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A new auxiliary function method for general constrained global optimization*

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Abstract

In this paper, we firstly propose a method to obtain an approximate feasible point for general constrained global optimization problems (with both inequality and equality constraints). Then we propose an auxiliary function method to obtain a global minimizer or an approximate global minimizer with a required precision for general global optimization problems by locally solving some unconstrained programming problems. Some numerical examples are reported to demonstrate the efficiency of the present optimization method.

Keywords. auxiliary function method, general constrained global optimization

Mathematics Subject Classification 2000. 90C26, 90C30

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1 Introduction

We consider the following general constrained global optimization problem:

$$
\begin{aligned}
(P) \qquad & \min \quad f(x) && (1.1)\\
& \text{s.t.} \quad g_i(x) \le 0, \ i = 1, \ldots, m \\
& \qquad\quad h_j(x) = 0, j = 1, \ldots, l && (1.2)\\
& \qquad\quad x \in X,
\end{aligned}
$$

where $f : X \to \mathbb{R}$, $g_i, h_j : X \to \mathbb{R}, i = 1, 2, \ldots, m, j = 1, \ldots, l$ are continuously differentiable on X, and X is an open box or a closed box.

Generally speaking, it is difficult to solve problem (P), even to find a feasible point of (P) due to the presence of equality constraints, see [1] and [2]. We know that for unconstrained global optimization problem without special structural property, the auxiliary function methods (such as filled function methods) has attracted extensive attention in the last two decades, see, e.g. [3]-[7], [9], [11]-[14]. In [10], the authors also proposed a filled function for inequality constrained global optimization problems (for (P) without equality constraints).

Due to the involved equality constraints in (P), it is very difficult to use the existing auxiliary function methods including the existing filled function methods to obtain an exact global minimizer or an approximate global minimizer since the interior of the feasible set is usually empty. In this paper, we will first propose a method to find an ϵ-approximate feasible solution via locally solving an smooth unconstrained optimization problem, where ϵ is any preset positive number, then we will construct an auxiliary function and introduce a new auxiliary function method to search for a global minimizer or an approximate global minimizer for problem (P). Throughout the paper, we use $\|x\|$ to represent the l_2 norm of x in $\mathbb{R}^n$, i.e. $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$, where $x = (x_1, \ldots, x_n)^T$.

The rest of this paper is organized as follows. In Section 2, a method is proposed to find an approximate feasible point for problem (P). In Section 3, an auxiliary function is introduced to improve the obtained approximate feasible point for problem (P) by locally solving some unconstrained optimization problems. A new auxiliary function method is proposed in Section 4 to obtain a global minimizer or an approximate global minimizer for problem (P). Finally, some illustrative numerical examples are given in Section 5.

2 A method for finding an approximate feasible point

For a given $\epsilon > 0$, let

$$\begin{aligned} S &= \{x \in X \mid g_i(x) \le 0, h_j(x) = 0,\ i = 1,\dots,m, j = 1,\dots,l\}, &(2.1)\\ S_\epsilon &= \{x \in X \mid g_i(x) \le \epsilon, |h_j(x)| \le \epsilon,\ i = 1,\dots,m, j = 1,\dots,l\}. &(2.2)\end{aligned}$$

Definition 2.1. *A point $\bar{x} \in X$ is said to be an ϵ-approximate feasible solution to problem (P), if $\bar{x} \in S_\epsilon$.*

Definition 2.2. *A point $x^* \in X$ is said to be an ϵ-approximate global minimizer of problem (P), if $x^* \in S_\epsilon$ and $f(x) \ge f(x^*) - \epsilon, \forall x \in S_\epsilon$.*

For any $r > 0$, let

$$\psi_r(t) = \begin{cases} \frac{2}{r}t - 1, & \text{if} \quad t \ge r \\ \frac{(t-r)^2}{r^2} + \frac{2}{r}t - 1, & \text{if } 0 < t < r \\ 0, & \text{if } t \le 0 \end{cases} \tag{2.3}$$

and

$$\phi(t) = \begin{cases} 1, & \text{if} \quad t \ge 1 \\ -2t^3 + 3t^2, & \text{if } 0 < t < 1 \\ 0, & \text{if } t \le 0 \end{cases}. \tag{2.4}$$

