COPYRIGHT NOTICE

FedUni ResearchOnline
http://researchonline.ballarat.edu.au

This is the submitted for peer-review version of the following article:

Mammadov, M. (2014). Turnpike theorem for an infinite horizon optimal control problem with time delay. *SIAM Journal on Control and Optimization 52*(1): 420-438

Which has been published in final form at:
http://dx.doi.org/10.1137/130926808

© 2014, Society for Industrial and Applied Mathematics.

This is the author's version of the work. It is posted here with permission of the publisher for your personal use. No further distribution is permitted.

SIAM J. CONTROL OPTIM.
Vol. 0, No. 0, pp. 000–000
© XXXX Society for Industrial and Applied Mathematics

TURNPIKE THEOREM FOR AN INFINITE HORIZON OPTIMAL CONTROL PROBLEM WITH TIME DELAY*

MUSA A. MAMMADOV†

Abstract. An optimal control problem for systems described by a special class of nonlinear differential equations with time delay is considered. The cost functional adopted could be considered as an analogue of the terminal functional defined over an infinite time horizon. The existence of optimal solutions as well as the asymptotic stability of optimal trajectories (that is, the turnpike property) are established under some quite mild restrictions on the nonlinearities of the functions involved in the description of the problem. Such mild restrictions on the nonlinearities allowed us to apply these results to a blood cell production model.

Key words. optimal control, turnpike property, time delay systems, asymptotic stability, blood cell model

AMS subject classifications. 49J99, 92C37

DOI. 10.1137/130926808

1. Introduction.

In this paper the problem of asymptotic stability of optimal trajectories in nonlinear optimal control problems with time delay is investigated. Such problems are tightly related to several recently developed mathematical models having important practical applications. The phenomenon of stability of optimal trajectories is often called the *turnpike property* after Chapter 12 in [8]. This property states that, regardless of initial conditions, all *optimal* trajectories spend most of the time within a small neighborhood of some *optimal stationary point* when the planning period is long enough. For a classification of different definitions for the turnpike property, we refer to [5, 31, 39, 53], as well as [6] for the so-called exponential turnpike property. Possible applications in Markov games can be found in a recent study [22].

A number of powerful theoretical approaches have been suggested for the study of the turnpike property for both continuous and discrete systems. Some convexity assumptions are sufficient for discrete systems [31, 39]; however, rather restrictive assumptions are usually required for continuous time systems. We briefly mention here some approaches developed for continuous time systems.

The majority of approaches in the literature involve optimal control problems with (discounted and undiscounted) *integral functionals* (see [5, 53] and references therein). Among the most successful approaches developed we mention here the approach developed by Rockafellar [46, 47] that applies related techniques from convex analysis and the "direct" approach developed by Scheinkman, Brock, and collaborators (see, for example, [30]) that applies the Maximum principle and then reduces the main problem to the study of stability of ordinary differential equations with unknown terminal values for costate variables. Several other approaches in this area have been developed, including those considering a special class of problems (e.g., [14, 43, 50, 54]). An interesting class of control problems considered in [9, 10] involves

*Received by the editors June 28, 2013; accepted for publication (in revised form) December 5, 2013; published electronically DATE.
http://www.siam.org/journals/sicon/x-x/92680.html
†Centre for Informatics and Applied Optimization, University of Ballarat, Victoria 3353, Australia, and National ICT Australia, Sydney NSW 2015, Australia (m.mammadov@ballarat.edu.au, musa.mammadov@nicta.com.au).

long-run average cost functions where the asymptotic behavior of optimal solutions is defined in terms of a probability measure.

It should be noted that quite restrictive assumptions are used in these studies, especially when considering *discounted* integral functionals. In this sense, Rockafellar's approach is the most successful one, as can be seen from a recent publication [48] where for a special convex problem (Ramsey's problem) the turnpike property is established without any additional restrictive assumptions.

Much stronger results are obtained for a special class of terminal functionals defined as a lower limit at infinity of utility functions. This approach is introduced in [34] (see [37] for more references), where stability results were established for some classes of nonconvex problems with applications to environment pollution models. Later, this class of terminal functionals was used to establish the turnpike theory in terms of statistical convergence [36, 44], where the convergence of optimal trajectories to some stationary point is proved in the sense of "almost" convergence while ordinary convergence may not be true.

In this paper the turnpike property is established for an optimal control problem described by time delay systems. Such systems find various applications in biology and medicine [3, 12, 24, 40, 41], where the time delay arises naturally, for instance, as a time lag required for cells to start dividing after they are activated. Some other applications are in epidemiology and population dynamics [13, 23], laser optics [29, 33], power systems and neural networks [2, 13, 25, 49], physiological processes [16, 3, 33], life sciences and economics [12, 27], and other natural sciences [12, 29]. A comprehensive modern exposition of recent theoretical achievements and various applications can be found in monographs [7, 15, 23].

Note that many aspects of optimal control problems involving time delay systems have been studied in the literature for many decades (see, for example, [45]); however, the turnpike theory for these systems could be considered in its initial stage. This is partly due to the highly complexity of such studies generated by the time delay factor. To the best of our knowledge, there is only one successful approach developed for continuous systems with delay. This approach, developed in [4] for a discounted integral functional, considers the following system with *infinite* time delay:

$$\dot{x}(t) = g(x(t), u(t)) + \int_{-\infty}^{t} r(t-s) f(x(s)) ds \quad \text{a.e. } t \geq 0, \tag{1.1}$$

Under some quite mild assumptions, the existence of catching up optimal solutions as well as the convergence of these solutions to a unique optimal steady state are proved. Later, this approach was further developed by [51], where the existence of overtaking optimal solutions was established without boundedness assumptions, and in [52] the turnpike property was established for solutions on finite intervals.

It is important to note that one of the assumptions in [4] applies the boundedness of $r(t),\ t \in \mathbb{R}^+$. For example, function r defined by

$$r(t) = \begin{cases} \infty & \text{if } t = \tau, \\ 0 & \text{if } t \geq 0, t \neq \tau, \end{cases} \tag{1.2}$$

does not satisfy this assumption. Consequently, the results of [4] are no longer applicable to the case of singular time delay in the form $f(x(t-\tau))$ (instead of infinite time delay in (1.1)).

The first turnpike results for singular time delay are established in our recent papers [17, 38] by considering the above mentioned terminal functionals. Similar results for discrete optimal control models are presented in [18]. This paper extends these results to a much broader class of systems in terms of nonlinearities of the functions involved in the description of the system. In particular, function f, related to the time delay terms (see system (2.1) below), is not assumed to be increasing. This is the case, for example, in the blood cell production model from [28] that is considered in section 5.

The rest of this article is organized as follows. In the next section we formulate the main problem, notation, and assumptions that are used throughout the paper. The existence of optimal trajectories along with some preliminary results are provided in section 3. The main theorem, the turnpike property, is proved in section 4. In section 5 two practical applications are considered: the generalized Ramsey model with time delay and the blood cell production model. Note that it is a new initiative that for the former model an optimal control problem is introduced by taking some parameters of the model as time-dependent control variables.

2. Problem formulation.

Consider the following system:

$$\dot{x}(t) = u(t)f(x(t-\tau)) - g(x(t)) \quad \text{a.e. } t \geq 0, \tag{2.1}$$

$$x(t) = x_0(t),\ t \in [-\tau, 0], \quad u(t) \in [0,1]\ \forall t. \tag{2.2}$$

Here $f, g:\ \mathbb{R}^+ \to \mathbb{R}^+$ are continuous on the positive semiaxis $\mathbb{R}^+ = [0, \infty)$, $u(t)$ is a measurable control function with values in the interval [0,1], $\tau > 0$ is the delay, and initial function x_0 is nonnegative and continuous.

Given initial function $\mathbf{x_0} = x_0(t), t \in [-\tau, 0]$, and control $\mathbf{u} = u(t), t \geq 0$, an absolutely continuous function $\mathbf{x} = x(t), t \geq 0$, satisfying (2.1)–(2.2), is called a solution or a trajectory. We assume that for every initial function $\mathbf{x_0}$ and control $\mathbf{u}$, there exists a unique solution denoted by $\mathbf{x} = \mathbf{x}(\mathbf{x_0}, \mathbf{u})$.

