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A SCALED BOUNDARY POLYGON FORMULATION FOR ELASTO-PLASTIC ANALYSES

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Abstract

This study presents a novel scaled boundary polygon formulation to model elasto-plastic material responses in structures. The polygons have flexible mesh generation capabilities and are more accurate than standard finite elements, especially for problems with cracks and notches. Shape functions of arbitrary n -sided polygons are constructed using the scaled boundary finite element method. These shape functions are conforming and linearly complete. When modeling a crack, strain singularities are analytically modeled without enrichment. Standard finite element procedures are used to formulate the stiffness matrix and residual load vector. The nonlinear material constitutive matrix and the internal stresses are approximated locally in each polygon by a polynomial function. The stiffness matrix and the residual load vector are matrix power integrals that can be evaluated analytically even when a strain singularity is present. Standard nonlinear equation solvers e.g. the modified Newton-Raphson algorithm are used to obtain the nonlinear response of the structure. The proposed formulation is validated using five numerical benchmarks.

Keywords: scaled boundary finite element method, elasto-plastic, material nonlinear, polygon element, finite element method

1. Introduction

The finite element method (FEM) is a versatile computational tool that is widely used in engineering practice today. Contrary to analytical approaches, which attempt to solve boundary value problems by considering the domain of interest as a single entity, the FEM introduces the concept of discretization. A domain of complex geometry is partitioned into a finite number of subdomains of simplex shapes, termed finite elements. Typically, the shapes of two-dimensional finite elements are triangles or quadrilaterals i.e. polygons with 3 and 4 sides, respectively.

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Polygon finite elements with arbitrary number of sides was developed by Wachspress [1] in the mid 1970's. However, their application have not been explored for many years until Meyer et al. [2] and Floater et al. [3] generalized them to arbitrary shaped polygons using the concept of barycentric and mean-valued coordinates. Sukumar and Tabarraei [4] showed that conforming polygon elements can be formulated using Voronoi diagrams and natural neighbor shape functions. They also reported the development of conforming polygon elements using Wachspress basis functions based on barycentric coordinates. The development of adaptive h - and p -versions of polygon elements was reported by Milbradt and Pick [5]. The extension of polygon elements for fracture analysis [6, 7] have also been recently pursued by enriching the basis functions as in the extended finite element method (XFEM) [8]. An alternative method to construct polygonal shape functions based on the maximum entropy approach [9, 10] was also proposed by Sukumar [11, 12]. The shape functions of the polygon elements reported in [1, 2, 3, 4, 11, 12] are rational polynomials and require special integration techniques e.g. Natarajan et al. [13] or quadrature rules e.g. Mousavi et al. [14] to resolve their respective stiffness matrices.

Other alternatives have also been explored to develop polygon finite elements. One of these is based on the assumed stress hybrid method of Pian [15]. The polygon finite elements that are developed using this approach include the hybrid polygon finite element [16] and the Voronoi-cell FEM [17, 18, 19]. The Voronoi-cell FEM has been applied to solve many challenging engineering problems such as elasto-plastic and thermo-elastic response of heterogeneous materials [17, 18] and cohesive crack propagation [19]. Liu et al. [20] developed the smoothed FEM, which introduced the concept of strain smoothing to remove spurious modes from the finite element formulation. This resulted in a stable element formulation that is free from locking. The implementation of the smoothed FEM was further generalized for polygons of arbitrary number of sides [21, 22]. The smoothed FEM has been shown to be very efficient in modeling problems with cracks [23, 24, 25, 26] and is good candidate for plastic analysis [27, 28].

Polygons with more than four sides involve more nodes in their interpolation compared with a standard finite element. They generally exhibit superior solution accuracy. Additional benefits of using polygons include flexibility in meshing complex geometries such as that in biomechanics applications [29] and modeling polycrystals [16, 17], where it is desirable for each crystal to be meshed with a polygon conforming to its natural shape. In topology optimization, Talischi et al. [30] showed that polygons possess higher degree of geometric isotropy and eliminates the mesh dependency resulting from fixed mesh representation with standard triangles or quadrilaterals. These advantages have further motivated the use of polygon finite elements as an alternative to standard finite elements using triangles or quadrilaterals.

The scaled boundary finite element method (SBFEM) provides an alternative technique to construct polygon elements [31]. This method was developed by Song and Wolf [32] in the late 1990s and has, since its introduction, been applied to many problems in engineering such as wave propagation [33, 34],

unbounded media [35, 36, 37, 38], diffusion [39], fluid-structure interaction [40], fracture [41, 42, 43, 44, 45, 46, 47], acoustics [48] electro-statics [49] and thin plates [50]. For these problems, the SBFEM was shown to be more efficient compared with the standard FEM. In unbounded media problems [35, 36, 37, 38], the SBFEM can satisfy the radiation condition at infinity exactly. This significantly reduces the number of degrees-of-freedom (DOF) in the computational model. Wave propagation problems can be solved efficiently using a local transmitting boundary condition that is based on the continued fraction solution of the SBFEM's dynamic stiffness matrix [34, 33] without the need to evaluate convolution integrals. For thin plate analysis, the SBFEM with spectral elements [50] was shown to be locking-free and converges more rapidly compared with other plate elements reported in the literature. The SBFEM has some unique features in fracture analysis. Unlike the FEM, it does not need local mesh refinement in the vicinity of the crack tip or special singular elements [51]. Unlike the the extended FEM (XFEM) [8, 52, 53] or meshfree methods [54, 55], the SBFEM does not require crack tip enrichment and the special integration techniques associated with them. Any kind of stress singularity including power logarithms and the T -stress can be modeled accurately with the SBFEM because they are expressed analytically in its solutions [44, 56]. These features have motivated the development of various SBFEM-based crack propagation modeling methodologies [45, 57, 58, 59, 60, 61] that reported significant reduction in the number of DOF, higher accuracy and simple re-meshing and mesh mapping algorithms.

Application of the SBFEM mentioned above has so far been limited to linear elastic materials only. In problems where material non-linearity is involved, a coupled approach is usually adopted. Here, the FEM [35, 36] or the meshless method [37, 38] is used to model the portion of the computational domain exhibiting material non-linearity. To the authors' knowledge, the SBFEM has so far been developed for only simple nonlinear partial differential equations [62] i.e. two-dimensional nonlinear Poisson type problem where the conductivity is a fixed function of the state variables. A homotopy analysis method was proposed to transform the original nonlinear problem into an infinite number of linear Poisson type problems that is than solved by the SBFEM. A different approach is adopted in this paper.

The novel polygon-based SBFEM formulation described in this paper is based on the recent polygon-based SBFEM for linear elastic [31] and nonlinear cohesive crack propagation [63]. This technique is flexible in meshing complex geometries, and the use of polygons to discretize the computational domain naturally complements the SBFEM. It uses the scaled boundary shape functions described in [64, 65] and is applicable to polygons of any number of sides. Only the boundary of the polygon is discretized with one-dimensional elements using Gauss-Lobatto-Lagrange shape functions [66, 65]. Within a polygon, the shape functions are obtained analytically. Higher order shape functions can be easily constructed for a polygon element compared with the other types of polygon elements reported in the literature [1, 2, 3, 4, 11, 12]. For example, the polygonal shape functions that are evaluated by the maximum entropy approach require

the solution of an optimization problem [12].

The scaled boundary shape functions naturally include the strain singularities at a crack or notch. In this respect, the formulation developed in this paper shares some similarities with the hybrid-stress super singular wedge-tip element developed by Chen and Sze [67]. In both methodologies, an eigenvalue problem was first formulated to determine the order of stress singularities and angular dependencies of the stress and displacement field around the crack tip. In the present formulation, however, the stiffness matrix is directly obtained from the eigenvectors of the eigenvalue problem. Any kind of strain singularity (if present) is automatically represented in the formulation. In the hybrid-stress super singular wedge-tip element, an asymptotic solution is first obtained from the eigenvalue solution. This solution is used to enrich the hybrid-stress formulation to enable it to model stress singularities. In the present formulation, higher order shape functions can be conveniently constructed by simply increasing the order of the shape functions on the polygon boundaries without any change to the formulation. In the hybrid-stress super singular wedge-tip element, the construction of higher order elements is not straightforward. Moreover, the shape functions have to be properly selected to satisfy the inter-element compatibility.

The key step in the developed formulation that enables it to model material nonlinearity is to use the scaled boundary shape functions to interpolate the displacement field. Conceptually, this operation involves simply replacing the standard finite element shape functions with the scaled boundary shape functions in the element formulation. Therefore, standard finite element procedures are adopted for the elasto-plastic analysis. To simplify the integration for the stiffness matrix and internal load vector, the elasto-plastic constitutive matrix and the internal stresses are approximated by polynomial functions over a polygon. Numerical integration is required only on the polygon boundary. Integration within a polygon can be performed analytically. Once the system of nonlinear equations in each polygon has been assembled, standard nonlinear equation solvers e.g. Newton-Raphson algorithm, are used to obtain the elasto-plastic response of the structure.

In this paper the basic principles of the SBFEM are first described in Section 2. The scaled boundary polygon shape functions are introduced so that the displacement field can be expressed in terms of nodal displacements. Section 3 describes how the elasto-plastic constitutive matrix is approximated in a polygon. Section 4 describes the scaled boundary polygon formulation for elasto-plastic analysis. An algorithm for numerical implementation is described in Section 5. In Section 6, the developed formulation is validated using several numerical benchmarks. Section 7 summarizes the major conclusions that can be drawn from this study.

2. Scaled boundary polygon shape functions

2.1. Polygon representation of the scaled boundary finite element method

A domain, Ω , can be discretized with a mesh of arbitrary n -sided polygons. A polygon mesh can be generated from Delaunay triangulation [31] or Voronoi

diagrams [4]. The former approach is adopted in this paper. Any polygon can be modeled as a SBFEM subdomain as long as its geometry satisfies the SBFEM scaling requirement [68], i.e. any point on the polygon boundary should be directly visible from the scaling center. This requirement can be satisfied for any convex polygon and some concave polygons. The polygon elements generated from a Delaunay triangulation [31] automatically satisfy this requirement.

