ (x,y)\notin \operatorname{gph} F}} \frac{d(y, F(x))}{d_h(x, F^{-1}(y))}$$

$$\geq \lim_{\delta\downarrow 0} \inf_{\substack{(x,y)\in B_{2\delta}(\bar{x},\bar{y}) \\ (x,y)\notin \operatorname{gph} F}} \frac{d(y, F(x))}{d_h(x, F^{-1}(y))} = r_H[F](\bar{x}, \bar{y}).$$

□

Formulas (20) – (23) applied to multifunction (26) lead to the following definitions:

$$|\nabla F|_{y,H}(x) := \limsup_{u\to x,\, u\neq x,\, u-x\in H} \frac{[f_y(x) - f_y(u)]_+}{d(u, x)}, \tag{27}$$

$$|\nabla F|^{\diamond}_{y,H}(x) := \sup_{u\neq x,\, u-x\in H} \frac{[f_y(x) - f_y(u)]_+}{d(u, x)}, \tag{28}$$

$$\overline{|\nabla F|}^{>}_{H}(\bar{x}, \bar{y}) := \liminf_{(x,y)\to(\bar{x},\bar{y}),\, f_y(x)\downarrow 0} |\nabla F|_{y,H}(x), \tag{29}$$

$$\overline{|\nabla F|}^{\diamond}_{H}(\bar{x}, \bar{y}) := \liminf_{(x,y)\to(\bar{x},\bar{y}),\, f_y(x)\downarrow 0} |\nabla F|^{\diamond}_{y,H}(x), \tag{30}$$

where f_y is defined by (10).

The next theorem is a consequence of Theorem 5.

Theorem 8. *Let X and Y be a Banach space and a metric space respectively, $F : X \rightrightarrows Y$ be a closed multifunction and $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Suppose H is a subspace of X. Then*

$$r_H[F](\bar{x}, \bar{y}) = \overline{|\nabla F|}^{\diamond}_{H}(\bar{x}, \bar{y}) \geq \overline{|\nabla F|}^{>}_{H}(\bar{x}, \bar{y}).$$

If, additionally, Y is a normed linear space, then the last inequality holds as equality.

Corollary 9. *Let X and Y be a Banach space and a metric space respectively, $F : X \rightrightarrows Y$ be a closed multifunction and $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Suppose H is a subspace of X. Consider the following conditions:*

(i) *F is metrically regular along H near $(\bar{x}, \bar{y})$;*
(ii) $\overline{|\nabla F|}^{\diamond}_{H}(\bar{x}, \bar{y}) > 0$*;*
(iii) $\overline{|\nabla F|}^{>}_{H}(\bar{x}, \bar{y}) > 0$.

Then (iii) $\Rightarrow$ (ii) $\Leftrightarrow$ (i).

Moreover, the following assertions are true:

(a) *if* (24) *holds with some $\tau > 0$, U and V, then $\tau^{-1} \le \overline{|\nabla F|}^{\diamond}_{H}(\bar{x}, \bar{y})$;*
(b) *if $0 < \tau^{-1} < \overline{|\nabla F|}^{\diamond}_{H}(\bar{x}, \bar{y})$, then* (24) *holds with some U and V.*

If, additionally, Y is a normed linear space, then $\overline{|\nabla F|}^{\diamond}_{H}(\bar{x}, \bar{y})$ in (a) *and* (b) *above can be replaced by $\overline{|\nabla F|}^{>}_{H}(\bar{x}, \bar{y})$.*