We do not address in detail the question of global existence of solutions to (2.1), (2.2); the results are well known and readily available in the literature for different types of systems. We only note that (2.1) has a special structure where terms $x(t-\tau)$ and $x(t)$ are separated. In this case the existence, uniqueness, and continuation of solutions follow from related results for systems without time delay. Indeed, on each interval $[0, \tau],\ [\tau, 2\tau], \ldots,$ (2.1) becomes a differential equation without delay. Therefore, for example, if g is Lipschitz continuous, then given any $\mathbf{x_0}, \mathbf{u}$, problem (2.1), (2.2) has a unique solution (see, e.g., Chapter 1, section 1.11, Theorem (Existence), and Chapter 2, section 2.21, Theorem (Uniqueness) in [41]).

These kinds of equations has attracted significant interest in recent years due to their frequent appearance in a wide range of applications. They serve as mathematical models describing various real-life phenomena in mathematical biology, population dynamics and physiology, electrical circuits and laser optics, economics, life sciences, and other fields. See [1, 23, 28, 33] for a partial list of applications and further details.

Many economic models lead to differential delay equations of the form (2.1); a partial list of economic applications is given in [11, 20, 27]. In these applications, x stands for the capital and $\tau > 0$ is the length of the production (investment) cycle. The component $f(x(t-\tau))$ describes a general commodity being produced at time t and $g(x(t))$ stands for the "amortization" of the capital. After each cycle of production a certain part of the commodity (capital) is used for the investment, while the remaining part is consumed. This decision is controlled by parameter $u(t) \in [0,1]$. We shall assume that at any time $t > 0$, the part $u(t)f(x(t-\tau))$ is assigned for production purposes (investment), while the remaining part $[1-u(t)]f(x(t-\tau))$ is consumed. From (2.1) we have

$$[1-u(t)]f(x(t-\tau)) = f(x(t-\tau)) - g(x(t)) - \dot{x}(t) \quad \text{a.e. } t \geq 0.$$

Given any solution $\mathbf{x} = \mathbf{x}(\mathbf{x_0}, \mathbf{u})$, we define consumption $c(t)$ as

$$c(t) = f(x(t-\tau)) - g(x(t)) - \dot{x}(t) \quad \forall t, \text{ where } \dot{x}(t) \text{ exists.}$$

The optimality is defined by the following functional:

$$\text{Maximize: } C(\mathbf{x_0}, \mathbf{u}) \triangleq \liminf_{t\to\infty,\ \dot{x}(t)\,\text{exists}} [f(x(t-\tau)) - g(x(t)) - \dot{x}(t)]. \tag{2.3}$$

Functional (2.3) aims to maximize the lower level of consumption when t goes to infinity. It can be considered as an analogue of the terminal functional defined for an infinite time horizon. We refer to [37] for more information about the results on stability of optimal trajectories in terms of similar functionals for systems without time delay.

Given initial function $\mathbf{x_0}$, control $\mathbf{u}$ is called optimal if for any control $\tilde{\mathbf{u}}$ the inequality

$$C(\mathbf{x_0}, \mathbf{u}) \geq C(\mathbf{x_0}, \tilde{\mathbf{u}})$$

holds. In this case the corresponding solution $\mathbf{x}$ to (2.1) will be called an optimal solution.

Problem (2.1)–(2.3) is derived based on economic interpretations. However, it might be suitable to describe problems from other areas as well. An example is the blood cell production model considered in section 5.

There is a significant body of theoretical research on differential delay equation (2.1) in the past 20 or 30 years. They address various aspects of the dynamics in such equations including complicated behavior and chaos, among others. However, most of it deals with the case of linear function g; that is, when $g(x) = bx$, $b > 0$. Papers [19, 32] represent a partial list of related references.

It should be noted that even for a very simple case when considering a constant control function, trajectories of (2.1) may have quite complicated structures, depending on initial functions. However, the trajectories that provide maximum possible value to the functional (2.3) might be well structured; that is, the turnpike property may still hold. Indeed, under some conditions it will be proved that given any initial function x_0, there is an optimal solution to problem (2.1)–(2.3) and all optimal trajectories converge to some unique stationary point as time goes to infinity.

The main conditions that will be used throughout the paper are provided below.

Assumption A.

- $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous.
- $g(0) = 0$ and $g(x)$ is strictly increasing; i.e., $g(x_1) < g(x_2)$ for all $0 \leq x_1 < x_2$.
- There is $X > 0$ such that

$$f(x) > g(x) > 0 \ \ \forall x \in (0, X) \quad \text{and} \quad f(x) < g(x) \ \ \forall x > X. \tag{2.4}$$

In addition, the Lipschitz continuity of g will be assumed in the main results of this paper. However, it is important to note that no other conditions will be imposed on the nonlinearity of f. In particular f does not need to be increasing. This is the major point in this paper that allows us to apply the results obtained to some practical models in medicine and biology.

Denote

$$c^* = \max\{f(x) - g(x) : \ x \in \mathbb{R}^+\} \tag{2.5}$$

and define the set of optimal stationary points

$$\mathfrak{T} = \{x^* \in \mathbb{R}^+ : \ f(x^*) - g(x^*) = c^*\}. \tag{2.6}$$

From (2.4) it follows that $\mathfrak{T} \subset [0, X)$ and $c^* > 0$. Moreover, $f(x^*) > 0$ for each point $x^* \in \mathfrak{T}$.

In terms of applications in economics, value c^* will be interpreted as a maximum steady consumption that could be achieved in problem (2.1)–(2.3). Accordingly, each point in $\mathfrak{T}$ is as a steady state (stationary point) guaranteing consumption c^*. Clearly, $\mathfrak{T}$ is a closed set and it may contain more than one point.

Given a stationary point x^* and corresponding constant initial function $x_0(t) \equiv x^*$, the function $x(t) \equiv x^*$ is a solution to (2.1) corresponding to the constant control $u(t) \equiv u^*$ with

$$u^* = 1 - \frac{c^*}{f(x^*)} = \frac{g(x^*)}{f(x^*)} < 1. \tag{2.7}$$

3. The existence of optimal trajectories. We start with some preliminary results about the structure of trajectories of system (2.1). Denote

$$M^f = \max_{x \in [0,X]} f(x), \quad M^g = \sup_{x \in \mathbb{R}^+} g(x). \tag{3.1}$$

Clearly, $M^f < \infty$ and, since g is increasing, $M^g = \lim_{x\to\infty} g(x)$.

PROPOSITION 3.1. *Suppose that Assumption* A *holds. Then solutions* $\mathbf{x} = x(t)$, $t \geq 0$, *to* (2.1) *are bounded; that is, there is a number* $M < \infty$ *such that*

$$\limsup_{t\to\infty} x(t) \leq M \quad \forall \mathbf{x}.$$

In addition, if f is increasing, then

$$\limsup_{t\to\infty} x(t) \leq X \quad \forall \mathbf{x}.$$

Proof. Take any initial function $\mathbf{x_0}$ and control $\mathbf{u}$. Let $\mathbf{x} = x(t)$ be the corresponding solution.

1. First we show that $x(t)$ is bounded. Consider two cases related to the numbers M^f and M^g defined in (3.1).

1a. Let $M^f \geq M^g$.

From Assumption A it follows that for all $x \geq X$ the inequality

$$f(x) \leq g(x) < M^g \leq M^f$$

holds. Then $f(x) \leq M^f$ and $g(x) \leq M^f$ are satisfied for all $x \geq 0$. From (2.1) we have

$$\dot{x}(t) \leq u(t) f(x(t-\tau)) - g(x(t)) \leq M^f \qquad \text{a.e. } t \geq 0.$$

On the other hand, $\dot{x}(t) \geq -g(x(t)) \geq -M^f$ a.e. $t \geq 0$, and therefore,

$$|\dot{x}(t)| \leq M^f \quad \text{a.e. } t \geq 0.$$

This means that there is a number $M' < \infty$ such that

$$|x(t) - x(t-\tau)| \leq M', \quad \forall t. \tag{3.2}$$

Now assume to the contrary that $x(t)$ is unbounded. Then there is a sequence t_n such that

$$\dot{x}(t_n) \geq 0, \; x(t_n) \to \infty \quad \text{and} \quad x(t_n - \tau) \leq x(t_n). \tag{3.3}$$

From (3.2) it follows $x(t_n - \tau) \to \infty$ and, therefore, $x(t_n - \tau) > X$ for sufficiently large n. In this case taking into account $u(t_n) \leq 1$, $x(t_n - \tau) \leq x(t_n)$ and Assumption A, we have

$$\begin{aligned} \dot{x}(t_n) &\leq u(t_n) f(x(t_n - \tau)) - g(x(t_n)) \\ &\leq f(x(t_n - \tau)) - g(x(t_n)) \leq f(x(t_n - \tau)) - g(x(t_n - \tau)) < 0. \end{aligned}$$

This contradicts $\dot{x}(t_n) \geq 0$ in (3.3). This means that $x(t)$ should be bounded.