Fig. 1a shows a polygon modeled by the SBFEM. A scaling center is defined at the geometric center of the polygon. Each edge on a polygon is discretized using one-dimensional line elements with local coordinates $-1 \leq \eta \leq 1$. The shape functions of these elements can be of any order. The order of the shape functions do not complicate the mesh generation process because it only involves adding additional nodes on the element edges.

The Cartesian coordinates of a point on a line element with M nodes are

$$x_b(\eta) = \mathbf{N}(\eta)\mathbf{x}_b \quad (1)$$

$$y_b(\eta) = \mathbf{N}(\eta)\mathbf{y}_b \quad (2)$$

where $\mathbf{x}_b = [x_1 \ x_2 \ x_3 \ \dots \ x_M]^T$ and $\mathbf{y}_b = [y_1 \ y_2 \ y_3 \ \dots \ y_M]^T$ are the nodal coordinates. $\mathbf{N}(\eta) = [N_1(\eta), N_2(\eta), N_3(\eta), \dots, N_M(\eta)]^T$ is the shape function vector of the line elements on the polygon boundary. Standard one-dimensional Gauss-Lobatto-Lagrange shape functions [66] can be used.

The radial direction from the scaling center to the boundary is described by a dimensionless radial coordinate ξ with $\xi = 0$ at the scaling center and $\xi = 1$ at the boundary. The geometry of the domain is described by scaling the boundary along ξ . The Cartesian coordinates of a point inside the domain with the origin at the scaling center are given by the scaled boundary transformation equations [68]

$$x(\eta) = \xi \mathbf{N}(\eta)\mathbf{x}_b \quad (3)$$

$$y(\eta) = \xi \mathbf{N}(\eta)\mathbf{y}_b \quad (4)$$

A polygon with a crack or a notch can be modeled by selecting the scaling center at the crack or notch tip. The straight notch/crack edges are formed by scaling the nodes A and B on the boundary and are not discretized (see Fig. 1b). The line elements do not form a closed loop. Similarly, the straight material interfaces passing through the scaling center can be modeled without discretization. The material sector covered by scaling an element on the boundary assumes the properties of the element as is shown in Fig. 1b.

2.2. General solution of the scaled boundary finite element method

For a sector covered by a line element on a polygon's boundary (see Fig. 1a), the displacement field $\mathbf{u}(\xi, \eta)$ at any point in SBFEM coordinates can be written as

$$\mathbf{u}(\xi, \eta) = \mathbf{N}_u(\eta)\mathbf{u}(\xi) \quad (5)$$

where $\mathbf{u}(\xi)$ are radial displacement functions along a line connecting the scaling center and a node at the boundary and $\mathbf{N}_u(\eta)$ is the shape function matrix

$$\mathbf{N}_u(\eta) = \begin{bmatrix} N_1(\eta) & 0 & N_2(\eta) & 0 & \dots & 0 & N_M(\eta) & 0 \\ 0 & N_1(\eta) & 0 & N_2(\eta) & \dots & 0 & 0 & N_M(\eta) \end{bmatrix} \quad (6)$$

The radial displacements functions $\mathbf{u}(\xi)$ are the solution the SBFEM equation in displacement [68]

$$\mathbf{E}_0 \xi^2 \mathbf{u}(\xi)_{,\xi\xi} + (\mathbf{E}_0 - \mathbf{E}_1 + \mathbf{E}_1^T) \xi \mathbf{u}(\xi)_{,\xi} - \mathbf{E}_2 \mathbf{u}(\xi) + \mathbf{F}(\xi) = 0 \quad (7)$$

which describes the equilibrium in the ξ direction. In Eq. (7), $\mathbf{E}_i$, $i = 0, 1, 2$ are coefficient matrices that depend on the geometry and material properties and $\mathbf{F}(\xi)$ is a load vector that includes contributions from side-face tractions, body and thermal loads [68]. For every polygon, the $\mathbf{E}_i$'s are assembled element by element just as in the FEM.

When $\mathbf{F}(\xi) = 0$, the solution for Eq. (7) is obtained by introducing the variable, $\Theta(\xi)$ [68]

$$\Theta(\xi) = \begin{Bmatrix} \mathbf{u}(\xi) \\ \mathbf{q}(\xi) \end{Bmatrix} \quad (8)$$

This enables Eq. (7) to be transformed into a first order ordinary differential equation with twice the number of unknowns in a polygon as

$$\xi \Theta(\xi)_{,\xi} = -\mathbf{Z} \Theta(\xi) \quad (9)$$

where $\mathbf{Z}$ is a Hamiltonian matrix

$$\mathbf{Z} = \begin{bmatrix} \mathbf{E}_0^{-1} \mathbf{E}_1^T & -\mathbf{E}_0^{-1} \\ \mathbf{E}_1 \mathbf{E}_0^{-1} \mathbf{E}_1^T - \mathbf{E}_2 & -\mathbf{E}_1 \mathbf{E}_0^{-1} \end{bmatrix} \quad (10)$$

and $\mathbf{q}(\xi)$ is the internal nodal force.

A block diagonal Schur decomposition of $\mathbf{Z}$ in Eq. (9) is performed. The Schur block and transformation matrices necessary for a polygon are expressed as

$$\mathbf{Z} \begin{bmatrix} \Psi_u \\ \Psi_q \end{bmatrix} = \begin{bmatrix} \Psi_u \\ \Psi_q \end{bmatrix} \mathbf{S}_n \quad (11)$$

The diagonal entries of $\mathbf{S}_n$ are equal to the real parts of the eigenvalues. They include two zeros (representing the two modes of translational rigid body motion) and negative numbers. Ψ_u and Ψ_q are transformation matrices corresponding to the modal displacements and forces, respectively. For a polygon with $0 \leq \xi \leq 1$, the solution of Eq. (9) is

$$\mathbf{u}(\xi) = \Psi_u \xi^{-\mathbf{S}_n} \mathbf{c}_n \quad (12)$$

$$\mathbf{q}(\xi) = \Psi_q \xi^{-\mathbf{S}_n} \mathbf{c}_n \quad (13)$$

where the integration constants $\mathbf{c}_n$ are extracted from the nodal displacements on the polygon boundary $\mathbf{u}_b = \mathbf{u}(\xi = 1)$ as [68]

$$\mathbf{c}_n = \Psi_u^{-1} \mathbf{u}_b \quad (14)$$

2.3. Scaled boundary shape functions

The scaled boundary shape functions can be derived by first substituting Eq. (14) into Eq. (12) so that the radial displacement functions $\mathbf{u}(\xi)$ can be written as

$$\mathbf{u}(\xi) = \mathbf{\Psi}_u \xi^{-\mathbf{S}_n} \mathbf{\Psi}_u^{-1} \mathbf{u}_b \quad (15)$$

Substituting Eq. (15) into Eq. (5), the displacement field $\mathbf{u}(\xi, \eta)$ in a sector covered by a line element on the polygon boundary (see Fig. 1a) can be expressed in terms of $\mathbf{u}_b$ as

$$\mathbf{u}(\xi, \eta) = \mathbf{N}_u(\eta) \mathbf{\Psi}_u \xi^{-\mathbf{S}_n} \mathbf{\Psi}_u^{-1} \mathbf{u}_b \quad (16)$$

Eq. (16) is evaluated using part of the modes in $\mathbf{\Psi}_u$ corresponding to the degrees-of-freedom of the line element only. The product

$$\mathbf{\Phi}(\xi, \eta) = \mathbf{N}_u(\eta) \mathbf{\Psi}_u \xi^{-\mathbf{S}_n} \mathbf{\Psi}_u^{-1} \quad (17)$$

is defined as the scaled boundary shape function.

These shape functions are continuous inside the polygon and across adjacent polygons provided both polygons have the same element on the common edge. $\mathbf{\Phi}(\xi, \eta)$ can reproduce all rigid body modes and constant strain terms for uncracked polygons [69, 65]. For cracked polygons, it was shown in [65] that $\mathbf{\Phi}(\xi, \eta)$ can reproduce the leading terms of the William's eigenfunction expansion. Figs. 2a and 2b show the scaled boundary shape functions for a normal polygon and a cracked polygon, respectively.

2.4. Stress and strain fields in terms of scaled boundary shape functions

In the SBFEM, the strain field $\epsilon(\xi, \eta)$ is given by [68]

$$\epsilon(\xi, \eta) = \mathbf{B}_1(\eta) \mathbf{u}(\xi)_{,\xi} + \xi^{-1} \mathbf{B}_2(\eta) \mathbf{u}(\xi) \quad (18)$$

with the strain-displacement matrices [68]

$$\mathbf{B}_1(\eta) = \frac{1}{|\mathbf{J}(\eta)|} \begin{bmatrix} y_b(\eta)_{,\eta} & 0 \\ 0 & -x_b(\eta)_{,\eta} \\ -x_b(\eta)_{,\eta} & y_b(\eta)_{,\eta} \end{bmatrix} \mathbf{N}_u(\eta) \quad (19)$$

$$\mathbf{B}_2(\eta) = \frac{1}{|\mathbf{J}(\eta)|} \begin{bmatrix} -y_b(\eta) & 0 \\ 0 & x_b(\eta) \\ x_b(\eta) & -y_b(\eta) \end{bmatrix} \mathbf{N}_u(\eta)_{,\eta} \quad (20)$$

Taking the first order derivative of Eq. (15) with respect to ξ and substituting into Eq. (18), $\epsilon(\xi, \eta)$ can be expressed as

$$\epsilon(\xi, \eta) = (\mathbf{B}_1(\eta) \mathbf{\Psi}_u [-\mathbf{S}_n] + \mathbf{B}_2(\eta) \mathbf{\Psi}_u) \xi^{-\mathbf{S}_n - \mathbf{I}} \mathbf{\Psi}_u^{-1} \mathbf{u}_b \quad (21)$$