Obviously, $\psi_r(t)$ and $\phi(t)$ are continuously differentiable on R. Figure 2.1 shows the behavior of function $\psi_r(t)$ for $r = 0.1, 0.2$ and $r = 0.4$, and Figure 2.2 shows the behavior of function $\phi(t)$, where $\phi(t)$ is just the function $f_{r,c}(t)$ with $c = 1$ given in reference [10], but function $\psi_r(t)$ is different from any other auxiliary functions given in existing references, such as, in reference [8], the similar auxiliary function is given as following: $L_\epsilon(\xi) = \begin{cases} \xi, & \xi < -\epsilon \\ -(\xi - \epsilon)^{\frac{1}{2}/4\epsilon}, & -\epsilon \le \xi \le \epsilon \\ 0, & \xi > \epsilon \end{cases}$; in reference [10], the similar auxiliary function is given as following: $f_{r,c}(t) := \begin{cases} c, & t \ge 0 \\ -\dfrac{2c}{r^3}t^3 - \dfrac{3c}{r^2}t^2 + c, & -r < t \le 0 \\ 0, & t \le -r \end{cases}$.

If X is an open box, let $X = \prod_{i=1}^n (c_i, d_i)$; if X is a closed box, let $X = \prod_{i=1}^n [c_i, d_i]$. Let $\bar{c} := (c_1 - 1, \dots, c_n - 1)^T$, then $\bar{c} \in R^n \setminus X$. Let

$$G_r(x) = \frac{1}{\|x - \bar{c}\|} \cdot \phi\Big(\sum_{j=1}^{l} \psi_{\frac{3r^2}{4}}(h_j^2(x) - \frac{r^2}{4}) + \sum_{i=1}^{m} \psi_{\frac{r}{2}}(g_i(x) - \frac{r}{2})\Big), \tag{2.5}$$

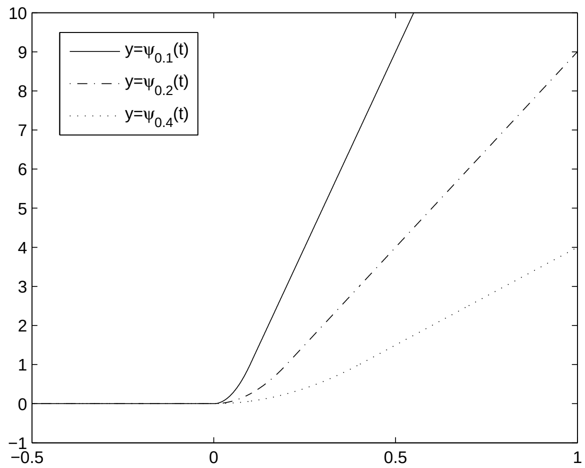

Figure 2.1: The behavior of $\psi_r(t)$

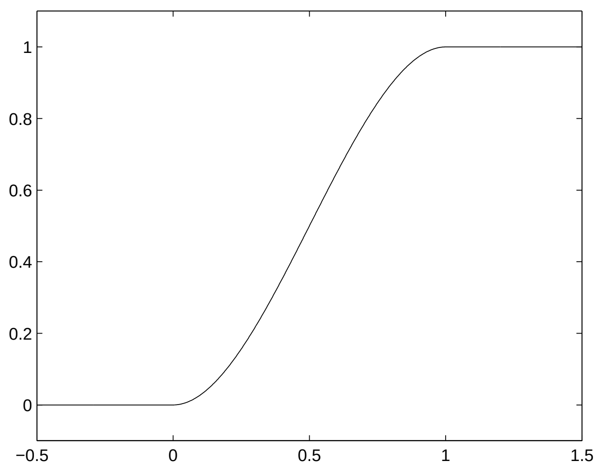

Figure 2.2: The behavior of $\phi(t)$

where $r > 0$ is a parameter. Then $G_r(x)$ is continuously differentiable on X. Here $\sum_{j=1}^{l} \psi_{\frac{3r^2}{4}}(h_j^2(x) - \frac{r^2}{4})$ is used to penalize the points who do not satisfy the equality constraints $h_j(x) = 0, j = 1, \dots, l$; $\sum_{i=1}^{m} \psi_{\frac{r}{2}}(g_i(x) - \frac{r}{2})$ is used to penalize the points who do not satisfy the inequality constraints $g_i(x) \leq 0, i = 1, \dots, m$.