1b. Now consider the case $M^f < M^g$, and on the contrary assume that $x(t)$ is unbounded.

By Assumption A there is $\tilde{x} > X$ and a positive number a such that

$$g(x) \geq M^f + a \quad \forall x \geq \tilde{x}.$$

Since $x(t)$ is unbounded, for any $x' \geq \tilde{x}$ there is t' (not necessarily unique) for which the equality $x(t') = x'$ is satisfied. Let $t' = \arg\min\{t \geq 0 : \; x(t') = x'\}$; that is, the following hold:

$$x(t') = x' \quad \text{and} \quad x(t) < x(t') \quad \forall t < t'.$$

Then, we can construct an increasing sequence t_n such that

$$t_n \to t', \quad \dot{x}(t_n) \geq 0 \quad \text{and} \quad x(t_n - \tau) \leq x(t_n) \quad \forall t_n.$$

Clearly, t_n can be chosen so large that the inequality

$$g(x(t_n)) \geq g(x') - a/2 \geq (M^f + a) - a/2 \geq M^f + a/2 \tag{3.4}$$

holds. Consider two cases:

(i) Let $x(t_n - \tau) > X$, where X is defined in Assumption A. Then, since $x(t_n - \tau) \leq x(t_n)$, from Assumption A we have

$$f(x(t_n - \tau)) < g(x(t_n - \tau)) \leq g(x(t_n)).$$

Thus

$$\dot{x}(t_n) = u(t_n) f(x(t_n - \tau)) - g(x(t_n)) \leq f(x(t_n - \tau)) - g(x(t_n)) < 0.$$

This is a contradiction.

(ii) Let $x(t_n - \tau) \le X$. In this case $f(x(t_n - \tau)) \le M^f$. Then, for t_n satisfying (3.4) we have

$$\dot{x}(t_n) \le f(x(t_n - \tau)) - g(x(t_n)) \le M^f - (M^f + a/2) = -a/2 < 0.$$

Again, this is a contradiction.

Therefore, we have proved that $x(t)$ is bounded.

2. Now assume that f is increasing. Let $\xi \triangleq \limsup x(t)$. Then, there is a sequence $t_n \to \infty$ such that

$$\dot{x}(t_n) \ge -\frac{1}{n}, \quad x(t_n) \to \xi, \quad x(t_n - \tau) \to \eta \le \xi. \tag{3.5}$$

For example, such a sequence could be generated as follows. Take any $k \in \{1, 2, \ldots\}$ and let $\tilde{s}$ such that $x(\tilde{s}) \in (\xi - \frac{1}{k}, \xi + \frac{1}{k})$. Then we show there is $s_k \ge \tilde{s}$ satisfying

$$\dot{x}(s_k) \ge -\frac{1}{k} \quad \text{and} \quad x(s_k) \in \left[\xi - \frac{1}{k}, \xi + \frac{1}{k}\right]. \tag{3.6}$$

Indeed, if this is not true, then $\dot{x}(s_k) < -\frac{1}{k} < 0$ for almost all $s \ge \tilde{s}$ satisfying $x(s) \in [\xi - \frac{1}{k}, \xi + \frac{1}{k}]$, which means that $x(t)$ leaves the interval $[\xi - \frac{1}{k}, \xi + \frac{1}{k}]$. In this case, since $x(t)$ is continuous and ξ is a limit point, there exist $s_1, s_2 > \tilde{s}$ defined by

$$s_1 = \arg\min \left\{ t \ge \tilde{s} : \ x(t) = \xi - \frac{1}{2k} \right\} \text{ and } s_2 = \arg\max \left\{ t \in (\tilde{s}, s_1) : \ x(t) = \xi - \frac{1}{k} \right\}.$$

Thus $x(s_2) < x(s_1)$ and there exists $s_k \in (s_2, s_1)$ satisfying $\dot{x}(s_k) > 0 \ge -\frac{1}{k}$ and consequently (3.6).

Now consider sequence s_k generated by (3.6). Clearly $x(s_k) \to \xi$. Moreover, we can chose a subsequent $t_n = s_{k_n}$ such that $x(s_{k_n} - \tau)$ also converges, say, to some point η. Then $\dot{x}(t_n) \ge -\frac{1}{k_n} \ge -\frac{1}{n}$ that leads to (3.5).

Thus, there exists a sequence t_n satisfying (3.5). Since $u(t_n) \le 1$

$$\dot{x}(t_n) = u(t_n) f(x(t_n - \tau)) - g(x(t_n)) \le f(x(t_n - \tau)) - g(x(t_n))$$

and we have $0 \le f(\eta) - g(\xi)$. As f is increasing, $f(\eta) \le f(\xi)$ and thus $f(\xi) \ge g(\xi)$. Then from Assumption A it follows that $\xi \le X$; that is, $\limsup x(t) \le X$.

Proposition 3.1 is proved. ∎

Proposition 3.2. *Suppose that Assumption* A *holds. Then for every solution* $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, \mathbf{u})$ *to* (2.1) *the following inequality holds:*

$$C(\mathbf{x}_0, \mathbf{u}) \le c^*.$$

Proof. Take any initial function $\mathbf{x}_0$ and control $\mathbf{u}$. Consider corresponding solution $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, \mathbf{u}) = x(t), t \ge 0$. Recall that

$$C(\mathbf{x}_0, \mathbf{u}) = \liminf_{t \to \infty, \ \dot{x}(t) \text{ exists}} [f(x(t - \tau)) - g(x(t)) - \dot{x}(t)]. \tag{3.7}$$

Denote $\xi = \limsup_{t \to \infty} x(t) < \infty$, and let $t_n \to \infty$ be a sequence satisfying (3.5) considered in the proof of Proposition 3.1. From (3.7) we have

$$C(\mathbf{x}_0, \mathbf{u}) \le \lim_{n \to \infty} \left[f(x(t_n - \tau)) - g(x(t_n)) + \frac{1}{n} \right] \le f(\eta) - g(\xi).$$

Here $\eta \le \xi$ and therefore $g(\eta) \le g(\xi)$. Then from definition of c^* we obtain

$$C(\mathbf{x}_0, \mathbf{u}) \le f(\eta) - g(\eta) \le c^*.$$

Proposition 3.2 is proved. □

PROPOSITION 3.3. *Suppose that Assumption* A *holds and g is Lipschitz continuous. Then for any given nonzero continuous initial function $\mathbf{x_0}$ and any optimal stationary point $x^* \in \mathfrak{T}$ there is a control $\mathbf{u}$ such that corresponding solution $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, \mathbf{u})$ to* (2.1) *converges to x^* and*

$$C(\mathbf{x}_0, \mathbf{u}) = c^*$$

holds.

Proof. Take any optimal stationary point $x^* \in \mathfrak{T}$ and any nonzero initial function $\mathbf{x_0} = x_0(t)$, $t \in [-\tau, 0]$. We will construct required control $\mathbf{u}$ in three steps.

1. Taking $u(t) = 1$, from (2.1) we have

$$\dot{x}(t) = f(x(t-\tau)) - g(x(t)), \quad t \ge 0.$$

Denote the solution to this equation by $\tilde{x}(t)$. Since $x_0(t)$ is continuous and nonzero, $x_0(t) > 0$ is satisfied on some interval belonging to $[-\tau, 0]$. Thus there is $\tilde{t} > 0$ such that $\tilde{x}(\tilde{t}) > 0$. We set $\tilde{u}(t) = 1$ for $t \in [0, \tilde{t}]$ with the corresponding solution $\tilde{x}(t)$.

