Monitoring Progress in Algebra in a CAS Active Context: Symbol Sense, Algebraic Insight and Algebraic Expectation

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The purpose of this paper is to provide researchers with a shared framework, terminology and tool to improve the coherence of research into learning mathematics with CAS and to assist its findings to accumulate into a significant body of knowledge. Experience with calculators in arithmetic led to a framework for number sense. There is an obvious parallel for algebra, where the development of algebraic insight to monitor symbolic work will assume high importance. We present a framework for algebraic insight then explore one aspect, algebraic expectation, in detail. Just as estimation is a valued skill for monitoring arithmetic calculations, we suggest that expectation should be a focus in teaching algebra, especially when symbolic technology is available. Through typical examples, we demonstrate the value of the algebraic insight framework for monitoring students' work with CAS.

1. INTRODUCTION

Faced with the increasing availability of Computer Algebra Systems (CAS) for doing, teaching and learning mathematics, both teachers and researchers question what algebra should be taught. They fear that students will merely replace the memorisation of algebraic manipulation routines with the memorisation of calculator specific button sequences. In a CAS active context, what aspects of algebra are important and what aspects should we be monitoring in order to judge a student's progression in developing facility with algebra? We propose that the answer to this question is the construct we call 'algebraic insight' and we have organised the components of this construct in a framework that is useful as a guide to both teachers and researchers monitoring students' progress. Pierce (2002) reported the details of a study which provided the context for the development of this framework and in a related paper, Pierce and Stacey (2002), we described the place of this construct when teaching algebra. In this paper we emphasise the need for such a framework for research, explain the components of algebraic insight and then illustrate the use of the framework in monitoring individual student's algebra progress. First we situate our thinking in terms of three key papers from the 1990s related to this topic.

2. NUMBER SENSE AND SYMBOL SENSE

The last twenty-five years has seen the increasing availability and affordability of technology that will carry out routine mathematical processes in educational settings: first arithmetic calculators and now, graphical calculators and symbolic manipulators. Once students became able to use four function calculators to do arithmetic, researchers and educators saw more clearly than before, that there is more to arithmetic than calculation. Now symbolic manipulators, in the form of Computer Algebra Systems, are doing the same for algebra.

After a decade or more of experience of teaching with four function calculators, the consensus amongst educators was that a new emphasis for arithmetic could be placed on the understanding and ability to plan, monitor, estimate and interpret arithmetic calculations. This ability, described by the term *number sense* is well summarised by McIntosh, Reys and Reys (1992) who gave a comprehensive definition of number sense in a framework (see Appendix 1) that organised its component parts. McIntosh et al said that this framework was not an exhaustive listing of all possible components of number sense (an impossible task) but ‘an attempt to articulate a structure which clarifies, organises, and interrelates some of the generally agreed upon components of number sense’ (p5). We aim to put forward an equivalent framework for algebra.

While students working with the three representations of functions offered by CAS still need number sense, they also require equivalent abilities for working with algebraic symbols and graphs which, by analogy, have been called *symbol sense and graph sense.* There have been two important attempts to describe symbol sense, by Fey (1990) and Arcavi (1994). First, Fey (1990), who was reflecting on the impact of technology teaching and learning mathematics, suggested five basic abilities (see Appendix 2) that are each part of the thinking that enables a mathematician to recognise equivalent expressions or form an expectation of the nature of the result of a problem. The construct put forward in this paper as *algebraic insight* encompasses the abilities touched on by Fey (1990) but expands these to a more comprehensive set. Second, Arcavi (1994) proposed a very general interpretation of symbol sense (see Appendix 3), which applies across the problem solving process when formulating a problem algebraically, solving the mathematically formulated problem and interpreting the results in terms of the original problem. More recently Boero (2001) has described *algebraic anticipation*, the ability not only to apply standard patterns of transformations but also to

> …foresee some aspects of the final shape of the object to be transformed related to the goal to be reached, and some possibilities of transformation. This ‘anticipation’ allows planning and continuous feedback. In the case of transformations after formalisation, anticipation is based on some peculiar properties of the external algebraic expression. (p99).

Boero’s algebraic anticipation is also related to both symbol sense and algebraic insight, and appears to be similar to the concept described below as algebraic expectation.