Example 10. Consider again the mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x) = (x_1 + x_2,\, x_1 - x_2)$ where $x = (x_1, x_2)$. As established in Examples 1 and 4, it is metrically regular near $(0, 0)$. We are going to show that it is not metrically regular near $(0, 0)$ along the subspace $H = \mathbb{R} \times \{0\}$. For simplicity, we assume that $\mathbb{R}^2$ is equipped with the maximum type norm. Take $x = (0, \alpha)$ with $\alpha \ne 0$ and $y = (0, 0)$. Then $f_y(x) = \|-\alpha, \alpha\| = |\alpha|$ and, for any $h = (\beta, 0) \in H$, $f_y(x + h) = \|-(\alpha + \beta), \alpha - \beta\| = \max\{|\alpha + \beta|, |\alpha - \beta|\} \ge |\alpha|$. Hence, $\overline{|\nabla F|}^{\diamond}_{H}(0, 0) = \overline{|\nabla F|}^{>}_{H}(0, 0) = |\nabla F|^{\diamond}_{y,H}(x) = |\nabla F|_{y,H}(x) = 0$. The claimed assertion follows from Corollary 9.

5. Metric multi-regularity

Let $F : X \rightrightarrows Y$ be a mapping between a normed linear space X and the product of $n \ge 1$ metric spaces $Y = Y_1 \times Y_2 \times \ldots \times Y_n$. Throughout this section we assume that F can be represented as $F = (F_1, F_2, \ldots, F_n)$, where each F_i is a mapping from X into Y_i. This means that for any $x \in X$ its image $F(x)$ under F is the product of the images:

$$F(x) = F_1(x) \times F_2(x) \times \ldots \times F_n(x). \tag{31}$$

If F is single-valued this assumption is fulfilled automatically.

Let $\bar{x} \in X$ and $\bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n) \in F(\bar{x})$.

Besides considering the metric regularity of F, one can also examine this property componentwise. The next proposition which strengthens [9, Proposition 5.2 (ii)] shows that the metric regularity of F implies the metric regularity of all its components.

Proposition 11. $r[F](\bar{x}, \bar{y}) \le \min_{1 \le i \le n} r[F_i](\bar{x}, \bar{y}_i)$.

Proof. If $r[F](\bar{x}, \bar{y}) = 0$, the inequality holds true trivially. Let $r[F](\bar{x}, \bar{y}) > 0$. Take any neighbourhoods U of $\bar{x}$ and $V = V_1 \times V_2 \times \ldots \times V_n$ of $\bar{y}$. By definition (16), taking a smaller U if necessary, we can ensure that $F(x) \cap V \ne \emptyset$ for all $x \in U$. Take any i, $1 \le i \le n$, any $x \in U$ and any $y_i \in V_i$. For all

$j \neq i$ take some $y_j \in F_j(x) \cap V_j$ and compose $y = (y_1, y_2, \ldots, y_n)$. Then $y \in V$, $d(y, F(x)) = d(y_i, F_i(x))$ and $d(x, F^{-1}(y)) = d(x, F_i^{-1}(y_i))$. By the definition (16), $r[F](\bar{x}, \bar{y}) \leq r[F_i](\bar{x}, \bar{y}_i)$. Since this inequality is valid for any i, the assertion has been proved. □

The inequality in Proposition 11 can be strict [9, Example 5.3].

There is another way of dealing with mappings into product spaces. The following local regularity property of F near $(\bar{x}, \bar{y})$, taking into account the behaviour of its components, can be of interest.

F is called *metrically multi-regular* [8] at $(\bar{x}, \bar{y})$ if there exists a $\tau > 0$ and neighbourhoods U of $\bar{x}$ and V_i of $\bar{y}_i$, $i = 1, 2, \ldots, n$, such that

$$d(0, \bigcap_{i=1}^{n} (F_i^{-1}(y_i) - x_i)) \leq \tau \max_{1 \leq i \leq n} d(y_i, F_i(x_i)),$$
$$\forall x_i \in U,\ y_i \in V_i,\ i = 1, 2, \ldots, n. \quad (32)$$

Obviously, when $n = 1$, the above property coincides with the conventional one. When $n > 1$, this property is stronger than the metric regularity which corresponds to taking $x_i = \bar{x}$, $i = 1, 2, \ldots, n$, in the above definition.