2. Continue control $\tilde{u}(t)$ by setting $\tilde{u}(t) = 0$ for $t > \tilde{t}$. We have

$$\dot{x}(t) = -g(x(t)), \quad t \ge \tilde{t}, \quad x(\tilde{t}) = \tilde{x}(\tilde{t}).$$

Let $\tilde{x}(t)$ be the solution to this equation. Since g is Lipschitz continuous and $x \ge 0$, we have $g(x) \le Lx \ for\, all\ x \ge 0$, and therefore

$$\dot{\tilde{x}}(t) \ge -L\tilde{x}(t), \quad t \ge \tilde{t}.$$

Applying Gronwall's inequality we obtain

$$\tilde{x}(t) \ge \tilde{x}(\tilde{t}) \exp(-Lt) > 0, \quad t \ge \tilde{t}.$$

From Assumption A we have $\dot{\tilde{x}}(t) = -g(\tilde{x}(t)) < 0$ for all $t \ge \tilde{t}$. Then it is not difficult to observe that $\tilde{x}(t) \to 0$. Therefore, there are $\tilde{x} > 0$ and sufficiently large $t_1 > \tau$ such that

$$\tilde{x}(t) < \tilde{x} \quad \forall t \in [t_1 - \tau, t_1]. \tag{3.8}$$

Thus, on the interval $t \in (\tilde{t}, t_1]$, we define $\tilde{u}(t) = 0$ with the corresponding solution $\tilde{x}(t)$ satisfying (3.8).

3. Next, we continue the above solution for $t \ge t_1$ by taking

$$\tilde{u}(t) = \min\left\{1, \frac{g(x^*)}{f(\tilde{x}(t-\tau))}\right\}, \quad t \ge t_1, \tag{3.9}$$

with the corresponding solution $\tilde{x}(t)$ satisfying

$$\dot{\tilde{x}}(t) = \tilde{u}(t) f(\tilde{x}(t-\tau)) - g(\tilde{x}(t)), \quad t \ge t_1. \tag{3.10}$$

Clearly, $\tilde{u}(t) \in [0, 1]$ for all $t \ge t_1$, which means that $\tilde{u}(t)$ is a feasible control. By Proposition 3.1 solution $\tilde{x}(t)$ is bounded and therefore it is defined on $[t_1, \infty)$.

Given any $t \geq t_1$, there are two possibilities:

- If $f(\tilde{x}(t-\tau)) \geq g(x^*)$, then $\tilde{u}(t) = \frac{g(x^*)}{f(\tilde{x}(t-\tau))}$ and

$$\dot{\tilde{x}}(t) = g(x^*) - g(\tilde{x}(t)). \tag{3.11}$$

- If $f(\tilde{x}(t-\tau)) < g(x^*)$, then $\tilde{u}(t) = 1$ and

$$\dot{\tilde{x}}(t) = f(\tilde{x}(t-\tau)) - g(\tilde{x}(t)) < g(x^*) - g(\tilde{x}(t)). \tag{3.12}$$

Therefore

$$\dot{\tilde{x}}(t) \leq g(x^*) - g(\tilde{x}(t)) \quad \forall t \geq t_1. \tag{3.13}$$

Denote $p_1 = \liminf \tilde{x}(t)$ and $p_2 = \limsup \tilde{x}(t)$.

3.a. By the definition of p_2 there is a sequence $t_n \to \infty$ such that

$$\dot{\tilde{x}}(t_n) \geq 0 \ \ \forall n \quad \text{ and } \quad \tilde{x}(t_n) \to p_2.$$

Then from (3.13) we have $0 \leq g(x^*) - g(p_2)$. Since g is increasing it follows that $p_2 \leq x^*$.

3.b. Suppose that $p_1 < x^*$. Then there is a sequence $t_n \to \infty$ such that

$$\dot{\tilde{x}}(t_n) \leq 0, \quad \tilde{x}(t_n) \to p_1 \quad \text{ and } \quad \tilde{x}(t_n - \tau) \to x' \geq p_1. \tag{3.14}$$

If the set of indices $\{n : \ f(\tilde{x}(t_n - \tau)) \geq g(x^*)\}$ is infinite, then from the relation (see (3.11))

$$\dot{\tilde{x}}(t_n) = g(x^*) - g(\tilde{x}(t_n))$$

we have $g(x^*) - g(p_1) \leq 0$ or $p_1 \geq x^*$, which is a contradiction.

If the set $\{n : \ f(\tilde{x}(t_n - \tau)) \geq g(x^*)\}$ is finite, then $\{n : \ f(\tilde{x}(t_n - \tau)) < g(x^*)\}$ is infinite. In this case from relation (see (3.12))

$$\dot{\tilde{x}}(t_n) = f(\tilde{x}(t_n - \tau)) - g(\tilde{x}(t))$$

we have $f(x') - g(p_1) \leq 0$ or $g(p_1) \geq f(x')$. From 3.a we know that $\limsup \tilde{x}(t) \leq x^*$. Then, $x' \leq x^* < X$ and from Assumption A it follows $f(x') > g(x')$. Therefore, $g(p_1) \geq f(x') > g(x')$, which leads to $p_1 > x'$. This contradicts the last relation in (3.14).

Therefore, the assumption $p_1 < x^*$ leads to a contradiction. Then we have $p_1 = \liminf \tilde{x}(t) \geq x^*$ and $p_2 = \limsup \tilde{x}(t) \leq x^*$, which means that solution $\tilde{x}(t)$ converges to $x^* : \tilde{x}(t) \to x^*$ as $t \to \infty$.

Now consider the control function $\tilde{u}(t)$. Since $f(x^*) > g(x^*)$ and $\tilde{x}(t) \to x^*$, from (3.9) it follows that $\tilde{u}(t) < 1$ for sufficiently large t, and $\tilde{u}(t) \to \frac{g(x^*)}{f(x^*)}$ as $t \to \infty$. Then

$$\lim_{t\to\infty} \dot{\tilde{x}}(t) = \lim_{t\to\infty} [\tilde{u}(t) f(\tilde{x}(t-\tau)) - g(\tilde{x}(t))] = 0.$$

Denote the control $\tilde{u}(t)$ and solution $\tilde{x}(t)$ constructed in this way by $\mathbf{u}$ and $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, \mathbf{u})$, respectively. The value of the functional in this case is

$$C(\mathbf{x}_0, \mathbf{u}) = \lim_{t\to\infty} [f(\tilde{x}(t-\tau)) - g(\tilde{x}(t)) - \dot{\tilde{x}}(t)] = f(x^*) - g(x^*) = c^*.$$

Proposition 3.3 is proved. □

From Propositions 3.2 and 3.3 we conclude the following theorem about the existence of optimal solutions.

THEOREM 3.4 (existence). *Suppose that Assumption* A *holds and g is Lipschitz continuous. Then given any nonzero continuous initial function $\mathbf{x_0}$ there exists an optimal control $\mathbf{u}$ such that corresponding solution $\mathbf{x} = x(t)$ converges to some optimal stationary point and the objective function achieves its maximum possible value c^*:*

$$C(\mathbf{x_0}, \mathbf{u}) = c^* \quad \text{and} \quad x(t) \to x^* \in \mathfrak{T} \text{ as } t \to \infty.$$

4. Stability of optimal solutions. Theorem 3.4 shows that given any nonzero continuous initial function $\mathbf{x_0}$ there is an optimal control $\mathbf{u}$ such that the objective function achieves its maximum possible value c^*. Moreover, one of such optimal solutions converges to some optimal stationary point. The aim of this section is to prove that all the optimal solutions are stable in a sense that each of them converges to some stationary point; that is, the turnpike property holds for the problem (2.1)–(2.3).

First we present the following proposition.

PROPOSITION 4.1. *Suppose that Assumption* A *holds. Let $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, \mathbf{u}) = x(t), t \geq 0$, be a solution to* (2.1) *such that $C(\mathbf{x}_0, \mathbf{u}) = c^*$. Then for every $\varepsilon > 0$ there is $T_\varepsilon < +\infty$ such that*

$$\dot{x}(t) \leq g(x(t-\tau)) - g(x(t)) + \varepsilon \quad \text{a.e. } t \geq T_\varepsilon. \tag{4.1}$$

Moreover

$$\limsup_{t\to\infty} x(t) \in \mathfrak{T}. \tag{4.2}$$

Here $\mathfrak{T}$ is the set of optimal stationary points defined by (2.6).

Proof. We have

$$C(\mathbf{x}_0, \mathbf{u}) = \liminf_{t\to\infty,\ \dot{x}(t)\text{ exists}} [f(x(t-\tau)) - g(x(t)) - \dot{x}(t)] = c^*.$$

Then, for every $\varepsilon > 0$ there is $T_\varepsilon < +\infty$ such that

$$f(x(t-\tau)) - g(x(t)) - \dot{x}(t) \geq c^* \quad \text{a.e. } t \geq T_\varepsilon.$$

By the definition of c^* in (2.5) we have

$$f(x(t-\tau)) - g(x(t-\tau)) \leq c^*.$$

From the last two inequalities we obtain (4.1). Now we show (4.2).