Introducing a strain mode $\mathbf{\Psi}_\epsilon(\eta)$ with

$$\mathbf{\Psi}_\epsilon(\eta) = (\mathbf{B}_1(\eta) \mathbf{\Psi}_u [-\mathbf{S}_n] + \mathbf{B}_2(\eta) \mathbf{\Psi}_u) \quad (22)$$

Eq. (21) can be simplified as

$$\boldsymbol{\epsilon}(\xi, \eta) = \boldsymbol{\Psi}_\epsilon(\eta) \xi^{-\mathbf{S}_n - \mathbf{I}} \boldsymbol{\Psi}_u^{-1} \mathbf{u}_b \quad (23)$$

The matrix product

$$\mathbf{B}(\xi, \eta) = \boldsymbol{\Psi}_\epsilon(\eta) \xi^{-\mathbf{S}_n - \mathbf{I}} \boldsymbol{\Psi}_u^{-1} \quad (24)$$

is the scaled boundary strain-displacement matrix. Therefore, the strain field can be written as

$$\boldsymbol{\epsilon}(\xi, \eta) = \mathbf{B}(\xi, \eta) \mathbf{u}_b \quad (25)$$

Note that the strain singularity at the notch/crack is naturally included when some of the eigenvalues of $\mathbf{S}_n$ satisfy $-1 \leq \text{Re}(\lambda(\mathbf{S}_n)) \leq 0$.

In elasto-plasticity theory, the incremental strain field can be decomposed into an elastic and a plastic component

$$\Delta \boldsymbol{\epsilon} = \Delta \boldsymbol{\epsilon}_e + \Delta \boldsymbol{\epsilon}_p \quad (26)$$

The plastic strain increment can be computed from the plastic flow rule. Assuming associative plasticity, it is expressed as

$$\Delta \boldsymbol{\epsilon}_p = \frac{\partial F}{\partial \boldsymbol{\sigma}} \Delta \lambda \quad (27)$$

where $F \equiv F(\boldsymbol{\sigma}, \kappa)$ is a yield function that depends on the current stress state $\boldsymbol{\sigma}$ and the hardening parameter κ . $\Delta \lambda$ is the plastic multiplier. For isotropic hardening material under von Mises yield criterion for example the yield function can be expressed as

$$F(\boldsymbol{\sigma}, \kappa(\boldsymbol{\epsilon}_p)) = \sqrt{\frac{3}{2} J_2} - \sigma_f(\boldsymbol{\epsilon}_p) = 0 \quad (28)$$

where J_2 is the second deviatoric stress invariant and $\sigma_f(\boldsymbol{\epsilon}_p)$ is the instantaneous plastic yield limit.

Substituting the Hooke's law, $\Delta \boldsymbol{\sigma} = \mathbf{D} \Delta \boldsymbol{\epsilon}_e$, and Eq. (27) in Eq. (26), the incremental stress field $\Delta \boldsymbol{\sigma}$ can be written as

$$\Delta \boldsymbol{\sigma} = \mathbf{D}_{ep} \Delta \boldsymbol{\epsilon} \quad (29)$$

where

$$\mathbf{D}_{ep} = \mathbf{D} - \mathbf{D} \left(\frac{\partial F}{\partial \boldsymbol{\sigma}} \right) \left(\frac{\partial F}{\partial \boldsymbol{\sigma}} \right)^T \mathbf{D} \left(A + \left(\frac{\partial F}{\partial \boldsymbol{\sigma}} \right)^T \mathbf{D} \left(\frac{\partial F}{\partial \boldsymbol{\sigma}} \right) \right)^{-1} \quad (30)$$

is the elasto-plastic constitutive matrix and

$$A = -\frac{1}{\Delta \lambda} \frac{\partial F}{\partial \kappa} d\kappa \quad (31)$$

is the hardening modulus.

Substituting Eq. (25) formulated for the incremental strain field $\Delta \boldsymbol{\epsilon}$, the incremental stress field is expressed as

$$\Delta \boldsymbol{\sigma}(\xi, \eta) = \mathbf{D}_{ep} \mathbf{B}(\xi, \eta) \Delta \mathbf{u}_b \quad (32)$$

where $\Delta \mathbf{u}_b$ is the nodal displacement increment.

3. Elasto-plastic constitutive matrix

When a material yields, the elasto-plastic constitutive matrix $\mathbf{D}_{\text{ep}}$ is not constant within a polygon. In order to simplify the integration, the variation of $\mathbf{D}_{\text{ep}}$ is approximated by a polynomial in terms of scaled boundary coordinates, (ξ, η) .

To achieve this, the variation of each component in $\mathbf{D}_{\text{ep}}$ in the polygon is first approximated by a polynomial function of the Cartesian coordinates, (x, y)

$$\mathbf{D}_{\text{ep}}(x, y) = \mathbf{D}_0 + \mathbf{D}_1x + \mathbf{D}_2y + \mathbf{D}_3x^2 + \mathbf{D}_4xy + \mathbf{D}_5y^2 + \dots \quad (33)$$

The coefficients, $\mathbf{D}_i$, $i = 1, 2, 3, \dots$, are determined by a least squares fit over each polygon. The Gaussian integration points can be used for this purpose. Additional fitting points along the ξ coordinate can also be included to compute the fitting function (see Fig. 3). This assumption requires polygons of sufficiently small size so that the variation of $\mathbf{D}_{\text{ep}}$ in the plastic zone can be accurately represented.

Substituting Eqs. (3) and (4) into Eq. (33) and combining the terms with the same order of ξ , the elasto-plastic constitutive matrix $\mathbf{D}_{\text{ep}}(\xi, \eta) \equiv \mathbf{D}_{\text{ep}}(x, y)$ is expressed as

$$\mathbf{D}_{\text{ep}}(\xi, \eta) = \mathbf{D}^{(0)} + \mathbf{D}^{(1)}(\eta)\xi + \mathbf{D}^{(2)}(\eta)\xi^2 + \dots = \sum_{k=0} \mathbf{D}^{(k)}(\eta)\xi^k \quad (34)$$

4. Scaled boundary polygon formulation for elasto-plasticity

The primary reason for introducing the scaled boundary shape functions in Section (2.3) is to rewrite the displacement interpolation as a product of the shape function matrix with the nodal displacement vector i.e. Eq. (16). This form is similar to the displacement interpolation in standard finite elements [70]. Similarly, the strain field $\boldsymbol{\epsilon}(\xi, \eta)$ can be expressed as the product of a strain-displacement matrix with the nodal displacement vector i.e. Eq. (25). When $\mathbf{u}(\xi, \eta)$ and $\boldsymbol{\epsilon}(\xi, \eta)$, are expressed in terms of their nodal displacements as in Eqs. (16) and (25) each polygon can be treated as an “element” in the finite element sense and standard finite element procedures to handle material nonlinearity can be used.

The scaled boundary polygon formulation for elasto-plasticity can be derived starting from the principle of virtual work [71], which is written in SBFEM coordinates as

$$\int_{\Omega} \delta \boldsymbol{\epsilon}^T \Delta \boldsymbol{\sigma}(\xi, \eta) d\Omega = \int_{\Gamma} \delta \mathbf{u}^T \mathbf{f}_t d\Gamma + \int_{\Omega} \delta \mathbf{u}^T \mathbf{f}_b d\Omega - \int_{\Omega} \delta \boldsymbol{\epsilon}^T \boldsymbol{\sigma}(\xi, \eta) d\Omega \quad (35)$$

where $\Delta \boldsymbol{\sigma}(\xi, \eta)$ is the incremental stress field, $\mathbf{f}_b$ is the body force intensity and $\mathbf{f}_t$ is the surface traction, $\delta \boldsymbol{\epsilon}(\xi, \eta)$ is the virtual strain field and $\delta \mathbf{u}(\xi, \eta)$ is the virtual displacement field. Substituting Eq. (16), (25) and (32) into Eq. (35) for

the virtual displacements, virtual strains and incremental stresses, respectively, and invoking the arbitrariness of the virtual displacements, yields

$$\left(\int_{\Omega} \mathbf{B}^T(\xi, \eta) \mathbf{D}_{\text{ep}} \mathbf{B}(\xi, \eta) d\Omega \right) \Delta \mathbf{u}_b = \int_{\Gamma} \mathbf{\Phi}^T(\xi, \eta) \mathbf{f}_t d\Gamma + \int_{\Omega} \mathbf{\Phi}^T(\xi, \eta) \mathbf{f}_b d\Omega - \int_{\Omega} \mathbf{B}^T(\xi, \eta) \boldsymbol{\sigma}(\xi, \eta) d\Omega \quad (36)$$

The term within the parentheses on the left-hand-side of Eq. (36) is the elasto-plastic stiffness matrix of the polygon $\mathbf{K}_{\text{ep}}$. The first two terms on the right-hand-side are the external load vector $\mathbf{R}_{\text{ext}}$ and the last term on the right-hand-side of Eq. (36) is the internal load vector $\mathbf{R}_{\text{int}}$. Eq. (36) can, therefore, be simplified as

$$\mathbf{K}_{\text{ep}} \Delta \mathbf{u}_b = \mathbf{R}_{\text{ext}} - \mathbf{R}_{\text{int}} \quad (37)$$

Eq. (37) is a system of nonlinear equations that can be assembled polygon-by-polygon as in the FEM, leading to

$$\left(\sum_{i=1}^{nPol} \mathbf{K}_{\text{ep}} \right) \Delta \mathbf{U}_b = \sum_{i=1}^{nPol} (\mathbf{R}_{\text{ext}} - \mathbf{R}_{\text{int}}) \quad (38)$$

where $\Delta \mathbf{U}_b$ is the global incremental displacement vector. Standard nonlinear solution techniques e.g., the modified Newton-Raphson iterations can be used to solve Eq. (38).