Algebraic insight impacts on ‘solving’ symbolically formulated problems

Figure 1 illustrates where algebraic insight is located with respect to Arcavi’s more general ‘symbol sense’, in terms of a basic model of the problem solving process. Symbol sense is involved in the formulating, solving and interpreting stages. Algebraic insight, designed to capture the insight that a student needs to work with algebraic symbols in transformational activity (Kieran, 1996), is relevant only in the “solving” phase of the problem solving process.

Figure 1 also shows that algebraic insight is relevant whether or not one is using CAS, but it is the CAS environment that has highlighted its importance and in turn, it is a concept that is especially relevant for studying students’ progress when using CAS. Many of the symbol sense abilities described by Arcavi (1994) are unaffected by the availability of CAS since CAS only performs manipulations and calculations facilitated by algorithmic routines; it does not set up a model, nor decide how best to solve a problem; nor does it interpret results.

INSERT FIGURE 1 MODEL FOR PROBLEM SOLVING HERE

Figure 1 A model of problem solving showing the places of symbol sense and algebraic insight (Pierce & Stacey , 2002).

Figure 2 demonstrates the relationships between the various constructs in this paper from a set-theoretic viewpoint. The whole picture is divided into three sections: number sense, graph sense and symbol sense (as described by Arcavi) relevant to dealing with algebraic work in symbolic, graphical and numeric representations. The right hand side of figure 2 indicates that algebraic insight is mostly but not entirely within symbol sense and that it is divided into two parts, algebraic expectation and the ability to link the symbolic to other representations, which will be described below.

INSERT FIGURE 2 (CIRCLE) HERE

Figure 2 The place of algebraic insight and its components within the senses needed when working with CAS.

The framework presented below organises and exemplifies these constructs; then its use as a guide for analysing students' work and hence monitoring their progress is explored through typical examples observed when students work with CAS. The framework we propose is based on both our experience of students and that reported in other studies. The next section emphasises the need for such a framework in research.

3. THE NEED FOR AN AGREED FRAMEWORK

One of the purposes of proposing and delineating the construct of *algebraic insight* is for use when researching the links between algebraic knowledge and the use of CAS, especially its symbolic manipulation facility. Algebraic insight is a useful concept to guide teaching (as will be illustrated in our examples below) but our main intention for this paper is that a careful delineation will provide researchers with a shared framework, terminology and tool to improve the coherence of research and to assist its findings to accumulate into a significant body of knowledge.

In this, we are motivated by recent proposals that education researchers should more frequently use the same variables, theoretical constructs and measures across different research studies. The USA National Research Council's *Committee on Scientific Principles for Educational Research* (Shavelson and Towne, 2002) has identified this as a priority to support the accumulation of research-based knowledge in education. The RAND Mathematics Study Panel (2003) goes further when it proposes that US government agencies might consider enhancing the infrastructure for research and development by "a special effort to assemble and, where necessary, develop measurement instruments and technology that could be widely used by researchers, and thus enhance the opportunities for comparing and contrasting findings of various research efforts" (p 70). Making a similar point, Burkhardt and Schoenfeld (2003) call for "a requirement to justify *not* using established instruments and methods" in order to increase comparability of studies. Shavelson and Towne, RAND, and Burkhardt and Schoenfeld all call for shared instruments as well as shared constructs. Space limitations preclude presentation of instruments to assess algebraic insight here, but suggestions with reports of results from students can be found in Ball, Stacey and Pierce (2001), Pierce (2002) and Ball, Pierce and Stacey (2003).

4. A FOCUS WITHIN SYMBOL SENSE: ALGEBRAIC INSIGHT

We see competence in algebraic insight as having two aspects: *algebraic expectation* and *ability to link representations*. Since the use of graphical calculators has already raised much discussion of the thinking involved in *linking representations* (see for example Dick, 1992) its place as part of algebraic insight is established in the framework but it is not discussed in depth in this paper where we focus instead on describing algebraic expectation: the thinking that must accompany the formal symbolic operations and transformational activity of algebra even when CAS is available.

The term *algebraic expectation* is used here to name the thinking process that takes place when an experienced mathematician considers the nature of the result they expect to obtain as the outcome of some algebraic (and symbolic) process. For example, it takes place when a mathematician looks at two expressions and decides, without doing any explicit calculations or manipulations, whether they are likely to be equivalent.