Denote $p = \limsup_{t\to\infty} x(t)$. As in (3.5), there is a sequence $t_n \to \infty$ such that

$$\dot{x}(t_n) \geq -\frac{1}{n}, \quad x(t_n) \to p, \quad x(t_n - \tau) \to \eta \leq p. \tag{4.3}$$

We have

$$c^* \leq \liminf_{n\to\infty} [f(x(t_n-\tau)) - g(x(t_n)) - \dot{x}(t_n)] \leq f(\eta) - g(p). \tag{4.4}$$

Now, if $p > \eta$, then $g(p) > g(\eta)$ and therefore $c^* < f(\eta) - g(\eta)$, which is a contradiction. If $p = \eta$, then $c^* \leq f(p) - g(p)$, which yields $p \in \mathfrak{T}$.

Proposition 4.1 is proved. ∎

Proposition 4.1 describes the structure of optimal solutions $\mathbf{x}$ satisfying $C(\mathbf{x}_0, \mathbf{u}) = c^*$ when t is sufficiently large. In what follows we consider the case when the set of optimal stationary points $\mathfrak{T}$ has an empty interior. In this case the convergence of optimal solutions to some steady state will be proved. Such a property of optimal solutions is called the turnpike property [21, 37, 53]. We note that it is quite difficult to prove the turnpike property in the case when the set $\mathfrak{T}$ contains more than one optimal stationary points, even in the absence of time delay (see, for example, [35]).

THEOREM 4.2 (turnpike property). *Suppose that Assumption* A *holds, g is Lipschitz continuous, and the set of optimal stationary points $\mathfrak{T}$ has an empty interior. Then given any nonzero continuous initial function $\mathbf{x_0}$ and any optimal control $\mathbf{u}$ in the problem* (2.1)–(2.3)*, there is a stationary point $x^*_{\mathbf{x_0},\mathbf{u}} \in \mathfrak{T}$ such that corresponding optimal solution $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, \mathbf{u}) = x(t), t \geq 0$, converges to that point; that is,*

$$\lim_{t\to\infty} x(t) = x^*_{\mathbf{x_0},\mathbf{u}} \in \mathfrak{T}. \tag{4.5}$$

Proof. Take any optimal solution $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, \mathbf{u}) = x(t), t \geq 0$, to problem (2.1)–(2.3). By Theorem 3.4 we know that $C(\mathbf{x}_0, \mathbf{u}) = c^*$. Denote

$$q := \liminf_{t\to\infty} x(t) \quad \text{and} \quad p := \limsup_{t\to\infty} x(t).$$

From Proposition 4.1 it follows that $p \in \mathfrak{T}$. If $q = p$, then (4.5) is true. Consider the case $q < p$.

Since g is Lipschitz continuous, there is $L < \infty$ such that

$$|g(p) - g(x)| \leq L|p - x| \quad \forall x \geq 0. \tag{4.6}$$

Take any integer number K satisfying $K > (2L + 1)\,\tau$. Clearly $K \geq 1$.

Let $\eta > 0$ be an arbitrary small number such that

$$\eta < \frac{p - q}{2^{K+1}}. \tag{4.7}$$

Denote $\delta = 2^{K-1}\eta$. Clearly $\delta < \frac{p-q}{4}$ and $q < p - 2\delta < p - \delta < p$.

By the definition of p and q, there are sequences $t_n^1 < s_n < t_n^2$ such that $t_n^1 \to \infty$ and the following hold:

- $x(t_n^1) = x(t_n^2) = p - \delta$ *for all* $n = 1, 2, \ldots$;
- $x(t) < p - \delta$ *for all* $t \in (t_n^1, t_n^2)$;
- $x(s_n) = p - 2\delta$;
- $x(t) \in (p - 2\delta, p - \delta)$ *for all* $t \in (s_n, t_n^2)$.

Since $x(t) < p - \delta < p$ *for all* $t \in (s_n, t_n^2)$, and g is increasing, from (4.6) we have

$$g(p) - g(x(t)) \leq L(p - x(t)) \quad \forall t \in (s_n, t_n^2). \tag{4.8}$$

Take any small number $\varepsilon \in (0, \frac{\eta}{2})$. Clearly $\varepsilon < \frac{\delta}{2^K}$ or

$$2^K \frac{\varepsilon}{\delta} < 1. \tag{4.9}$$

Since g is continuous and strictly increasing, by definition of p given number ε there is $T_\varepsilon^1 < \infty$ such that the inequality $g(x(t - \tau)) < g(p) + \varepsilon$ is satisfied for all $t \geq T_\varepsilon^1$. Then from (4.1), by assuming T_ε is large enough to satisfy $T_\varepsilon \geq T_\varepsilon^1$, we obtain that

$$\dot{x}(t) \leq g(p) - g(x(t)) + 2\varepsilon \quad \text{a.e. } t \geq T_\varepsilon. \tag{4.10}$$

A. Now we take any n such that $t_n^1 > T_\varepsilon$ and perform the following $K \geq 1$ steps.

A.1. Since $p - x(t) < 2\delta$ for all $t \in (s_n, t_n^2)$, from (4.8) and (4.10) we obtain

$$\dot{x}(t) \leq L(p - x(t)) + 2\varepsilon \leq 2L\delta + 2\varepsilon \quad \text{a.e. } t \in (s_n, t_n^2). \tag{4.11}$$

Denote $a_1 = s_n$, $b_1 = t_n^2$. Then

$$\delta = x(b_1) - x(a_1) = \int_{a_1}^{b_1} \dot{x}(t)dt \leq (2L\delta + 2\varepsilon)(b_1 - a_1)$$

or

$$b_1 - a_1 \geq \frac{1}{2L + 2^1(\varepsilon/\delta)}.$$

From (4.9) we have $2^1(\varepsilon/\delta) \leq 2^K(\varepsilon/\delta) < 1$ and therefore

$$b_1 - a_1 \geq \frac{1}{2L + 1}. \tag{4.12}$$

A.2. (if $K \geq 2$) Consider $p - \frac{\delta}{2} \in (p - \delta, p)$ and let $a_2 \geq b_1$ and $b_2 > a_2$ be chosen so that $x(a_2) = p - \delta,\ x(b_2) = p - \frac{\delta}{2}$, and

$$x(t) \in \left(p - \delta, p - \frac{\delta}{2}\right) \ \forall t \in (a_2, b_2) \quad \text{and} \quad x(t) \leq p - \frac{\delta}{2} \ \forall t \in [b_1, a_2].$$

Note that since $x(t)$ is continuous and p is a limit point, such points a_2, b_2 exist and are given by

$$b_2 = \arg\min\{t : t > b_1,\ x(t) = p - \delta/2\}, \quad a_2 = \arg\max\{t : t \in (b_1, b_2),\ x(t) = p - \delta\}.$$

Now, similar to (4.11), since $p - x(t) < \delta$ for $t \in (a_2, b_2)$ we have

$$\dot{x}(t) \leq L\delta + 2\varepsilon \quad \text{a.e. } t \in (a_2, b_2).$$

Then

$$\frac{\delta}{2} = x(b_2) - x(a_2) = \int_{a_2}^{b_2} \dot{x}(t)dt \leq (L\delta + 2\varepsilon)(b_2 - a_2)$$

or

$$b_2 - b_1 \geq b_2 - a_2 \geq \frac{1}{2L + 2^2(\varepsilon/\delta)}.$$

Now, again from (4.9) we have $2^2(\varepsilon/\delta) \leq 2^K(\varepsilon/\delta) < 1$ and therefore

$$b_2 - b_1 \geq b_2 - a_2 \geq \frac{1}{2L + 1}.$$

We can continue this process by constructing a_i, b_i, $i = 2, \ldots, K$, such that

$$b_{i-1} \leq a_i < b_i,\ x(a_i) = p - \frac{\delta}{2^{i-2}},\ x(b_i) = p - \frac{\delta}{2^{i-1}},$$

and

$$b_i - b_{i-1} \geq b_i - a_i \geq \frac{1}{2L+1} \quad \forall i = 2, \dots, K,$$

(4.13)

$$x(t) \in \left(p - \frac{\delta}{2^{i-2}}, p - \frac{\delta}{2^{i-1}}\right) \quad \forall t \in (a_i, b_i), \quad \text{and} \quad x(t) \leq p - \frac{\delta}{2^{i-1}} \quad \forall t \in [b_{i-1}, a_i].$$

Therefore we obtain

$$b_K - a_1 = (b_1 - a_1) + \sum_{i=2,\dots,K} (b_i - b_{i-1}) \geq \sum_{i=1,\dots,K} \frac{1}{2L+1} \geq \frac{K}{2L+1} > \tau.$$

B. Thus we have shown that for any number n, for which $t_n^1 > T_\varepsilon$, there is a point $\bar{s}_n \doteq b_K$ such that

$$x(\bar{s}_n) = p - \frac{\delta}{2^{K-1}} = p - \eta.$$

Moreover $x(s_n) = p - \delta$ and $\bar{s}_n - s_n = b_K - a_1 > \tau$. Therefore $\bar{s}_n - \tau \in (s_n, \bar{s}_n)$ and (4.13) yields $x(\bar{s}_n - \tau) < p - \eta$.