A multifunction $F : X \rightrightarrows Y$ of the type (31) can be used, for instance, to define a system of *generalized equations*:

$$0_{Y_i} \in F_i(x), \quad i = 1, 2, \ldots, n. \quad (33)$$

If $\bar{x}$ is a solution of (33), then metric multi-regularity of F at $(\bar{x}, 0)$ means the existence of a joint "stabilizing" action satisfying an "error bound" type estimate when both the right-hand sides and variables of each of the generalized equations are perturbed independently.

The following constant corresponds to the above metric multi-regularity property:

$$\hat{r}[F](\bar{x}, \bar{y}) := \liminf_{\substack{(x_i, y_i) \to (\bar{x}, \bar{y}_i),\, i = 1, 2, \ldots, n \\ (y_1, \ldots, y_n) \notin F_1(x_1) \times \ldots \times F_n(x_n)}} \frac{\max\limits_{1 \leq i \leq n} d(y_i, F_i(x_i))}{d(0, \bigcap\limits_{i=1}^{n} (F_i^{-1}(y_i) - x_i))}. \quad (34)$$

Its relationship with (16) is straightforward:

$$\hat{r}[F](\bar{x}, \bar{y}) \leq r[F](\bar{x}, \bar{y}),$$

where the equality holds if $n = 1$.

The metric multi-regularity property can be treated in the framework of the metric regularity along a subspace examined above. Indeed, let $Z = X^n$ and $z = (x_1, x_2, \ldots, x_n) \in Z$. One can consider multifunction $\Phi : Z \rightrightarrows Y$ defined by

$$\Phi(z) = F_1(x_1) \times F_2(x_2) \times \ldots \times F_n(x_n). \quad (35)$$

Note that each "component" of Φ in the above formula depends on its own argument.

In the space Z, one can consider the diagonal subspace

$$H = \{(x_1, x_2, \ldots, x_n) \in X^n |\, x_1 = x_2 = \ldots = x_n\}. \quad (36)$$

Evidently, $\Phi(z) = F(x)$ if $z = (x, x, \ldots, x) \in H$, and $(\bar{z}, \bar{y}) \in \operatorname{gph}\Phi$, where $\bar{z} = (\bar{x}, \bar{x}, \ldots, \bar{x})$.

The next proposition shows that the metric regularity of Φ near $(\bar{z}, \bar{y})$ along H is exactly the metric multi-regularity of F near $(\bar{x}, \bar{y})$ (cf. [9, Proposition 5.5 (iv)]).

Proposition 12. *Let multifunction $\Phi : Z \rightrightarrows Y$ and subspace H of Z be defined by* (35) *and* (36) *respectively. Then $\hat{r}[F](\bar{x}, \bar{y}) = r_H[\Phi](\bar{z}, \bar{y})$.*

Proof. It follows immediately from definition (35) that, for any $z = (x_1, x_2, \ldots, x_n) \in Z$ and $y = (y_1, y_2, \ldots, y_n) \in Y$, one has

$$d(y, \Phi(z)) = \max_{1 \le i \le n} d(y_i, F_i(x_i)),$$
$$\Phi^{-1}(y) = F_1^{-1}(y_1) \times F_2^{-1}(y_2) \times \ldots \times F_n^{-1}(y_n),$$
$$d_H(z, \Phi^{-1}(y)) = d(0, \bigcap_{i=1}^{n} (F_i^{-1}(y_i) - x_i)).$$

The assertion follows by comparing definitions (25) and (34). $\square$

Formulas (27) – (30) applied to multifunction (35) and subspace (36) lead to the following definitions where $\hat{y} = (y_1, y_2, \ldots, y_n) \in Y$:

$$f_y^i(x) := d(y, F_i(x)), \quad x \in X, \ y \in Y_i, \tag{37}$$

$$f_{\hat{y}}(x_1, \ldots, x_n) := \max_{1 \le i \le n} f_{y_i}^i(x_i), \tag{38}$$

$$|\nabla F|_{\hat{y}}(x_1, \ldots, x_n) := \limsup_{0 \ne u \to 0_X} \frac{[f_{\hat{y}}(x_1, \ldots, x_n) - f_{\hat{y}}(x_1 + u, \ldots, x_n + u)]_+}{\|u\|}, \tag{39}$$