4.1. Evaluation of elasto-plastic stiffness matrix

From Eq. (36) and (37), the elasto-plastic stiffness matrix of a polygon $\mathbf{K}_{\text{ep}}$ is

$$\mathbf{K}_{\text{ep}} = \int_{\Omega} \mathbf{B}^T(\xi, \eta) \mathbf{D}_{\text{ep}} \mathbf{B}(\xi, \eta) d\Omega \quad (39)$$

Using the polynomial approximation of $\mathbf{D}_{\text{ep}}$, the integral can be evaluated numerically in the η -direction and then analytically in the ξ -direction.

Substituting $\mathbf{B}(\xi, \eta)$ in Eq. (24) and $\mathbf{D}_{\text{ep}}$ in Eq. (34) into Eq. (39) and using the relation $d\Omega = |\mathbf{J}(\eta)| \xi d\eta d\xi$ [61], results in

$$\mathbf{K}_{\text{ep}} = \boldsymbol{\Psi}_u^{-T} \sum_{k=0} \left(\int_0^1 \xi^{-\mathbf{S}_n^T - \mathbf{I}} \mathbf{Y}^{(k)} \xi^{-\mathbf{S}_n + k\mathbf{I}} d\xi \right) \boldsymbol{\Psi}_u^{-1} \quad (40)$$

where the matrix $\mathbf{Y}^{(k)}$ is

$$\mathbf{Y}^{(k)} = \int_{-1}^1 \boldsymbol{\Psi}_{\epsilon}^T(\eta) \mathbf{D}^{(k)}(\eta) \boldsymbol{\Psi}_{\epsilon}(\eta) |\mathbf{J}(\eta)| d\eta \quad (41)$$

Eq. (41) can be numerically integrated using standard Gaussian/Gauss-Lobatto techniques for each term k . It is assembled element by element on the polygon boundary as in the FEM.

For each term in Eq. (40), the integration with respect to ξ is defined as

$$\mathbf{X}^{(k)} = \int_0^1 \xi^{-\mathbf{S}_n^T - \mathbf{I}} \mathbf{Y}^{(k)} \xi^{-\mathbf{S}_n + k\mathbf{I}} d\xi \quad (42)$$

Using the properties of matrix power functions [62] and integration by parts, $\mathbf{X}^{(k)}$ is the solution of the Lyapunov equation

$$(-\mathbf{S}_n + 0.5k\mathbf{I})^T \mathbf{X}^{(k)} + \mathbf{X}^{(k)} (-\mathbf{S}_n + 0.5k\mathbf{I}) = \mathbf{Y}^{(k)} \quad (43)$$

As the coefficient matrix $-\mathbf{S}_n + 0.5k\mathbf{I}$ is in Schur form, only a back substitution is required. The elasto-plastic stiffness matrix of a polygon $\mathbf{K}_{\text{ep}}$ can be evaluated once $\mathbf{X}^{(k)}$ has been determined for all the terms. Eq. (40) is therefore, simplified as

$$\mathbf{K}_{\text{ep}} = \mathbf{\Psi}_u^{-T} \left(\sum_{k=0} \mathbf{X}^{(k)} \right) \mathbf{\Psi}_u^{-1} \quad (44)$$

Compared with the polygon elements based on rational polynomials [1, 2, 3, 4, 11, 12] that require special integration techniques [13, 14], the semi-analytical integration in the present formulation is simpler. Moreover, it requires only a one-dimensional numerical integration on the boundary of a polygon.

4.2. Evaluation of the external load vector

From Eq. (36), the external load vector $\mathbf{R}_{\text{ext}}$ in a polygon is

$$\mathbf{R}_{\text{ext}} = \int_{\Gamma} \mathbf{\Phi}^T(\xi, \eta) \mathbf{f}_t d\Gamma + \int_{\Omega} \mathbf{\Phi}^T(\xi, \eta) \mathbf{f}_b d\Omega \quad (45)$$

The first term on the right-hand-side of Eq. (45) is the distributed load on the boundary. It can be further simplified, considering that at the polygon boundary ($\xi = 1$) $\mathbf{\Phi}(\xi, \eta) = \mathbf{N}_u(\eta)$ applies. This results in

$$\int_{\Gamma} \mathbf{\Phi}^T(\xi, \eta) \mathbf{f}_t d\Gamma = \int_{-1}^1 \mathbf{N}_u^T(\eta) |\mathbf{J}(\eta)| \mathbf{f}_t d\eta \quad (46)$$

This expression is the same as in the standard FEM.

The second term on the right-hand-side of Eq. (45) is the body load vector. For the case of a constant body load, this term is expressed by using Eq. (17) and the relation $d\Omega = |\mathbf{J}(\eta)| \xi d\eta d\xi$ as

$$\int_{\Omega} \mathbf{\Phi}^T(\xi, \eta) \mathbf{f}_b d\Omega = \int_0^1 \int_{-1}^1 (\mathbf{N}_u(\eta) \mathbf{\Psi}_u \xi^{-\mathbf{S}_n} \mathbf{\Psi}_u^{-1})^T |\mathbf{J}(\eta)| d\eta d\xi \mathbf{f}_b \quad (47)$$

Eq. (47) can be integrated numerically in the η -direction and analytically in the ξ -direction, resulting in

$$\int_{\Omega} \mathbf{\Phi}^T(\xi, \eta) \mathbf{f}_b d\Omega = \mathbf{\Psi}_u^{-T} (\mathbf{S}_n + 2\mathbf{I})^{-1} \mathbf{\Psi}_u^T \left(\int_{-1}^1 \mathbf{N}_u^T(\eta) |\mathbf{J}(\eta)| d\eta \right) \mathbf{f}_b \quad (48)$$

4.3. Evaluation of the internal load vector

From Eq. (37), the load vector due to the internal stresses is

$$\mathbf{R}_{\text{int}} = \int_{\Omega} \mathbf{B}^T(\xi, \eta) \boldsymbol{\sigma}(\xi, \eta) d\Omega \quad (49)$$

To evaluate this term, the stress distribution $\boldsymbol{\sigma}(x, y)$ is approximated by a polynomial function

$$\boldsymbol{\sigma}(x, y) = \mathbf{h}_0 + \mathbf{h}_1 x + \mathbf{h}_2 y + \mathbf{h}_3 x^2 + \mathbf{h}_4 xy + \mathbf{h}_5 y^2 + \dots \quad (50)$$

As with $\mathbf{D}_{\text{ep}}$, the coefficients, $\mathbf{h}_i$, $i = 1, 2, 3, \dots$, are determined by a least squares fit over each polygon using the Gaussian/Gauss-Lobatto integration points at the boundary (and if desired, at several intervals of ξ (see Fig. 3)).

The stresses at these points can be determine using standard finite element procedures such as the normal correction algorithm proposed by Owen and Hinton [71] or the return mapping algorithm of Simo and Hughes [72]. In this paper, the former approach, which is an explicit integration scheme, is adopted. The stepping is controlled by dividing the excess stress into m sections where m is the nearest integer given by [71]

$$m = 8\sigma_{Y^\circ}^{-1}(\bar{\sigma}_e^r - \sigma_Y) + 1 \quad (51)$$

In the Eq. (51), $\bar{\sigma}_e^r - \sigma_Y$ is the excess stress and σ_{Y° is the initial yield stress.

Using Eqs. (3) and (4), Eq. (50) can be transformed into the scaled boundary coordinates, (ξ, η) , as

$$\boldsymbol{\sigma}(\xi, \eta) = \mathbf{h}^{(0)} + \mathbf{h}^{(1)}(\eta)\xi + \mathbf{h}^{(2)}(\eta)\xi^2 + \dots = \sum_{k=0} \mathbf{h}^{(k)}(\eta)\xi^k \quad (52)$$

The orders of the curve fitting functions for $\boldsymbol{\sigma}(\xi, \eta)$, in Eq. (52) and for $\mathbf{D}_{\text{ep}}(\xi, \eta)$ in Eq. (34) do not necessarily have to be equal to each other. Substituting Eq. (24) and Eq. (52) into Eq. (49) and transforming the domain integral to the scaled boundary coordinates results in

$$\mathbf{R}_{\text{int}} = \boldsymbol{\Psi}_u^{-T} \sum_{k=0} \left(\int_0^1 \xi^{-\mathbf{S}_n^T + k\mathbf{I}} \mathbf{r}^{(k)} d\xi \right) \quad (53)$$

where the vector $\mathbf{r}^{(k)}$ is

$$\mathbf{r}^{(k)} = \int_{-1}^1 \boldsymbol{\Psi}_\epsilon^T(\eta) \mathbf{h}^{(k)}(\eta) |\mathbf{J}(\eta)| d\eta \quad (54)$$

Evaluating the integration in Eq. (53) analytically leads to

$$\mathbf{R}_{\text{int}} = \boldsymbol{\Psi}_u^{-T} \sum_{k=0} \left((-\mathbf{S}_n^T + (k+1)\mathbf{I})^{-1} \mathbf{r}^{(k)} \right) \quad (55)$$

5. Computational procedure

The scaled boundary polygon formulation has been implemented in Matlab [73]. Throughout the entire solution process, the matrices Ψ_u , S_n and $\Psi_\epsilon(\eta)$ do not change. They are computed only once at the beginning of the simulation and then stored in the system memory to be accessed during each load step. A modified Newton-Raphson algorithm [70] is used to solve the system of nonlinear equations. The relative norm of residual forces χ is used to determine if the solution has converged. A convergence tolerance of $\chi = 0.001$ is chosen. The algorithm for the formulation of the elasto-plastic stiffness matrix and residual load vector is given briefly as follows:

1. Loop over all polygons in the computational domain:
 - (a) Compute the modal displacements Ψ_u and block diagonal Schur matrix S_n (Eq. (11))
 - (b) Compute the strain modes $\Psi_\epsilon(\eta)$ at the integration points and curve fitting locations using Eq. (22)
2. Initialize the total displacement vector $U_b^{(J)} = 0$ at $J = 1$
3. For load step $J = 1, 2, \dots$:
 - (a) Loop over all polygons in the computational domain to evaluate the elasto-plastic stiffness matrix K_{ep} and the external load vector R_{ext} :
 - i. Compute the elasto-plastic constitutive matrix, D_{ep} at the curve fitting locations using Eq. (30)
 - ii. Perform curve fitting to obtain the coefficients D_k in Eq. (33)
 - iii. Construct its corresponding approximation in the scaled boundary coordinates, i.e., $D^{(k)}$ in Eq. (34)
 - iv. For each $D^{(k)}$, compute the matrices $Y^{(k)}$ at the integration points using Eq. (41)
 - v. Integrate analytically in ξ to obtain $X^{(k)}$ using Eq. (42)
 - vi. Evaluate the elasto-plastic stiffness matrix K_{ep} using Eq. (44)
 - vii. Evaluate the external load vector R_{ext} using Eq. (45)
 - (b) Assemble stiffness matrices and external load vectors of each polygon i.e., $\sum_{i=1}^{nPol} K_{ep}$ and $\sum_{i=1}^{nPol} R_{ext}$ in Eq. (38)
 - (c) Initialize the incremental displacement vector $\Delta U_b^{(j)} = 0$ at $j = 1$
 - (d) For each equilibrium iteration, $j = 1, 2, \dots$:
 - i. Loop over all polygons in the computational domain to evaluate the internal load vector $R_{int}^{(j)}$:
 - A. Compute the elastic-plastic stress vector $\sigma(\xi, \eta)$ at the fitting locations
 - B. Perform curve fitting to obtain the coefficients h_k in Eq. (50)
 - C. Construct its corresponding approximation in the scaled boundary coordinates, i.e., $h^{(k)}$ in Eq. (52)
 - D. For each $h^{(k)}$, compute the vectors $r^{(k)}$ at the integration points using Eq. (54)

- E. Integrate analytically in ξ to obtain the internal load vector $\mathbf{R}_{\text{int}}^{(j)}$ using Eq. (55)
- ii. Assemble the internal load vectors of each polygon, $\sum_{i=1}^{nPol} \mathbf{R}_{\text{int}}^{(j)}$, in Eq. (38)
- iii. Compute $d\mathbf{U}_b^{(j)}$ due to the residual force, $\sum_{i=1}^{nPol} (\mathbf{R}_{\text{ext}} - \mathbf{R}_{\text{int}}^{(j)})$, using Eq. (38)
- iv. Update the incremental displacements $\Delta\mathbf{U}_b^{(j+1)} = \Delta\mathbf{U}_b^{(j)} + d\mathbf{U}_b^{(j)}$
- v. Compute the norm of the relative residual load vector χ
- vi. If $\chi > \chi_{\text{tol}}$, set $j = j + 1$ and return to Step i.
- vii. Accumulate the incremental nodal displacements into the total displacement vector, $\mathbf{U}_b^{(J+1)} = \mathbf{U}_b^{(J)} + \Delta\mathbf{U}_b^{(j+1)}$, increment the load step number $J = J + 1$ and go to Step (a)

6. Numerical examples

Five numerical benchmarks are modeled using the proposed formulation. In all the problems presented in this section, the elasto-plastic constitutive matrix, $\mathbf{D}_{\text{ep}}$ and internal stresses $\boldsymbol{\sigma}(\xi, \eta)$ in Eq. (49), are curve fitted with a complete polynomial function of order 2. The locations used to compute $\mathbf{D}_{\text{ep}}(\xi, \eta)$ and $\boldsymbol{\sigma}(\xi, \eta)$ are the Gaussian integration points at $\xi=1$ and $\xi=0.5$. A tolerance value of $\chi_{\text{tol}} = 0.0001$ is used for the equilibrium iterations. All the simulations were performed on an Intel(R) Xeon(R) @3.33 GHz CPU desktop.

6.1. General tests

6.1.1. Elasto-plastic constant stress patch test

The 200mm $\times$ 200mm block subjected to a constant stress $p = 1000\text{MPa}$ and discretized by a patch of polygon elements shown in Fig. 4 is considered. The material properties of the block are Young's modulus $E = 206.9\text{GPa}$, Poisson's ratio $\nu = 0.29$, yield stress $\sigma_y = 450\text{MPa}$ and hardening modulus $H = 5\text{GPa}$. Plane strain conditions are assumed.

The stresses at the integration points shown in Fig. 4 at the end of the analysis is tabulated in Table 1. The results in Table 1 clearly indicate that all the stresses are reproduced exactly. The present formulation passes the elasto-plastic constant stress patch test.

6.1.2. Mesh distortion test

The 200mm $\times$ 200mm block in Section 6.1.1 is again considered. The patch is now discretized by four linear quadrilaterals as shown in Fig. 5. The parameters θ and α , as shown in Fig. 5a, are introduced to quantify the extent of mesh distortion. For this problem, $\theta = \pi/4$. The constant stress patch test is repeated for various values of α . The stresses at the integration points A , B and C are monitored and are reported in Table 2. The results indicate that the present formulation is insensitive to mesh distortion.

6.1.3. Shear locking test

The cantilever beam with dimensions and loading conditions shown in Fig. 6. The material properties of the beam are Young's modulus $E = 1000$ and Poisson's ratio $\nu = 0.3$. A set of consistent units is assumed. The beam is meshed using one 8-noded quadrilateral. Each line element on the quadrilateral is discretized by one quadratic element. The vertical displacement of the beam at point A is monitored whilst the aspect ratio of the beam is increased by increasing the length of the aspect ratio of the beam from 10 to 100 and 1000.

The vertical displacement of the beam is reported in Table 3. It is evident from the results in Table 3 that the formulation does not suffer from shear locking.

6.2. Thick cylinder subjected to internal pressure

This numerical benchmark is solved with three objectives in mind. The first objective is to validate the formulation for a standard elasto-plasticity problem in which the analytical solution is known. The second objective is to test the formulation for near incompressibility locking conditions. The third objective is to compare the performance of the developed formulation with standard finite elements in commercial softwares.

6.2.1. Validation with analytical solution

An infinitely long thick cylinder subjected to an internal pressure $p = 135.3\text{MPa}$ shown in Fig. 7 is considered. The material properties of the cylinder are Young's modulus $E = 205.8\text{GPa}$, Poisson's ratio $\nu = 0.3$ and yield stress $\sigma_y = 235.2\text{MPa}$. Plane strain conditions are assumed. The analytical solution for this problem is reported by Lubliner [74] for an elastic-perfectly plastic material with Tresca yield condition. The stress components in polar coordinates, (r, θ) , are

$$\sigma_{rr}(r) = \begin{cases} -\frac{\sigma_y}{2} \left[\left(\frac{c}{r}\right)^2 - \left(\frac{c}{r_o}\right)^2 \right], & r > c \\ -\frac{\sigma_y}{2} \left[1 - \left(\frac{c}{r_o}\right)^2 + \ln\left(\frac{c}{r}\right)^2 \right], & r < c \end{cases} \quad (56)$$

$$\sigma_{\theta\theta}(r) = \begin{cases} \frac{\sigma_y}{2} \left[\left(\frac{c}{r}\right)^2 + \left(\frac{c}{r_o}\right)^2 \right], & r > c \\ \frac{\sigma_y}{2} \left[1 + \left(\frac{c}{r_o}\right)^2 - \ln\left(\frac{c}{r}\right)^2 \right], & r < c \end{cases} \quad (57)$$

where r_o is the external radius of the cylinder and $c = 135.8\text{mm}$ is the radial distance from the origin to the exterior boundary of the plastic zone. The radial displacement is

$$u_r(r) = \frac{(1 - 2\nu)(1 + \nu)r\sigma_{rr}}{E} + \frac{(1 - \nu^2)c^2\sigma_y}{rE} \quad (58)$$

Two types of meshes are used to discretize the cylinder: (1) a $M \times N$ mesh of quadrilateral elements as shown in Fig. 8a and (2) an arbitrary n -sided polygon mesh as shown in Fig. 8b.

The $M \times N$ quadrilateral element mesh is first considered. Each quadrilateral in the mesh is treated as a SBFEM subdomain. The convergence of the nodal displacements and stresses at the integration points is investigated by: (1) refining the mesh through parameters M and N (h -refinement) and (2) increasing the order of shape function in each quadrilateral element (p -refinement).

In the h -refinement study, an edge of a quadrilateral is modeled with one quadratic element. Elasto-plastic analysis are carried out on the following 4×8 , 6×12 , 10×20 and 16×32 quadrilateral meshes. The nodal radial displacements u_r and von Mises stresses σ_{eq} at the integration points of every quadrilateral are monitored. Fig. 9a shows the percentage of relative error in the nodal displacement u_r of all the four meshes. Fig. 9b shows the relative errors in σ_{eq} at the integration points on the edges of every quadrilateral. It is evident from these figures that the errors in u_r and σ_{eq} decrease as the number of elements increases. The largest error in σ_{eq} for all the four meshes occurs at $r = c$, where the material behavior changed from elasto-plastic to fully elastic. At $r = c$, the stress field is not smooth and may not have been adequately represented by the continuous polynomial functions used to approximate $\mathbf{D}_{ep}$ and $\boldsymbol{\sigma}(\xi, \eta)$, in Eqs. (34) and (52), respectively. This contributes to the larger error observed in this region. It is to be noted that the errors are comparable to the tolerance χ_{tol} used for the equilibrium iterations.

In the p -refinement study, a 10×20 mesh is used. Elasto-plastic analysis are performed by changing the order of the elements on the edges of the quadrilaterals. Linear, quadratic and cubic elements are considered. The errors in u_r and σ_{eq} at the integration points, are shown in Figs. 10a and 10b, respectively. It is observed that the errors in u_r and σ_{eq} decrease when higher order elements are used.