This process can be repeated for any $\bar{n} > n$, satisfying $t_{\bar{n}}^1 > \max\{T_\varepsilon, \bar{s}_n\}$. In this way we can construct sequences of points $s_{n_m} < \bar{s}_{n_m} - \tau < \bar{s}_{n_m}$, $s_{n_m} \to \infty$, such that

$$x(s_{n_m}) = p - \delta,\ x(\bar{s}_{n_m}) = p - \eta,\ x(\bar{s}_{n_m} - \tau) < p - \eta, \tag{4.14}$$

and $x(t) < p - \eta$ for all $t \in (s_{n_m}, \bar{s}_{n_m})$.

Then, it is not difficult to observe that we can construct a sequence of points $\xi_{n_m} < \bar{s}_{n_m}$ such that

$$\bar{s}_{n_m} - \xi_{n_m} \to 0,\ s_{n_m} < \xi_{n_m},\ \dot{x}(\xi_{n_m}) \geq 0, \quad \text{and} \quad x(\xi_{n_m} - \tau) < p - \eta.$$

We note that the solution $x(t)$ to (2.1) is Lipschitz continuous and that is why $x(\xi_{n_m}) \to p - \eta$. Moreover, we can choose a convergent subsequence of the sequence $\{x(\xi_{n_m} - \tau)\}$. For simplicity let $x(\xi_{n_m} - \tau) \to \bar{p}$. Obviously, $\bar{p} \leq p - \eta$.

Thus from (2.3) we have

$$c^* = C(\mathbf{x}_0, \mathbf{u}) \leq \liminf_{m\to\infty} [f(x(\xi_{n_m} - \tau)) - g(x(\xi_{n_m})) - \dot{x}(\xi_{n_m})]$$

or

$$c^* \leq f(\bar{p}) - g(p - \eta).$$

Now, if $\bar{p} < p - \eta$, then $g(\bar{p}) < g(p - \eta)$ and we obtain a contradiction $c^* < f(\bar{p}) - g(\bar{p})$. Then the equality $\bar{p} = p - \eta$ should be satisfied. In this case from the last inequality we obtain $c^* = f(p - \eta) - g(p - \eta)$, which yields $p - \eta \in \mathfrak{T}$.

Since η is an arbitrary number satisfying (4.7), we have shown that

$$p - \eta \in \mathfrak{T} \quad \forall \eta \in \left(0, \frac{p - q}{2^{K-1}}\right).$$

This contradicts the assumption that $\mathfrak{T}$ has an empty interior.

Theorem 4.2 is proved. $\square$

5. Examples.

5.1. Generalized Ramsey model with time delay. This model is a special case of problem (2.1)–(2.3) when f is a production function in the form $f(x) = A\,x^{\alpha}$ with $A > 0$, $\alpha \in (0,1)$, and function g is linear: $g(x) = ax$ with $a > 0$. The corresponding system (2.1) assumes the form

$$\dot{x}(t) = u(t)A\,x^{\alpha}(t-\tau) - a\,x(t) \quad \text{a.e. } t \geq 0.$$

For any $u \in (0,1]$, this equation has a unique positive stationary point x^u given by $x^u = \left(\frac{A\,u}{a}\right)^{\frac{1}{1-\alpha}}$. An optimal stationary point is unique and corresponds to the control $u(t) = \alpha$. This optimal stationary point x^* and corresponding steady consumption c^* are defined by

$$x^* = \left(\frac{A\,\alpha}{a}\right)^{\frac{1}{1-\alpha}}, \quad c^* = A(1-\alpha)\left(\frac{A\,\alpha}{a}\right)^{\frac{\alpha}{1-\alpha}}.$$

It is not difficult to show that all the assumptions of Theorems 3.4 and 4.2 are satisfied. Then, given any nonzero initial function $x_0(t)$, there exists an optimal solution. Moreover, all optimal solutions converge to x^*.

5.2. Blood cell production model. This model was proposed in several papers [12, 26, 28] as a mathematical description of human blood cell production. It was used to describe and justify certain experimental data, in particular in the case of chronic myelogenous leukemia [28]. The equation reads

$$\dot{x}(t) = k\beta(x(t-\tau))x(t-\tau) - [\beta(x(t)) + \delta]x(t), \tag{5.1}$$

where the nonlinear function $\beta(x) = \beta_0/(1+x^n)$ is a monotone Hill function and $\beta_0, k = 2\,e^{-\gamma\tau}, n, \delta$ are all positive constants defined by the physiological process behind the equation (see [28] for more details). We assume that parameters of the model satisfy the following conditions:

$$\frac{(n-1)^2}{4n} < \frac{\delta}{\beta_0} < 1. \tag{5.2}$$

The stability of equilibria and the existence of periodic solutions depending on the parameters of the model have been well investigated. However, no optimal control problem based on this equation has been introduced yet. In this paper we formulate one such optimal control problem considering the factor $e^{-\gamma\tau}$ in the definition of parameter k and the coefficient β_0 as time-dependent control variables.

5.2.1. Introducing optimal control problem. The blood cell production model considered in [28] consists of two phases: resting phase $G(0)$ and proliferative phase P. Equation (5.1) related to the resting phase. In particular, function

$$x\beta(x) = \beta_0 \frac{x}{1+x^n}$$

describes the number of cells entering phase P in a unit time period. Parameter β_0 here is subject to discussion. It is numerically defined by the mean of data fitting; however, it can also be considered as a time-dependent control variable $\beta_0(t)$ with values in some interval, say, $[0, \beta_0]$, that controls the rate of cells entering phase P.

On the other hand, parameter $k = 2\,e^{-\gamma\tau}$ describes the attenuation in the cell proliferation due to apoptosis (programmed cell death). We can consider the factor $e^{-\gamma\tau} \in (0,1)$ in a more general setting. We assume that $k = 2\,u$, where $u \in [0,1]$ is a time-dependent control parameter describing the fraction of cells $x\beta(x)$ that are actually divided.

Therefore by employing two control parameters β_0 and u, model (5.1) can be represented in the form

$$\dot{x}(t) = u(t)\beta_0(t-\tau)\frac{2x(t-\tau)}{1+x^n(t-\tau)} - \left[\frac{\beta_0(t)}{1+x^n(t)} + \delta\right] x(t). \tag{5.3}$$

This equation describes the dynamics of the number of cells in the resting phase. However, the control variables β_0 and u are related to both phases $G(0)$ and P. It is quite natural to assume that the cell division process happens in an optimal way; that is, the control parameters β_0 and u follow some optimality criteria.

The following are two possible criteria that can be considered.

Criteria 1. Maximize the number of cells $\frac{\beta_0 x}{1+x^n}$ entering the proliferation phase.

Criteria 2. Minimize the number of dividing cells $u\frac{\beta_0 x}{1+x^n}$.

The first criteria relating to the resting phase is, in some sense, an attribute of living systems. It targets maximizing the number of cells in the resting phase by "sending" as many cells as possible to divide. The second criteria is related to the second phase P. It can be considered as a minimization of cost/energy required for division.

Such a multicriteria problem can be reduced to a single criteria to combine the interests of both phases. In this paper, we consider the simplest way of generating such a single criteria by taking their difference, for example, in the form

$$J(t|\mathbf{u},\beta_\mathbf{0}) = [1-u(t)]\frac{2\beta_0(t-\tau)x(t-\tau)}{1+x^n(t-\tau)}.$$

The aim would be to maximize $J(t|\mathbf{u},\beta_\mathbf{0})$ when t goes to infinity. In this way, we can formulate a control problem for system (5.3) by involving two control variables $u(t)$ and $\beta_0(t)$.