Algebraic expectation does not necessarily involve producing an approximate solution as in arithmetic “estimation” (there are differences in what can be done mentally with algebra and arithmetic) but rather noticing these ‘peculiar properties’, conventions, structure, and key features of an expression that determine features which may be expected in the solution. Students need to develop skill with scanning expressions for these clues that allow us to see and predict patterns, and make sense of symbolic operations.

5. A FRAMEWORK FOR ALGEBRAIC INSIGHT

The framework presented in table 1 aims to articulate a structure that clarifies, organises, and interrelates key elements of algebraic insight. It is not proposed as a catalogue of specific, itemised skills. The divisions within the framework are neither mutually exclusive nor exhaustive. It is an attempt to analyse what it is that ‘expert’ mathematicians do when they look at a result for an algebraic problem and say ‘there is a mistake here’ or ‘that looks OK’. This is the thinking used in what the problem solving literature (see for example, Schoenfeld, 1985) calls ‘monitoring’ or ‘control’. Mathematics teaching has, of necessity, focused a great deal of time and attention on algorithmic routines. Since CAS does these effectively, perhaps attention can be directed towards deliberately teaching these skills of *algebraic insight.*

As is shown in figure 2, and column 1 of table1, two aspects, *algebraic expectation* and *ability to link representations,* form a logical division of algebraic insight. Algebraic expectation focuses on the application of algebraic insight *within* the symbolic representation of mathematics. A simple example is ‘knowing to expect 0, 1 or 2 real solutions to a quadratic equation’. The ability to link representations deals with the students’ ability to move cognitively between symbolic (algebraic) representations and graphical or numeric representations. Such linking is also concerned with expectations, but expectations across representations. Algebraic insight will be shown when a student has expectations about graphs and tables that are linked to features of the symbolic representation; for example when a student asks and answers such questions as:

> What will the graph of the rule $y= x^2 +5$ look like? Should I expect it to cross the x-axis? If I am to construct a table of values for this rule, what might be a suitable increment to use?

The framework suggests three key elements of algebraic estimation and two key elements of ability to link representations. These elements are shown in column 2 of table 1 and each element is then illustrated in column 3 by typical *common instances*. Such common instances of the elements of each aspect may be seen when students demonstrate the abilities listed in the third column. The column 3 abilities do not form a definitive list but were selected from examples observed by the first author while teaching a functions and introductory calculus course for undergraduate students. In general, while the aspects and the elements of algebraic insight apply at any level, details of common instances will be specific to both age and stage.

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6. ALGEBRAIC INSIGHT – WHAT CAN WE LEARN FROM STUDENTS WORKING WITH CAS?

Algebraic insight, although important whether working by-hand or with technology, is brought into sharper focus when a CAS is available to perform routine processes. Observing and measuring students' algebraic insight is therefore an appropriate way to monitor students' progress in algebra when working in a CAS active context. We therefore illustrate our choice of key aspects and elements for algebraic insight, through examples of 'typical' students working with CAS in a functions and introductory calculus course. Again, because the ability to link representations has had considerable prominence in research on the use of graphical calculators, we have chosen examples that highlight algebraic expectation.

The action of using the symbolic module of a CAS to perform formal symbolic operations can be divided into four stages: first *choosing* whether to use CAS or not, second keying in or *entering* an expression, third *monitoring* the solution processes and finally *confirming* the solution(s). While it is possible that a student may use CAS to obtain correct solutions to some problems by merely pushing buttons in a prescribed or memorised sequence, constructive progress in mathematical understanding and the application of techniques to new problems will be facilitated when effective use of CAS (Pierce & Stacey, in press) is accompanied by well developed algebraic expectation.

Algebraic Expectation informs decisions about when to use CAS

Faced with a symbolically formulated problem, a student must make decisions about likely solution paths. If CAS is available one of the choices to be made is whether to use it or not. Just because it is available does not mean that its facility offers the most efficient means of solving a problem. Learning to use CAS effectively includes learning to make judicious choices about its use (Pierce & Stacey, in press.).