The cylinder is then discretized using polygon meshes. Three meshes having 61, 213 and 454 polygons are considered. Fig. 8b shows the mesh with 61 polygons. Quadratic elements are used to discretize the edges on the polygons boundaries. Elasto-plastic analysis are carried out with the polygon meshes considered. Fig. 11a shows the error in nodal displacement u_r . Fig. 11b shows the errors in σ_{eq} at the integration points. The results converge as the number of polygons increases. The errors of the polygon and quadrilateral meshes are comparable when the number of polygons and quadrilaterals are approximately the same. For example, the range of error is 0.0% – 0.3% for the 10×20 quadrilateral mesh, whereas it is –0.1% – 0.3% for the 213 polygon mesh.

Using the 213 polygon mesh, the simulation is repeated by increasing the order the elements used to discretize the polygon edges. Linear, quadratic and cubic elements are considered. Fig. 12a shows the relative error in nodal u_r . Fig. 12b shows the relative errors in σ_{eq} at the integration points. The errors in u_r and σ_{eq} decreases as the order the element increases.

Figs. 13a and 13b compares the distribution of σ_{rr} and $\sigma_{\theta\theta}$ obtained from the 10×20 quadrilateral mesh and the 213 polygon mesh with the analytical

solution. The results agree well with the analytical solution. Figs. 14a and 14b show the σ_{eq} contour plots obtained with the 10×20 quadrilateral mesh and the 213 polygon mesh at the end of the simulation. The stress distribution in the cylinder is symmetric and conforms to the the symmetry of the geometry and loading conditions of this problem.

6.2.2. Volumetric locking test

The 10×20 quadrilateral mesh is now considered. The elasto-plastic analysis is repeated considering the near incompressibility limit by increasing the value of the Poisson's ratio to 0.49, 0.499 and 0.4999. Figs. 15a and 15b show the percentage of error in nodal u_r and σ_{eq} at the integration points. The results in Figs. 15a and 15b indicate that the formulation does not suffer from volumetric locking in the near-incompressible limit.

It is of interest to note that for problems that are fully incompressible, the computation breaks down because the constitutive matrix $\mathbf{D}$ in Eq. 30 is not finite. For this particular case, an approach similar to that developed by Song and Wolf [75] will have to be used to formulate the coefficient matrices $\mathbf{E}_i$ in Eq. (7).

6.2.3. Comparison with standard finite elements

For the purpose of comparison with the standard finite elements, the elasto-plastic analysis is repeated using the $M \times N$ meshes described in Section 6.2.1. The quadratic 8-node element (QUAD8) is used for comparison. A tolerance value of $\chi_{tol} = 1 \times 10^{-6}$ is used for the equilibrium iterations.

Figs. 16a-16d compare the percentage of error in nodal u_r of the present formulation with QUAD8. The results in the figure suggests that the present formulation is generally more accurate than the QUAD8.

The computational time required for each mesh is reported in Table 4. As is expected, the computational time required by the proposed formulation is more than that required by QUAD8. The computational time for the stiffness matrix in the present formulation is more than QUAD8 because of additional time to compute Ψ_u , $\mathbf{S}_n$ and Ψ_ϵ before the analysis. However, this process accounts for approximately only 3.5% of the total computation time. The ratio of computational time required for an elasto-plastic analysis is approximately 1.9 compared with QUAD8. As is evident from Table 4, the bulk of the computational time is spent on the equilibrium iterations. The computational time is more in the present formulation because of the additional fitting points to bring to the yield surface and also the curve fitting operation to compute the stress distribution in Eq. (50).

Although the present formulation requires more computational time than the standard finite elements, it is more accurate. The computational efficiency of the present method is depicted in Fig. 17. The present formulation is more computationally efficient if higher accuracies are desired. Moreover, it has additional advantages such as the ability to assume any number of sides and asymptotic nature of its shape functions, which leads to flexibility in meshing and also remeshing during crack propagation as was demonstrated by Ooi et al. [31].

6.3. Perforated square plate

The perforated square plate subjected to the loading conditions shown in Fig. 18a is considered. This example was the benchmark problem used by Duster and Rank [76, 77], Duster et al. [78] to study the performance of the p -version FEM for elasto-plastic analysis. The material properties of the plate are: shear modulus $\mu = 80193.8\text{MPa}$, bulk modulus $\kappa = 164206.0\text{MPa}$ and yield stress $\sigma_y = 450\text{MPa}$. The material is elastic-perfectly plastic and the von Mises yielding criterion is adopted. Plane strain conditions are assumed. The reference solution for this problem was reported in [76] for $p = 300\text{MPa}$ using a fine mesh consisting of 5568 elements with seventh order shape functions having 546,755 degrees-of-freedom.

An elasto-plastic analysis is performed with the 193 polygon mesh shown in Fig. 18b. The error in the displacement components u_x and u_y at points A and B and the error in the stress component σ_{yy} at point C are monitored. The analysis is repeated with finer polygon meshes i.e., 364, 456, 644, 869 and 1139 polygons. Linear and quadratic shape functions are considered.

Table 5 shows that the errors in u_x , u_y and σ_{yy} at these points of interest reduce as the polygon mesh is gradually refined. Quadratic elements exhibit higher accuracy. For example, the 1139 polygon mesh with quadratic shape functions has an error of 1.08% in σ_{yy} whereas it is 7.96% for the same polygon mesh using linear shape functions. The error in u_x was 0.003% for a mesh with quadratic shape functions while it is 0.05% for the same mesh with linear shape functions.

Fig. 19a shows the σ_{eq} -contour plot obtained from the 456 polygon mesh at $p = 450\text{MPa}$. Fig. 19b shows the predicted plastic region, which agrees well with the p -version FEM results reported in [76].

6.4. Cook's Membrane

The Cook's membrane subjected to the loading conditions shown in Fig. 20a is considered. The material properties as reported in [79] are: Young's modulus $E = 20000$, Poisson's ratio $\nu = 0.3$, and yield stress $\sigma_y = 1$. A consistent set of units is assumed. The material is assumed to be elastic-perfectly plastic. The von Mises yielding condition is adopted. Plane stress conditions are assumed. The load-displacement responses from the three-field mixed FEM formulation of Bilotta et al. [79], and the finite element solution from ANSYS [80] are used for comparison.

The membrane is discretized using the polygon mesh shown in Fig. 20b. It has 167 polygons and 388 nodes. Linear elements are used to discretize the edges of the polygons. The results obtained from the polygon mesh are compared with those obtained with a constant strain triangular mesh having approximately the same number of degrees-of-freedom.

Fig. 21 shows the load-displacement response of the membrane. The load factor, λ , approaches the limiting value predicted by the three-field mixed formulation of Bilotta et al. [79] after which, convergence is no longer possible. There is little difference in the predicted load-displacement response between a

polygon mesh and a triangular element mesh having approximately the same number of nodes. Fig. 22 shows the σ_{eq} - contour plot of the membrane at $\lambda = 0.35$. The predicted yield region agrees well with that reported in [79].

6.5. Edge cracked specimen subjected to uniform tension

The edge cracked specimen subjected to a uniform tension $p = 1.47\text{MPa}$ shown in Fig. 23a is considered. The material properties of the specimen are Young's modulus $E = 700\text{MPa}$, Poisson's ratio $\nu = 0.2$ and yield stress $\sigma_y = 2.43\text{MPa}$. Plane strain conditions are assumed. The material is elastic-perfectly plastic. The von Mises yield criterion is adopted. The reference solution for this problem is obtained with ANSYS [80] using a fine mesh of 91,726 6-noded triangular elements.

Due to the symmetry of the geometry and boundary conditions, only half of the specimen is modeled. The performance of the formulation is investigated with uniform and non-uniform meshes. A uniform polygon mesh is generated using the procedures outlined in [31] by specifying a global element size for the triangular element mesh. For non-uniform polygon meshes, the plate geometry is first partitioned into sub-regions as shown in Fig. 23b. The sides of the plate is then divided as follows: (1) Lines *A* are divided into N segments with a 0.5 scale bias towards the location of the crack; (2) Lines *B* are divided into $2N$ uniform segments; and (3) Lines *C* are divided into $N/2$ uniform segments. The resulting triangular element mesh is used to generate the corresponding polygon mesh.

The uniform meshes are considered first. The polygon meshes shown in Fig. 24a-Fig. 24c are generated from triangular element meshes with sizes, $h = 0.014\text{mm}$, 0.010mm and 0.00075mm , respectively. Linear shape elements are used to discretise the polygon edges. In order to completely capture the angular variation of the stress around the crack, each edge on the cracked polygon is discretised with ten linear elements. Triangular element meshes having approximately the same number of nodes are used for comparison. Fig. 25a-Fig. 25d plot $\sigma_{\theta\theta}$ along r at $\theta = 0^\circ$ for the meshes shown in Fig. 24a-Fig. 24c. The numerical values of their errors are shown in Table 6.

The results in Table 6 indicate that the solutions converge to the reference solution as the mesh is refined. For approximately the same number of degrees-of-freedom, the polygon meshes are more accurate. For example, the error at $r = 0.625\text{mm}$ is -2.24% for the fine polygon mesh having 2128 degrees-of-freedom and 9.73% for the fine triangular element mesh with 2250 degrees-of-freedom. The results in Table 6 also indicate that the polygon meshes can be more accurate than the finer triangular element meshes. For example, the $\sigma_{\theta\theta}$ obtained from the medium polygon mesh shown in Fig. 18b is more accurate than the fine triangular element mesh in Fig. 24c.

Non-uniform meshes are considered next. The polygon meshes shown in Fig. 26a-Fig. 26c were generated from their corresponding triangular element meshes by specifying $N = 2, 4$ and 6 , respectively. Triangular element meshes having approximately the same number of nodes are used for comparison. The

plot of $\sigma_{\theta\theta}$ along r at $\theta = 0^\circ$ and their corresponding numerical values are shown in Fig. 26a-Fig. 26d and Table 7.