First we discuss some strategies for generating better control function $\beta_0(t)$ to maximize $J(t|\mathbf{u},\beta_\mathbf{0})$. Consider stationary points of (5.3) corresponding to stationary control $u(t) \equiv u$, $\beta_0(t) \equiv \beta_0$ with $u > 0.5$ and $\beta_0 > \delta$ (see condition (5.2)). We have

$$x = \left(\beta_0\frac{2u-1}{\delta} - 1\right)^{\frac{1}{n}}.$$

In this case, the value of $J(t|\mathbf{u},\beta_\mathbf{0})$ is

$$J(t|\mathbf{u},\beta_\mathbf{0}) = \frac{2\delta(1-u)}{2u-1}\left(\beta_0\frac{2u-1}{\delta} - 1\right)^{\frac{1}{n}}.$$

Clearly, in terms of maximization of functional $J(t|\mathbf{u},\beta_\mathbf{0})$ the choice of larger β_0 is preferable. In other words, it is clear that the optimal $\beta_0(t)$ could be expected to be at its maximal possible value. We do not provide any particular mathematical formulations to support this claim; however, it is true at least in terms of stationary states. Taking into account this factor, we can set $\beta_\mathbf{0}(t) \equiv \beta_0$, where β_0 is assumed to be the possible maximum value for this parameter. This in particular means that the number of cells entering the proliferative phase is controlled as a feedback control $\frac{\beta_0 x}{1+x^n}$.

Therefore, a trade-off situation is most likely related to the control parameter u rather than β_0. This is itself a very interesting observation. In the provided mathematical modeling, it means that a "real" control in the blood cell production process happens in the proliferative phase that in turn may depend on existing environment (nutritious, competition, etc.).

After fixing the control $\beta_0(t)$ at a constant level β_0, we can formulate the following optimal control problem:

$$\dot{x}(t) = u(t)\frac{2\beta_0\, x(t-\tau)}{1+x^n(t-\tau)} - \frac{\beta_0\, x(t)}{1+x^n(t)} - \delta\, x(t), \quad u(t) \in [0,1]; \tag{5.4}$$

$$\text{Maximize} \quad \liminf_{t\to\infty,\ \dot{x}(t)\,\text{exists}} \left[\frac{2\beta_0\, x(t-\tau)}{1+x^n(t-\tau)} - \frac{\beta_0\, x(t)}{1+x^n(t)} - \delta x(t) - \dot{x}(t)\right]. \tag{5.5}$$

In this case, problem (5.4), (5.5) reduces to problem (2.1)–(2.3) after denoting

$$f(x) = \frac{2\beta_0 x}{1+x^n} \quad \text{and} \quad g(x) = \frac{\beta_0 x}{1+x^n} + \delta x.$$

5.2.2. Turnpike property. First we check the assumptions of Theorems 3.4 and 4.2.

1. Consider function $g(x)$. Clearly $g(0) = 0$ and $g(x) > 0$ for all $x > 0$. We show that $\frac{d}{dx}g(x) > 0$ for all $x > 0$. We have

$$\frac{d}{dx}g(x) = \frac{\beta_0(1+x^n) - \beta_0 n x^n + \delta(1+x^n)^2}{(1+x^n)^2}.$$

Denote $\xi = 1 + x^n$. Then, the equation $\frac{d}{dx}g(x) = 0$ leads to

$$\delta\xi^2 - \beta_0(n-1)\xi + \beta_0 n = 0.$$

However, this equation does not have any positive solution ξ thanks to condition (5.2), which yields $\beta_0^2(n-1)^2 - 4\delta\beta_0 n < 0$.

Therefore, $\frac{d}{dx}g(x) > 0$; that is, $g(x)$ is strictly increasing. On the other hand, taking into account $\frac{d}{dx}g(0) = \beta_0 + \delta$ and $\lim_{x\to\infty}\frac{d}{dx}g(x) = \delta < \infty$, it is not difficult to observe that function $g(x)$ is Lipschitz continuous on $\mathbb{R}^+$.

2. Now consider function $f(x)$. Clearly f is continuous and $f(0) = 0$. Consider the difference

$$\triangle(x) \doteq f(x) - g(x) = \frac{\beta_0 x}{1+x^n} - \delta x.$$

Clearly $\triangle(0) = 0$. There is only one positive point X, for which $\triangle(X) = 0$, given by

$$X = \left(\frac{\beta_0}{\delta} - 1\right)^{\frac{1}{n}}. \tag{5.6}$$

From condition (5.2) it is clear that $X > 0$ is well defined. Moreover $\triangle(x) > 0$ if $x \in (0, X)$ and $\triangle(x) < 0$ if $x > X$. Therefore, condition A is satisfied.

3. Now we calculate optimal stationary points. It is not difficult to show that the equation $\frac{d}{dx}\triangle(x) = 0$ has only one positive solution x^* given by

$$x^* = (b-1)^{\frac{1}{n}}, \quad \text{where} \quad b = \frac{-\beta_0(n-1) + \sqrt{\beta_0^2(n-1)^2 + 4\beta_0\delta n}}{2\delta} > 1.$$

Thus there is only one optimal stationary point. The corresponding control value (the proportion of cells to divide) is

$$u^* = \frac{1}{2}\left(1 + \frac{\delta}{\beta_0}b\right) < 1.$$

Therefore, all the assumptions of Theorems 3.4 and 4.2 are satisfied. This means that if the parameters of the model (5.4) satisfy condition (5.2), then the turnpike property is true; that is, not depending on initial conditions, all optimal solutions converge to the unique stationary point x^*.

4. In [28] it is proved that system (5.1) may have periodic solutions that have been observed in chronic myelogenous leukemia. A unique positive stationary point of (5.1) is given by

$$x_* = \left[\frac{\beta_0}{\delta}(2e^{-\gamma\tau} - 1) - 1\right]^{\frac{1}{n}}.$$

Clearly, both stationary points x^* and x_* are in the interval $(0, X)$, where X is defined in (5.6). Moreover, x_* is also a stationary point of the system (5.4) corresponding to the control $u_* = e^{-\gamma\tau} < 1$.

If parameters γ and τ satisfy the relation

$$\gamma\tau = -\ln\left(\frac{1}{2} + \frac{1}{4}\left[\sqrt{(n-1)^2 + 4n\delta/\beta_0} - (n-1)\right]\right),$$

then the stationary point x_* of (5.1) coincides with the optimal stationary point x^* of the problem (5.4), (5.5).

The result obtained above (the turnpike property) shows that the existence of periodic solutions to (5.1) may be an indication of nonoptimal functioning in the proliferative phase.

Acknowledgments. The author thanks Prof. Anatoli Ivanov (Pennsylvania State University) for bringing this problem to my attention and Prof. Alan Perelson (Los Alamos National Laboratory) for his advice on taking the coefficient β_0 in the blood cell production model as a control parameter. The author also thanks the anonymous reviewers for their valuable comments and suggestions that greatly helped in improving the presentation.

REFERENCES

[1] M. Adimy and F. Crauste, *Mathematical model of hematopoiesis dynamics with growth factor-dependent apoptosis and proliferation regulations*, Math. Comput. Modelling, 49 (2009) pp. 2128–2137.

[2] U. An der Heiden, M. C. Mackey, and H. O. Walther, *Complex oscillations in a simple deterministic neuronal network*, Lectures in Appl. Math., 19 (1981), pp. 355–360.

[3] A. Beuter, L. Glass, M. C. Mackey, and M. S. Titcombe, *Nonlinear Dynamics in Physiology and Medicine*, Interdiscip. Appl. Math. 25, Springer, New York, 2003.