Students typically decide to use CAS for problems that they expect will be difficult or time-consuming (Pierce, 2002). Identification of objects (Framework code 1.2.1), recognition of simple factors (1.2.3) and identifying form (1.3.1) are common instances of algebraic expectation fundamental to such judgements. For example a student who, on reading $f(x) = \sin^2(2x+1) + \cos^2(2x+1)$, identified $2x+1$ as an object and recognised the form of the trigonometric identity, therefore realising that $f(x) = 1$, would hardly proceed by entering the original form of the function into CAS. Similarly a student recognising $(x+1)$ as an object and simple common factor would most likely simplify the expression $\frac{a(x+1)^5 + b(x+1)^2}{x+1}$ by hand since this would be faster than entering this long expression into a CAS. Entering expressions carries the risk of typographical and syntax errors along with a need for algebraic expectation, as illustrated by observing a typical student like Alice described below.

Tall and Thomas (1991) emphasised the importance of a structural view of algebra, and drew attention to the dual role of algebra expressions as both processes and objects. An important example of structure (1.2) occurs when composite functions, like those above, are used. Working with CAS is often greatly facilitated by adopting a structural view, so it supports and sometimes demands their use. For example, at the outset of a problem, it is often very helpful to define (or name) a function and use the name in further processes, for example: $f(g(x))$ if $g(x) = x + h$ and $f(x) = 5x^3 - 2x^2 + x + 7$. To show algebraic expectation a student would *not* need to perform the expansion of this new expression, but identify $(x+h)$ as an object that will replace each x. Students often make errors when creating such composite functions (for example, only replacing the first *x* of an expression with the new *f(x)* object) and so may be glad to assign the task of simplification to CAS. This allows us to monitor a deeper level of students' understanding that may have been masked by simple procedural errors, which, while not unimportant, often result from a lack of attention to detail or perhaps test anxiety. In any case many standard tests and classic items that test these procedural skills have already been developed and results of these studies are widely known.

Algebraic expectation is needed when entering expressions into CAS: Alice

When working by-hand a student can copy a question from a textbook but may make no progress towards the solution. It is often difficult to know what is blocking that student's progress but, sometimes, when they use CAS, their difficulties are more apparent to a teacher. Even in the initial stage of entering expressions the obstacles that students encounter are sometimes easier to identify when they are working with CAS. Apparently trivial errors, such as omitting a bracket, can often reveal fundamental mathematical difficulties as evidenced by the example of Alice's work, described in figure 3.

INSERT FIGURE 3 ALICE HERE

Figure 3 Alice needs to recognise equivalent expressions

Alice, was bewildered when she entered expressions into CAS and the resulting screen image did not match the printed or handwritten version in front of her. In a first instance, her error of omitting parentheses is not unexpected since as Booth (1988) noted, when working by hand, younger or less able students tend to ignore parentheses, evaluating everything from left to right. In order to enter such an expression correctly, Alice needed to first identify the structure of the expression

(Framework code 1.2.2) to recognise this as a ratio of two functions (x^2 +1) and (2-x) which must be made explicit for the machine. In addition, to understand the mismatch with the CAS output from her entry line, Alice needed to know that in mathematics there is an agreed convention for the order of operations (1.1.2) and that she should expect a CAS to be programmed to follow that convention. Thus, in the absence of suitably placed parentheses, her CAS performed the multiplications and divisions indicated before the additions and subtractions. When Alice had corrected her syntax she then needed to know the properties of operations (1.1.3) in order to recognise the equivalence of the two expressions $\frac{x^2+1}{2-x}$ and $\frac{-(x^2+1)}{x-2}$.

The variation on the first problem produced a further disturbing output for Alice. She needed to recognise these as equivalent expressions (1.1.3), provided $x \neq 1$, through recognition of simple factors (1.2.3) in the difference of two squares.

A new problem required Alice to use the expression $ae^x \sin \pi x$. As she worked on this problem, the designated 'e^x', 'π' and 'sin' buttons reminded her that letters can also be used to represent specific values and to name processes (1.1.1). Even the separate X key, accessed without using the 'alpha' mode reinforced the common convention that x would represent a variable while a would likely represent a constant value in that problem.