As with the case of the uniform mesh, for approximately the same number of degrees-of-freedom, the polygon meshes are more accurate. When the mesh is sufficiently fine, the polygon mesh can be more accurate than a finer triangular element mesh. This is observed from the results in Table 7 obtained with the medium polygon mesh shown in Fig. 26b and the fine triangular element mesh shown in Fig. 26c. Non-uniform polygon meshes are generally more accurate than the uniform polygon meshes. This is because the finer mesh in the plastic region of the non-uniform meshes allows the nonlinear material response to be modeled more accurately.

The distribution of $\sigma_{\theta\theta}$ along r at $\theta = 0^\circ$ in the near-tip region using the non-uniform polygon meshes is shown in Fig. 28. As finer polygon meshes are used, the stresses in the near-tip region approaches the reference solution. In particular, for the fine polygon mesh the predicted $\sigma_{\theta\theta}$ coincides with the reference solution for $r \geq 0.1\text{mm}$.

The averaged absolute error of the $\sigma_{\theta\theta}$ component at the points shown in Fig. 27a - Fig. 27c in terms of CPU times is illustrated in Fig. 29. The plot in Fig. 27 clearly indicates that the present formulation is more efficient than the standard finite element for elasto-plastic fracture problems. For the same amount of CPU time, the average absolute errors in $\sigma_{\theta\theta}$ obtained from the polygon meshes are consistently smaller than the finite element meshes.

Fig. 30 shows the von Mises stress contour plot of the medium uniform and non-uniform meshes at the end of the analysis.

6.6. Bi-metallic specimen with two edge cracks subjected to uniform tension

The copper-steel bi-metallic specimen with two edge cracks and subjected to a uniform tension, $p = 40\text{MPa}$ shown in Fig. 31 is considered. The material properties of copper are: Young's modulus $E_c = 110\text{GPa}$, Poisson's ratio $\nu_c = 0.34$, and yield strength $\sigma_{yc} = 70\text{MPa}$. The material properties of steel are: Young's modulus $E_s = 200\text{GPa}$, Poisson's ratio $\nu_s = 0.30$, and yield strength $\sigma_{ys} = 280\text{MPa}$. Plain strain conditions are assumed. Both material are elastic-perfectly plastic. The von Mises yield criterion is adopted. The reference solution for this problem is obtained from a mesh of 224,084 3-noded triangular elements using ANSYS [80].

Due to the symmetry of the geometry and loading conditions, only half of the specimen is modeled. Fig. 32a-Fig. 32c show the polygon meshes of the specimen and the corresponding 3-noded triangular element meshes, which have approximately the same numbers of degrees-of-freedom, used for comparison. Linear elements are used to model the edges of the polygons. The crack is modeled entirely using one polygon as shown in Fig. 1b. Each edge on the crack polygon is discretised into ten elements so that the angular variation of stresses around the crack can be captured accurately.

Fig. 33a-Fig. 33d show the distribution of the stress component, $\sigma_{yy}(r, \theta = \frac{3\pi}{4})$, where $r = 0$ is at the crack tip. Table 8 presents the numerical values at

selected locations. For approximately the same number of degrees-of-freedom, the polygon meshes are more accurate. For this problem, the results of the medium polygon mesh in Fig. 32b with 934 degrees-of-freedom are more accurate than those obtained from a fine triangular element mesh in Fig. 32c with 1454 degrees-of-freedom. Fig. 34 shows the von Mises stress contour plot of medium polygon mesh and the yield region at the end of the analysis. Only the copper region is observed to yield due to its lower yield strength.

7. Conclusions

A scaled boundary polygon formulation has been developed for elasto-plastic analysis of structures. Underpinning this formulation is the application of scaled boundary shape functions, which enable the displacements, strain and stress fields to be reformulated in terms of the nodal displacements as in the FEM. This operation of replacing the standard finite element shape functions with the polygon shape functions enable the SBFEM to model material nonlinearity. The scaled boundary shape functions are conforming and continuous within a polygon. For normal polygons, the shape functions are linearly complete. The construction of higher order scaled boundary shape functions is simpler compared with other polygon elements reported in the literature [1, 2, 3, 4, 11, 12]. These advantages together with the capability of exact geometry representation makes it interesting to further investigate the scaled boundary formulation in combination with the isogeometric analysis [81, 82, 83, 84, 85].

For polygons with cracks and notches, the derivatives of these shape functions naturally include the strain singularity at the crack/notch tip. This allows problems with cracks and notches to be modeled with higher accuracy without any enrichment functions or fine crack/notch tip meshes.

To simplify the integration of the stiffness matrix and the internal load vector, the elasto-plastic constitutive matrix and internal stresses in each polygon were approximated by polynomial functions. This implies that a sufficient number of polygons are required to discretise the computational domain so that the variation of the material constitutive matrices and the internal stress vector in the plastic zone can be sufficiently represented.

Standard FEM procedures were used to construct the stiffness matrix and residual load vector in each polygon. Numerical integration is required only on the boundaries of the polygons. Within the polygon, the stiffness matrix and internal load vector are power functions in the radial coordinate, ξ , and can be integrated analytically.

The formulation was successfully validated using several numerical benchmarks. The displacement and stress solutions of the scaled boundary polygons converged to the analytical and reference solutions reported in the literature as the meshes were refined. For problems with a crack, it was shown that the scaled boundary polygons were more accurate and computationally efficient compared with the standard FEM.

The developed formulation can be conceptually extended to three-dimensional problems. A polyhedra mesh of any number of faces can be generated from a

tetrahedral mesh following the procedure to generate a polygon mesh from a triangular element mesh [31] after some augmentation. The three-dimensional SBFEM equation in displacement [32] is then formulated for each polyhedron. The scaled boundary shape functions can be computed from the surface elements that discretizes the boundary of the polyhedra. Correspondingly, Eqs. (33) and (50) have to be polynomial functions in three dimensions. The stiffness matrix and residual load vectors can still be integrated analytically in the ξ direction following similar procedures in Sections 4.1 and 4.3. Plasticity can then be considered using standard finite element procedures.

The results of this study broadens the potential application of the SBFEM, which was previously limited to linear elastic materials only. The responses of nonlinear materials can now be modeled in a framework that is similar to that in the FEM.

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Tables

Table 1: Stresses at integration points for constant stress patch test

	σ_{xx} (MPa)	σ_{yy} (MPa)	τ_{xy} (MPa)
<i>A</i>	1000.0	0.0	0.0
<i>B</i>	1000.0	0.0	0.0
<i>C</i>	1000.0	0.0	0.0
<i>D</i>	1000.0	0.0	0.0
<i>E</i>	1000.0	0.0	0.0

Table 2: Stresses at integration points for mesh distortion test.

α	Location	σ_{xx} (MPa)	σ_{yy} (MPa)	τ_{xy} (MPa)
$20\sqrt{2}$	<i>A</i>	1000.0	0.0	0.0
	<i>B</i>	1000.0	0.0	0.0
	<i>C</i>	1000.0	0.0	0.0
$40\sqrt{2}$	<i>A</i>	1000.0	0.0	0.0
	<i>B</i>	1000.0	0.0	0.0
	<i>C</i>	1000.0	0.0	0.0
$60\sqrt{2}$	<i>A</i>	1000.0	0.0	0.0
	<i>B</i>	1000.0	0.0	0.0
	<i>C</i>	1000.0	0.0	0.0
$80\sqrt{2}$	<i>A</i>	1000.0	0.0	0.0
	<i>B</i>	1000.0	0.0	0.0
	<i>C</i>	1000.0	0.0	0.0

Table 3: Normalized displacements for shear locking test.

L/c	v_A/v_{anal}
10	1.000
100	1.000
1000	1.000

Table 4: Computational time for thick cylinder problem.

		Computational time (s)							
		4×8		6×12		10×20		16×32	
		QUAD8	Present	QUAD8	Present	QUAD8	Present	QUAD8	Present
Stiffness matrix $\mathbf{K}$	$\Psi_u, \mathbf{S}_n$	-	0.084	-	0.184	-	0.531	-	1.294
	$\Psi_\epsilon(\eta)$	-	0.045	-	0.105	-	0.291	-	0.738
	Step 1	0.039	0.022	0.081	0.05	0.264	0.136	0.564	0.357
	Step 2	0.005	0.004	0.009	0.008	0.031	0.021	0.054	0.066
	Step 3	0.004	0.004	0.009	0.008	0.022	0.022	0.055	0.067
	Step 4	0.005	0.004	0.009	0.008	0.023	0.022	0.055	0.066
	Step 5	0.005	0.004	0.009	0.008	0.023	0.021	0.055	0.065
	Step 6	0.004	0.004	0.009	0.008	0.023	0.023	0.055	0.066
	Step 7	0.005	0.004	0.009	0.008	0.022	0.022	0.055	0.066
	Step 8	0.015	0.023	0.025	0.036	0.049	0.081	0.102	0.177
	Step 9	0.017	0.029	0.027	0.049	0.083	0.149	0.162	0.375
	Step 10	0.018	0.033	0.045	0.087	0.118	0.228	0.257	0.547
Equilibrium iterations	Step 1	0.086	0.150	0.187	0.320	0.573	0.938	1.233	2.359
	Step 2	0.084	0.148	0.190	0.321	0.513	0.955	1.256	2.371
	Step 3	0.086	0.149	0.188	0.318	0.516	0.963	1.230	2.348
	Step 4	0.084	0.145	0.186	0.323	0.514	0.986	1.234	2.384
	Step 5	0.084	0.147	0.186	0.324	0.512	0.927	1.233	2.354
	Step 6	0.086	0.146	0.188	0.319	0.513	0.925	1.244	2.409
	Step 7	0.180	0.238	0.487	0.512	1.326	1.932	3.807	6.277
	Step 8	0.193	0.452	0.505	1.365	2.276	3.301	4.670	10.09
	Step 9	0.204	0.505	0.758	1.160	2.434	3.765	5.743	11.76
	Step 10	0.326	0.720	1.076	1.880	2.982	5.154	6.915	15.56
Total		1.531	3.191	4.182	7.693	12.82	22.22	29.96	63.82

Table 5: Percentage of error in u_x at node A, u_y at node B and σ_{yy} at node C in perforated plate example for $p = 300\text{MPa}$.