[4] D. A. Carlson, *The existence of catching-up optimal solutions for a class of infinite horizon optimal control problems with time delay*, SIAM J. Control Optim., 28 (1990), pp. 402–422.
[5] D. A. Carlson, A. B. Haurie, and A. Leizarowitz, *Infinite Horizon Optimal Control: Deterministic and Stochastic Systems*, 2nd ed., Springer-Verlag, Berlin, 1991.
[6] T. Damm, L. Grüne, M. Stieler, and K. Worthmann, *An exponential turnpike theorem for averaged optimal control*, SIAM J. Control Optim., to appear.
[7] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel, and H.-O. Walther, *Delay Equations: Functional-, Complex-, and Nonlinear Analysis*, Appl. Math. Sci. 110, Springer-Verlag, Berlin, 1995.
[8] R. Dorfman, P. A. Samuelson, and R. M. Solow, *Linear Programming and Economic Analysis*, McGraw-Hill, New York, 1958.
[9] V. Gaitsgory and M. Quincampoix, *Linear programming approach to deterministic infinite horizon optimal control problems with discounting*, SIAM J. Control Optim., 48 (2009), pp. 2480–2512.
[10] V. Gaitsgory and S. Rossomakhine, *Linear programming approach to deterministic long run average problems of optimal control*, SIAM J. Control Optim., 44 (2006), pp. 2006–2037.
[11] G. Gandolfo, *Economic Dynamics*, Springer-Verlag, Berlin, 1996.
[12] L. Glass and M. C. Mackey, *From Clocks to Chaos: The Rhythms of Life*, Princeton University Press, Princeton, NJ, 1988.
[13] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Springer, New York, 1992.
[14] D. E. Gusev and V. A. Yakubovich, *Turnpike theorem in the problem of continuous optimization with phase restrictions*, Systems Control Lett., 3 (1983), pp. 221–226.
[15] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Appl. Math. Sci. 99, Springer-Verlag, Berlin, 1993.
[16] U. Heiden *Delays in physiological systems*, J. Math. Biol., 8 (1979), pp. 345–364.
[17] A. F. Ivanov and M. A. Mammadov, *Global stability, periodic solutions, and optimal control in a nonlinear differential delay model*, Electron. J. Differential Equations, 19 (2010), pp. 177–188.
[18] A. F. Ivanov, M. A. Mammadov, and S. I. Trofimchuk, *Global stabilization in nonlinear discrete systems with time-delay*, J. Global Optim., 56 (2013), pp. 251–263.
[19] A. F. Ivanov and A. N. Sharkovsky, *Oscillations in singularly perturbed delay equations*, Dynamics Reported N. S., 1 (1991), pp. 164–224.
[20] A. F. Ivanov and A. V. Swishchuk, *Optimal control of stochastic differential delay equations with application in economics*, Internat. J. Qualitative Theory Differential Equations Appl., (2008), pp. 201–201.
[21] M. A. Khan and A. Piazza, *An overview of turnpike theory: Towards the discounted deterministic case*, Adv. Math. Econ., (2011), pp. 39–67.
[22] V. Kolokoltsov and W. Yang, *Turnpike theorems for Markov games*, Dyn. Games Appl., 2 (2012), pp. 294–312.
[23] Y. Kuang, *Delay Differential Equations: With Applications in Population Dynamics*, Math. Sci. Engrg. 191, Academic Press, San Diego, CA, 1993.
[24] Y. Kuang, J. D. Nagy, and J. J. Elser, *Biological stoichiometry of tumor dynamics: Mathematical models and analysis*, Discrete Contin. Dyn. Syst. Ser. B, 4 (2004), pp. 221–240.
[25] T. Lara, P. Ecimovciz, and J. Wu, *Delayed cellular neural networks: Model, applications, implementations, and dynamics*, Differ. Equ. Dyn. Syst., 3 (2002), pp. 71–97.
[26] M. C. Mackey, *Unified hypothesis of the origin of aplastic anemia and periodic hematopoiesis*, Blood, 51 (1978), pp. 941–956.
[27] M. C. Mackey, *Commodity price fluctuations: Price dependent delays and nonlinearities as explanatory factors*, J. Econom. Theory, 48 (1989), pp. 497–509.
[28] M. C. Mackey, C. Ou, L. Pujo-Menjouet, and J. Wu, *Periodic oscillations of blood cell populations in chronic myelogenous leukemia*, SIAM J. Math. Anal., 38 (2006), pp. 166–187.
[29] R. N. Madan, *Chua's Circuit: A Paradigm for Chaos*, World Scientific, Singapore, 1993.
[30] M. J. P. Magill and J. A. Scheinkman, *Stability of regular equilibria and the correspondence principle for symmetric variational problems*, Internat. Econ. Rev., (1979), pp. 297–315.
[31] V. L. Makarov and A. M. Rubinov, *Mathematical Theory of Economic Dynamics and Equilibria*, Springer, New York, 1977.
[32] J. Mallet-Paret and R. D. Nussbaum, *Global continuation and asymptotic behaviour for periodic solutions of a differential-delay equation*, Ann. Mat. Pura Appl., 145 (1986), pp. 33–128.

[33] J. Mallet-Paret and R. D. Nussbaum, *A differential-delay equation arising in optics and physiology*, SIAM J. Math. Anal., 20 (1989), pp. 249–292.
[34] M. A. Mamedov, *Asymptotical optimal paths in models with environment pollution being taken into account (in Russian)*, Optimization (Novosibirsk), 36 (1985), pp. 101–112.
[35] M. A. Mamedov, *A turnpike theorem for continuous-time control systems when the optimal stationary point is not unique*, Abstract Appl. Anal., 2003 (2003), pp. 631–650.
[36] M. A. Mamedov and S. Pehlivan, *Statistical cluster points and turnpike theorem in nonconvex problems*, J. Math. Anal. Appl., 256 (2001), pp. 686–693.
[37] M. A. Mammadov, *Turnpike theory: Stability of optimal trajectories*, in Encyclopedia of Optimization, 2nd ed., C. A., Floudas, and Pardalos, P. M., eds., Springer, New York, 2009.
[38] M. A. Mammadov and A. F. Ivanov, *Asymptotical stability of trajectories in optimal control problems with time delay*, in Proceedings of the Third Global Conference on Power Control and Optimization, Gold Coast, Australia, 2010.
[39] L. W. McKenzie, *Turnpike theory*, Econometrica, 44 (1976), pp. 841–865.
[40] P. W. Nelson, J. D. Murray, and A. S. Perelson, *A model of HIV-*1 *pathogenesis that includes an intracellular delay*, Math. Biosci., 163 (2000), pp. 201–215.
[41] P. W. Nelson and A. S. Perelson, *Mathematical analysis of delay differential equation models of HIV-*1 *infection*, Math. Biosci., 179 (2002), pp. 73–94.
[42] V. V. Nemickij and V. V. Stepanov, *Qualitative Theory of Differential Equations*. Courier Dover Publications, New York, 1989.
[43] A. I. Panasyuk and V. I. Panasyuk, *Asymptotic Turnpike Optimization of Control Systems*, Nauka i Tekhnika, Minsk, 1986.
[44] S. Pehlivan and M. A. Mamedov, *Statistical cluster points and turnpike*, Optimization, 48 (2000), pp. 91–106.
[45] J. P. Richard, *Time-delay systems: An overview of some recent advances and open problems*, Automatica, 39 (2003), pp. 1667–1694.
[46] R. T. Rockafellar, *Saddle points of Hamiltonian systems in convex problems of Lagrange*, J. Optim. Theory Appl., 12 (1973), pp. 367–390.
[47] R. T. Rockafellar, *Saddle points of Hamiltonian systems in convex Lagrange problems having a nonzero discount rate*, J. Econom. Theory, 12 (1976), pp. 71–113.
[48] R. T. Rockafellar, *Hamiltonian trajectories and saddle points in mathematical economics*, Control Cybernet., 38 (2009), pp. 1575–1588.
[49] X. M. Sun, J. Zhao, and D. J. Hill, *Stability and l2-gain analysis for switched delay systems: A delay-dependent method*, Automatica, 42 (2006), pp. 1769–1774.
[50] A. D. Tsvirkun and S. Yu. Yakovenko, *Lyapunov Functions and Turnpike Theory*, Lecture Notes in Control and Inform. Sci., 84, Springer-Verlag, Berlin, 1986, pp. 1002–1007.
[51] A. J. Zaslavski, *Solutions for a class of optimal control problems with time delay, Part* 1, J. Optim. Theory Appl., 91 (1996), pp. 155–184.
[52] A. J. Zaslavski, *Solutions for a class of optimal control problems with time delay, Part* 2, J. Optim. Theory Appl., 91 (1996), pp. 455–490.
[53] A. J. Zaslavski, *Turnpike Properties in the Calculus of Variations and Optimal Control*, Springer, New York, 2006.
[54] M. I. Zelikin, L. F. Zelikina, and R. Hildebrand, *Asymptotics of optimal synthesis for one class of extremal problems*, Proc. Steklov Inst. Math., (2001), pp. 87–115.