Mathematics teachers know that the meaning students assign to letter symbols is fundamental to their use of algebra. Many researchers, since the work of Küchemann (1981), have shown that there is a great deal of variety in these meanings and that a great deal of misunderstanding results from assigning inappropriate meaning to symbols. To show algebraic insight students must understand that letters can be used with different meanings in different contexts and be able to recognise which meaning is applicable for a particular expression. Working with CAS has helped to develop Alice's knowledge of the meaning of symbols (1.1.1).

Algebraic expectation applied when monitoring symbolic processes: Ben and Craig

Students need strategies for monitoring their work, and these are an essential part of algebraic expectation. First they need to be able to identify the inevitable typographical or keying errors which occur when entering an expression or function into CAS. Next they must monitor the algebraic processes and routines they choose to initiate through the use of CAS commands.

INSERT FIGURE 4 BEN AND CRAIG HERE

Figure 4 Ben and Craig demonstrate algebraic expectation

Ben and Craig, for example were working together to factorise polynomials. They had had experience with quadratics and cubics but this was the first time they were faced with a quartic expression, x^4-5x^2+4. A scenario of their working is set out in figure 4. Here we see algebraic expectation at work. Ben and Craig identified two key features of the expression; it was a polynomial (1.3.1) and the dominant term was x^4 (1.3.2). This informed their expectation that there were likely to be four factors. Their previous experience also led them to believe that the factors would be linear. In addition, Ben showed knowledge of the properties of operations (1.1.3) when he suggested that expanding Craig's result could help resolve the discrepancy. Expansion of Ben's result returned the original expression, whereas expansion of Craig's returned the expression 4x^4-5x^2+4. Neither of the boys, nor the two researchers present, had noticed the 4 on the far right of Craig's factorisation, which would have given them an easy clue to the explanation of the different factorisations. This illustrates how the limitations of current calculator screen size place additional demands on students' algebraic expectation.

Algebraic expectation supports confirmation of solutions

Linking form to solutions will inform expectation about the possible minimum and maximum number of solutions, the nature of solutions to be expected and the range over which a function may be defined. Identifying form for functions can mean noting that $x^4 + 4x^3 + 6x^2 + 4x + 1$ is a polynomial of degree 4, 2 + e^x is exponential and 4sin3*x* is trigonometric. At a second level it can mean seeing the symmetry of the coefficients of $x^4 + 4x^3 + 6x^2 + 4x + 1$ with descending powers of *x* suggesting a binomial expansion and that $e^{2x} + e^x - 2$ is a quadratic in e^x. Identifying form provides checks on the equivalence of two different expressions of solutions and whether all likely solutions to a problem have been found.

7. CONCLUSION

The illustrations outlined above make it clear that the availability of CAS does not mask students' understanding of algebra. Students' progress will be seen in their improved ability to choose appropriate routines and monitor their progress towards solutions. Such progression requires the development of algebraic insight.

Conceptualising the symbol sense required in the solving phase of problem solving (Figure 1) as consisting of the ability to link representations and algebraic expectation provides a framework for studying the likely effects of CAS use on curriculum, teaching and learning. Finally, the notion of algebraic expectation seems to capture particularly well the algebraic skill that parallels arithmetic estimation, and should thus not only be a useful focus for teaching but also a guide for monitoring students' progress.

We hope that the framework for algebraic expectation, presented in this paper as the insight to accompany formal symbolic operations, may provide a shared structure for curriculum planning, monitoring student progress and planning research.

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APPENDIX 1

Framework for considering number sense (first two columns only). (McIntosh et al, 1992, p4)

1	Knowledge of and facility with NUMBERS	1.1	Sense of orderliness of numbers
		1.2	Multiple representations of numbers
		1.3	Sense of relative and absolute magnitude of numbers
		1.4	System of benchmarks
2	Knowledge and facility with OPERATIONS	2.1	Understanding the effect of operations
		2.2	Understanding mathematical properties
		2.3	Understanding the relationship between operations
3	Applying knowledge of and facility with numbers and operations to COMPUTATIONAL SETTINGS	3.1	Understanding the relationship between problem context and the necessary computation
		3.2	Awareness that multiple strategies exist
		3.3	Inclination to utilize an efficient representation and/or method
		3.4	Inclination to review data and result for sensibility

APPENDIX 2

Fey's (1990) basic components of symbol sense

F1 Ability to scan an algebraic expression to make rough estimates of the patterns that would emerge or graphic representation …