$$|\nabla F|_{\hat{y}}^{\diamond}(x_1, \ldots, x_n) := \sup_{u \ne 0_X} \frac{[f_{\hat{y}}(x_1, \ldots, x_n) - f_{\hat{y}}(x_1 + u, \ldots, x_n + u)]_+}{\|u\|}, \tag{40}$$

$$\widehat{\overline{|\nabla F|}}{}^{>}(\bar{x}, \bar{y}) := \liminf_{\substack{(x_i, y_i) \to (\bar{x}, \bar{y}),\, i = 1, 2, \ldots, n \\ f_{\hat{y}}(x_1, \ldots, x_n) \downarrow 0}} |\nabla F|_{\hat{y}}(x_1, \ldots, x_n), \tag{41}$$

$$\widehat{\overline{|\nabla F|}}{}^{\diamond}(\bar{x}, \bar{y}) := \liminf_{\substack{(x_i, y_i) \to (\bar{x}, \bar{y}),\, i = 1, 2, \ldots, n \\ f_{\hat{y}}(x_1, \ldots, x_n) \downarrow 0}} |\nabla F|_{\hat{y}}^{\diamond}(x_1, \ldots, x_n). \tag{42}$$

Application of Theorem 8 to the setting of metric multi-regularity yields the following statement.

Theorem 13. *Let X be a Banach space and $Y = Y_1 \times Y_2 \times \ldots \times Y_n$ be the product of $n \ge 1$ metric spaces. Suppose that $F : X \rightrightarrows Y$ is a closed multifunction which can be represented as $F = (F_1, F_2, \ldots, F_n)$ where $F_i : X \rightrightarrows Y_i$, $i = 1, 2, \ldots, n$, and $\bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n) \in F(\bar{x})$. Then*

$$\hat{r}[F](\bar{x}, \bar{y}) = \widehat{\overline{|\nabla F|}}{}^{\diamond}(\bar{x}, \bar{y}) \ge \widehat{\overline{|\nabla F|}}{}^{>}(\bar{x}, \bar{y}).$$

If, additionally, Y is a normed linear space, then the last inequality holds as equality.

Corollary 14. *Let X be a Banach space and $Y = Y_1 \times Y_2 \times \ldots \times Y_n$ be the product of $n \geq 1$ metric spaces. Suppose that $F : X \rightrightarrows Y$ is a closed multifunction which can be represented as $F = (F_1, F_2, \ldots, F_n)$ where $F_i : X \rightrightarrows Y_i$, $i = 1, 2, \ldots, n$, and $\bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n) \in F(\bar{x})$. Consider the following conditions:*

(i) *F is metrically multi-regular near $(\bar{x}, \bar{y})$;*

(ii) $\widehat{\overline{|\nabla F|}}^{\diamond}(\bar{x}, \bar{y}) > 0$;

(iii) $\widehat{\overline{|\nabla F|}}^{>}(\bar{x}, \bar{y}) > 0$.

Then (iii) $\Rightarrow$ (ii) $\Leftrightarrow$ (i).

Moreover, the following assertions are true:

(a) *if* (32) *holds with some $\tau > 0$, U, $V_1,\ldots,V_n$, then $\tau^{-1} \leq \widehat{\overline{|\nabla F|}}^{\diamond}(\bar{x}, \bar{y})$;*

(b) *if $0 < \tau^{-1} < \widehat{\overline{|\nabla F|}}^{\diamond}(\bar{x}, \bar{y})$, then* (32) *holds with some U, $V_1,\ldots,V_n$.*

If, additionally, $Y_1, \ldots, Y_n$ are normed linear spaces, then $\widehat{\overline{|\nabla F|}}^{\diamond}(\bar{x}, \bar{y})$ in (a) *and* (b) *above can be replaced by $\widehat{\overline{|\nabla F|}}^{>}(\bar{x}, \bar{y})$.*

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