Proposition 2.1. *For any $r > 0$, x is a $\frac{r}{2}$-approximate feasible solution to problem (P) if and only if $G_r(x) = 0$.*

Proof. By definition, x is a $\frac{r}{2}$-approximate feasible solution of problem (P) if and only if

$$g_i(x) \leq \frac{r}{2}, |h_j(x)| \leq \frac{r}{2}, i = 1, \dots, m, j = 1, \dots, l,$$

which is equivalent to

$$\phi\Big(\sum_{j=1}^{l} \psi_{\frac{3r^2}{4}}(h_j^2(x) - \frac{r^2}{4}) + \sum_{i=1}^{m} \psi_{\frac{r}{2}}(g_i(x) - \frac{r}{2})\Big) = 0. \tag{2.6}$$

Obviously, (2.6) holds if and only if $G_r(x) = 0$. □

Note that x is not a $\frac{r}{2}$-approximate feasible solution to problem (P) if and only if $G_r(x) > 0$.

Consider the following problem $(A)_r$:

$$(A)_r \qquad \min \ G_r(x) \qquad \text{s.t.} \ x \in X,$$

where X is an open box or a closed box given in problem (P).

Theorem 2.1. *For any $r > 0$,*

a. if X is an open box, then any local minimizer $\bar{x}_r$ of problem $(A)_r$ satisfies that $\bar{x}_r \in S_r$, i.e., $\bar{x}_r$ is a r-approximate feasible point to problem (P);

b. if X is a close box, let $\bar{x}_r$ be a local minimizer of problem $(A)_r$, then one of the following results holds:

(1). $\bar{x}_r \in S_r$, i.e., $\bar{x}_r$ is a r-approximate feasible point to problem (P);

(2). $\bar{x}_r = d$, where $d = (d_1, \dots, d_n)^T$.

Proof. (a). For any $r > 0$, by (2.5), we know that

$$\begin{aligned}\nabla G_r(x) \quad = \quad & \frac{-(x-\bar{c})}{\|x-\bar{c}\|^3} \cdot \phi\Big(\sum_{j=1}^{l} \psi_{\frac{3r^2}{4}}(h_j^2(x) - \frac{r^2}{4}) + \sum_{i=1}^{m} \psi_{\frac{r}{2}}(g_i(x)) - \frac{r}{2}\Big) \\ & + \frac{1}{\|x-\bar{c}\|} \cdot \phi'\Big(\sum_{j=1}^{l} \psi_{\frac{3r^2}{4}}(h_j^2(x) - \frac{r^2}{4}) + \sum_{i=1}^{m} \psi_{\frac{r}{2}}(g_i(x) - \frac{r}{2})\Big) \\ & \cdot [\sum_{j=1}^{l} \psi'_{\frac{3r^2}{4}}(h_j^2(x) - \frac{r^2}{4}) 2h_j(x)\nabla h_j(x) + \sum_{i=1}^{m} \psi'_{\frac{r}{2}}(g_i(x) - \frac{r}{2})\nabla g_i(x)].\end{aligned}$$

Let $\bar{x}_r$ be a local minimizer of problem $(A)_r$. Then we have that $\nabla G_r(\bar{x}_r) = 0$ since X is an open box and $\bar{x}_r \neq c$ since $\bar{x}_r \in X$ and $c \notin X$. Suppose that

$\bar{x}_r \notin S_r$, then there exists one $j_r \in \{1,\dots,l\}$ such that $h_{j_r}^2(\bar{x}_r) > r^2$ or exists one $i_r \in \{1,\dots,m\}$ such that $g_{i_r}(\bar{x}_r) > r$. Then, we have that

$$\phi\Big(\sum_{j=1}^{l} \psi_{\frac{3r^2}{4}}(h_j^2(\bar{x}_r) - \frac{r^2}{4}) + \sum_{i=1}^{m} \psi_{\frac{r}{2}}(g_i(\bar{x}_r) - \frac{r}{2})\Big) = 1$$

and

$$\phi'\Big(\sum_{j=1}^{l} \psi_{\frac{3r^2}{4}}(h_j^2(\bar{x}_r) - \frac{r^2}{4}) + \sum_{i=1}^{m} \psi_{\frac{r}{2}}(g_i(\bar{x}_r) - \frac{r}{2}\Big) = 0.$$

Hence

$$\nabla G_r(\bar{x}_r) = \frac{-(\bar{x}_r - \bar{c})}{\|\bar{x}_r - \bar{c}\|^3} \neq 0,$$

which contradicts $\nabla G_r(\bar{x}_r) = 0$. Hence we have that $\bar{x}_r \in S_r$.