F2 Ability to make informed comparisons of order of magnitude for functions with rules of the form n_1, n_2, n_3,. n_k…

F3 Ability to scan a table of function values or a graph or to interpret verbally stated conditions, to identify the likely form of an algebraic rule that expresses the appropriate pattern…

F4 Ability to inspect algebraic operations and predict the form of the result, or as in arithmetic estimation, to inspect the result and judge the likelihood that it has been performed correctly…

F5 Ability to determine which of several equivalent forms might be most appropriate for answering particular questions… (Fey, 1990, pp 80-81 numbering added)

APPENDIX 3

Arcavi's (1994) summary of symbol sense

A1 An understanding of and aesthetic feel for the power of symbols: understanding how and when symbols can and should be used in order to display relationships, generalisations, and proofs which are otherwise hidden and invisible.

A2 A feeling for when to abandon symbols in favour of other approaches in order to make progress with a problem, or in order to find an easier or more elegant solution or representation.

A3 An ability to manipulate and to 'read' symbolic expressions as two complementary aspects of solving algebraic problems. Detached from the meaning or context of the problem and with the symbolic expression viewed globally, symbol handling can be relatively quick and efficient. On the other hand, the reading of the symbolic expressions towards meaning can add layers of connections and reasonableness to the results.

A4 The awareness that one can successfully engineer symbolic relationships that express the verbal or graphical information needed to make progress in a problem, and the ability to engineer those expressions.

A5 The ability to select a possible symbolic representation of a problem, and, if necessary, to have the courage, first, to recognise and heed one's dissatisfaction with that choice, and second, to be resourceful in searching for a better one as replacement.

A6 The realisation of the constant need to check symbol meanings while solving a problem, and to compare and contrast those meanings with one's own intuitions or with the expected outcome of the problem.

A7 Sensing the different 'roles' symbols can play in different contexts.

(Arcavi, 1994, p31, numbering added)

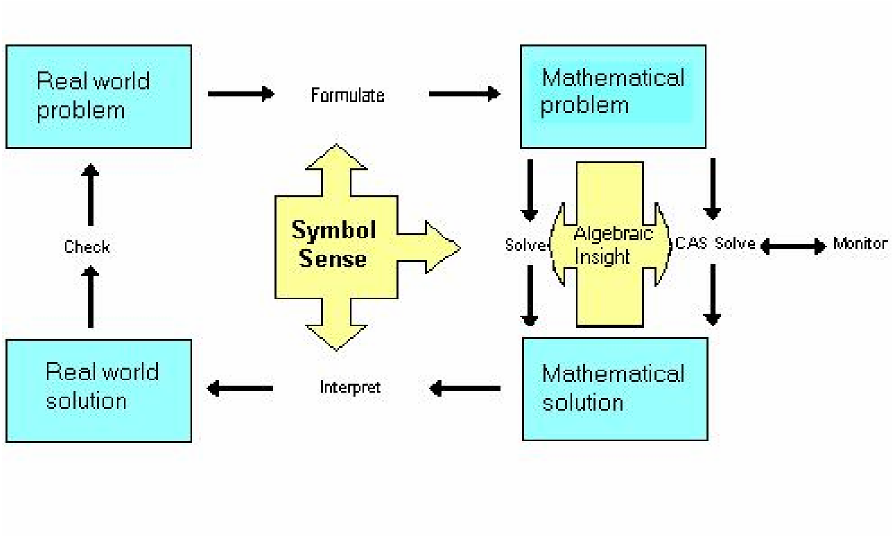

Figure 1 A model of problem solving showing the places of symbol sense and algebraic insight (Pierce & Stacey , 2002).