Polygons	Linear				Quadratic			
	DOF	% Error			DOF	% Error		
		u_x	u_y	σ_{yy}		u_x	u_y	σ_{yy}
193	852	0.2073	-0.1629	46.178	2088	-0.0509	0.0406	-19.970
361	1572	0.0633	-0.0495	17.247	3864	-0.0024	0.0032	-6.7562
456	1968	0.0582	-0.0454	14.844	4846	-0.0062	0.0056	-4.9631
644	2752	0.0635	-0.0496	11.589	6790	0.0019	-0.0010	-3.3882
869	3684	0.0488	-0.0380	9.6183	9104	0.0025	-0.0018	-2.1553
1139	4796	0.0509	-0.0397	7.9578	11868	0.0033	-0.0027	-1.0708

Table 6: Percentage of error in $\sigma_{\theta\theta}(r, 0)$ for edge cracked specimen problem: uniform meshes.

r (mm)	% Error in $\sigma_{\theta\theta}$			
	Triangular		Polygon	
	Fine	Coarse	Medium	Fine
0.625	9.73	-14.96	-5.62	-2.24
1.250	9.19	-3.36	-5.73	0.58
1.875	8.52	-4.99	1.99	3.29
2.500	7.34	2.07	1.55	4.66
3.125	6.23	1.56	1.51	2.36
3.750	5.56	1.67	2.46	2.66
4.375	5.13	1.17	1.20	1.90
5.000	4.94	1.22	1.17	1.79

Table 7: Percentage of error in $\sigma_{\theta\theta}(r, 0)$ for edge cracked specimen problem: non-uniform meshes.

r (mm)	% Error in $\sigma_{\theta\theta}$			
	Triangular		Polygon	
	Fine	Coarse	Medium	Fine
0.208	5.67	-12.77	4.49	2.28
0.417	4.04	-10.18	5.24	0.25
0.833	4.08	-7.30	3.17	0.69
1.250	4.31	-2.29	-0.83	0.00
1.667	4.41	-0.96	-1.13	-0.50
2.083	4.04	-1.46	-0.66	0.31
2.500	3.78	-1.25	-0.12	-0.27
3.333	2.72	-0.24	-0.48	-0.27
4.167	3.08	-1.52	-0.04	0.24
5.000	1.93	-0.52	-0.20	-0.19

Table 8: Percentage of error in $\sigma_{yy}(r, \frac{3\pi}{4})$ for bi-metallic specimen with central crack problem.

r (mm)	% Error in σ_{yy}			
	Triangular		Polygon	
	Fine	Coarse	Medium	Fine
0.375	-17.89	-15.03	-9.421	-0.116
1.500	-4.011	0.972	-7.969	-0.007
2.250	-2.571	-5.210	-2.074	0.003
3.000	-4.994	-4.188	-3.270	0.008
3.750	-9.855	-6.723	-5.559	-0.013
4.500	-6.968	-3.023	-2.570	-0.011
5.250	-4.748	-4.132	-2.873	-0.012
6.000	-4.911	-2.434	-2.946	0.003
7.620	-2.955	-3.152	-1.940	0.014
9.906	-2.285	-1.791	-1.152	-0.002
13.13	-2.420	-1.652	-0.946	-0.005
17.68	-1.803	-1.214	-1.152	-0.004

Figures

Figure 1: Polygon representation of the scaled boundary finite element method: (a) normal polygon, (b) polygon modelling a notch.

(a)

(b)

Figure 2: Scaled boundary shape functions: (a) regular pentagon and (b) cracked pentagon.

Figure 3: Location of points to compute fitting surfaces for $\mathbf{D}_{\text{ep}}(x, y)$ and $\sigma(x, y)$ for a polygon with linear elements along its boundary.

Figure 4: Elasto-plastic constant stress patch test.

Figure 5: Quadrilateral meshes for mesh distortion test: (a) $\alpha_D = 20$; (b) $\alpha_D = 40$; (c) $\alpha_D = 60$ and (d) $\alpha_D = 80$.

Figure 6: Cantilever beam subjected to bending moment for shear locking test.

Figure 7: Thick cylinder subjected to internal pressure p .

(a)

(b)

Figure 8: Scaled boundary finite element meshes of thick cylinder: (a) $M \times N$ mesh and (b) typical polygon mesh.

(a)

(b)

Figure 9: Percentage of error for thick cylinder problem using quadrilateral meshes with increasing mesh density: (a) nodal u_r and (b) σ_{eq} at integration points.

(a)

(b)

Figure 10: Percentage of error for thick cylinder problem using 10×20 quadrilateral mesh with increasing order of shape functions: (a) nodal u_r and (b) σ_{eq} at integration points.

(a)

(b)

Figure 11: Percentage of error for thick cylinder problem using polygon meshes with increasing mesh density: (a) nodal u_r and (b) σ_{eq} at integration points.

(a)

(b)

Figure 12: Percentage of error for thick cylinder problem using a 213 polygon mesh with increasing order of shape functions: (a) nodal u_r and (b) σ_{eq} at integration points.

(a)

(b)

Figure 13: Distribution of: (a) σ_{rr} and (b) $\sigma_{\theta\theta}$ in thick cylinder.

(a)

(b)

Figure 14: Plot of σ_{eq} for thick cylinder problem using: (a) 10×20 quadrilaterals and (b) 213 polygons.

(a)

(b)

Figure 15: Percentage of error for thick cylinder problem using a 10×20 unifrom mesh varying ν : (a) nodal u_r and (b) σ_{eq} at integration points.

(a)

(b)

(c)

(d)

Figure 16: Comparison of percentage of error in nodal u_r between the 8-noded quadrilateral element of ANSYS with the present formulation using: (a) 4×8 mesh; (b) 6×12 mesh; (c) 10×20 mesh and (d) 16×32 mesh.

Figure 17: Average absolute error in nodal u_r with respect to CPU times.

Figure 18: Perforated square plate subjected to uniform tension: (a) geometry of one quarter of the plate and (b) typical polygon mesh having 193 polygons.

(a)

(b)

Figure 19: Stress contour plot of perforated plate: (a) σ_{eq} -contour plot and (b) comparison of yield region with Duster and Rank [76].

(a)

(b)

Figure 20: Cooks membrane problem: (a) geometry and (b) polygon mesh.

Figure 21: Load-deflection response for Cook's membrane problem.

(a)

(b)

Figure 22: Stress contour plot of Cook's membrane problem: (a) σ_{eq} -contour plot and (b) comparison of yield region with Bilotta et al. [79].

Figure 23: Edge cracked specimen subjected to uniform tension: (a) geometry and (b) distribution of 3-noded triangular elements along each edge on half of the specimen.

(a) Polygon: 239 polygons, 612 nodes; Triangle: 1170 elements, 649 nodes

(b) Polygon: 331 polygons, 814 nodes; Triangle: 1492 elements, 818 nodes

(c) Polygon: 450 polygons, 1064 nodes; Triangle: 2028 elements, 1125 nodes

Figure 24: Uniform polygon meshes used to discretise the specimen and corresponding triangular element meshes used for comparison: (a) coarse mesh; (b) medium mesh and (c) fine mesh.

(a)

(b)

(c)

Figure 25: Distribution of $\sigma_{\theta\theta}$ along $(r, 0)$ in edge cracked specimen using (a) coarse; (b) medium and (c) fine uniform meshes.

(a) Polygon: 91 polygons, 253 nodes; Triangle: 548 elements, 325 nodes

(b) Polygon: 192 polygons, 471 nodes; Triangle: 918 elements, 500 nodes

(c) Polygon: 325 polygons, 753 nodes; Triangle: 1376 elements, 737 nodes

Figure 26: Non-uniform polygon meshes used to discretise the specimen and corresponding triangular element meshes used for comparison: (a) coarse mesh; (b) medium mesh and (c) fine mesh.

(a)

(b)

(c)

Figure 27: Distribution of $\sigma_{\theta\theta}$ along $(r, 0)$ in edge cracked specimen using: (a) coarse; (b) medium and (c) fine non-uniform meshes.

Figure 28: Distribution of $\sigma_{\theta\theta}$ along $(r, 0)$ in the near-tip region of the edge cracked specimen using non-uniform polygon meshes.

Figure 29: Average absolute error in $\sigma_{\theta\theta}$ with respect to CPU times.

(a)

(b)

Figure 30: von-Mises stress contour plot of edge cracked specimen using medium (a) uniform and (b) non-uniform polygon meshes.

Figure 31: Bi-material specimen with two edge cracks.

(a) Polygon: 129 polygons, 318 nodes; Triangle: 538 elements, 305 nodes

(b) Polygon: 200 polygons, 467 nodes; Triangle: 877 elements, 488 nodes

(c) Polygon: 320 polygons, 714 nodes; Triangle: 1340 elements, 727 nodes

Figure 32: Polygon meshes used to discretise the specimen and corresponding triangular element meshes used for comparison: (a) coarse mesh; (b) medium mesh and (c) fine mesh.

(a)

(b)

(c)

Figure 33: Distribution of σ_{yy} along $(r, \frac{3\pi}{4})$ in bi-metallic specimen with two edge cracks: (a) coarse mesh; (b) medium mesh and (c) fine mesh.

Figure 34: Stress contour plot of bi-metallic specimen with two edge cracks: (a) σ_{eq} -contour plot and (b) yield region.