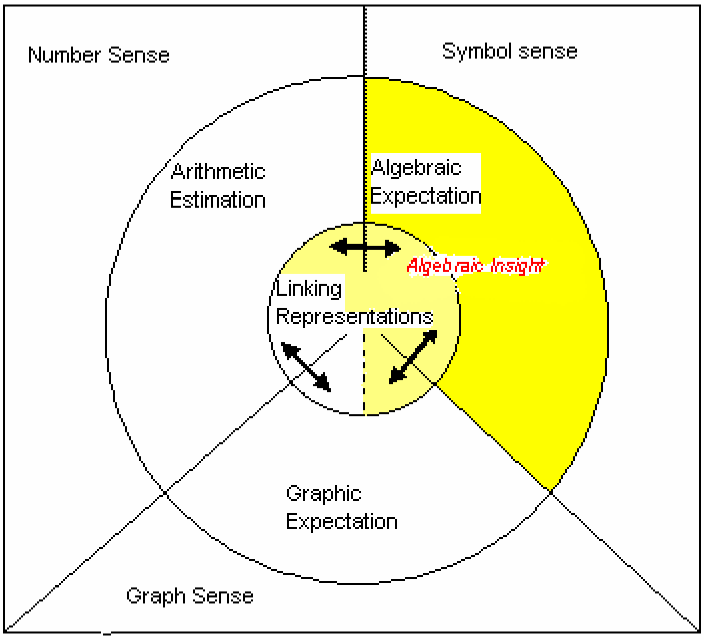

Figure 2 The place of algebraic insight and its components within the senses needed when working with CAS.

Aspects	Elements	Common Instances	
1. Algebraic Expectation	1.1 Recognition of conventions and basic properties	1.1.1	Know meaning of symbols
		1.1.2	Know order of operations
		1.1.3	Know properties of operations
	1.2 Identification of structure	1.2.1	Identify objects
		1.2.2	Identify strategic groups of components
		1.2.3	Recognise simple factors
	1.3 Identification of key features	1.3.1	Identify form
		1.3.2	Identify dominant term
		1.3.3	Link form with solution type
2. Ability to Link representations	2.1 Linking of symbolic and graphic representations	2.1.1	Link form with shape
		2.1.2	Link key features with likely position
		2.1.3	Link key features with intercepts and asymptotes
	2.2 Linking of symbolic and numeric representations	2.2.1	Link number patterns or type with form
		2.2.2	Link key features with suitable increment for table
		2.2.3	Link key features with critical intervals of table

Table 1 A Framework for algebraic insight

When working with the function $f(x)=\dfrac{x^2+1}{2-x}$ Alice typed the sequence x^2+1/2-x, resulting in the screen image $x^2+\dfrac{1}{2}-x$. She was bewildered.

Alice re-entered the expression using correct syntax and then wanted to check that she had been successful in correctly defining this function so she typed *f(x)* again in the entry line and pressed the enter key. Again, to her surprise the expression which appeared on the screen was not $\dfrac{x^2+1}{2-x}$ but $\dfrac{-(x^2+1)}{x-2}$.

Later, when working on a variation of this first problem, Alice correctly entered the expression $\dfrac{x^2-1}{x-1}$ as (x^2-1)/(x-1). The CAS showed the expression as she had hoped on the left of her screen but, on the right, just showed $x+1$. Again she was surprised.

Figure 3 Alice needs to recognise equivalent expressions

Ben and Craig have been asked to factorise the quartic expression x^4-5x^2+4. They have previous experience only with quadratics and cubic polynomials. Before entering anything, Ben predicted that there would be four factors and then they set to work, each using a CAS calculator.

Ben: "There will be four factors" (predicting before entering)

Both boys enter the expression on their separate machines and select the "factorise" command.

Ben's calculator shows: (x-*1*)(x+*1*)(x-*2*)(x+*2*)

Ben: "That's what I expected, 4 factors"

Craig's calculator shows: $\left(x^2+\frac{\sqrt{13}x}{2}+1\right)\left(x^2-\frac{\sqrt{13}x}{2}+1\right).4$

Craig (surprised): "Oh wow! How come I got that? I expected an answer like your's - not mine- because with quadratics it's two factors, with cubics it's three, therefore with that pattern with quartics it's four!"

Ben: (musing on why Craig's result might be correct even though there are only two factors) "If you look [at the expression] x^2 times x^2 is x^4"

Craig: "Yes, but it's not in simplest form, factorising simplifies."

Ben: "We could expand your expression to see if they are both correct."

Ben's expansion returned the original expression x^4-5x^2+4

Craig's expansion returned the expression 4x^4-5x^2+4.

This input error explained the discrepancy, although there is more for them to learn.

Figure 4 Ben and Craig demonstrate algebraic expectation