(b). Suppose that X is a closed box and let $X = \prod_{i=1}^{n}[a_i, b_i]$. Let $\bar{x}_r$ be a local minimizer of problem $(A)_r$. If $\bar{x}_r \notin S_r$ and $\bar{x}_r \neq d$, then there exists an integer number k_r such that $1 \leq k_r \leq n$ and $(\bar{x}_r)_{k_r} < d_{k_r}$, where $(\bar{x}_r)_{k_r}$ and d_{k_r} are the k_rth components of point $\bar{x}_r$ and d, respectively. Since $\bar{x}_r$ is a local minimizer of problem $(A)_r$,there exits a positive number λ_r such that $0 < \lambda_r \leq d_{k_r} - (\bar{x}_r)_{k_r}$ and

$$G_r(\bar{x}_r) \leq G_r(x), \forall x \in N(\bar{x}_r, \lambda_r), \tag{2.7}$$

where $N(\bar{x}_r, \lambda_r) = \{x \in X \mid \|x - \bar{x}_r\| \leq \lambda_r\}$. Let $\bar{y}_r := \begin{cases} (\bar{x}_r)_i, & i \neq k_r \\ (\bar{x}_r)_{k_r} + \lambda_r, & i = k_r \end{cases}$. Then, we can prove that $\bar{y}_r \in N(\bar{x}_r, \lambda_r)$ and $G_r(\bar{y}_r) < G_r(\bar{x})$, which contradicts (2.7).

In fact, by $(\bar{y}_r)k_r = (\bar{x}_r)_{k_r} + \lambda_r$ and by $0 < \lambda_r \leq d_{k_r} - (\bar{x}_r)_{k_r}$, we have that $(\bar{x}_r)_{k_r} \leq (\bar{y}_r)k_r \leq d_{k_r}$. Hence, $\bar{y}_r \in N(\bar{x}_r, \lambda_r)$. Furthermore, we have that

$$\begin{aligned} G_r(\bar{y}_r) &\leq \frac{1}{\|\bar{y}_r - \bar{c}\|} \\ &= \frac{1}{(\sum_{i \neq k_r} |(\bar{x}_r)_i - (\bar{c})_i|^2 + ((\bar{x}_r)_{k_r} - (\bar{c})_{k_r} + \lambda_r)^2)^{\frac{1}{2}}} \\ &< \frac{1}{(\sum_{i=1}^{n} |(\bar{x}_r)_i - (\bar{c})_i|^2)^{\frac{1}{2}}} \\ &= \frac{1}{\|\bar{x}_r - \bar{c}\|} \\ &= G_r(\bar{x}_r) \end{aligned}$$

since $\bar{x}_r \notin S_r$. □

Remark 2.1. *By the proof of Theorem 2.1, we know that for any $r > 0$, any stationary point of $G_r(x)$ on X (not necessarily local minimizer of problem $(A)_r$) is also a r-approximate feasible point to problem (P).*

3 A new auxiliary function for problem (P) and its properties

For a given $r > 0$, consider the following problem:

$$
\begin{array}{lll}
(P)_r & \min & f(x) \\
 & \text{s.t.} & x \in S_r.
\end{array}
$$

Let $x_r^* \in S_r$ be a local minimizer of problem $(P)_r$ and let

$$
\begin{aligned}
F_{r,x_r^*}(x) &= \frac{1}{\|x - x_r^*\|^2 + 1}\phi\Bigg(\psi_{\frac{r}{2}}\Big(f(x) - f(x_r^*) + \frac{r}{2}\Big) + \sum_{i=1}^{m}\psi_{\frac{r}{2}}\Big(g_i(x) - \frac{r}{2}\Big) \\
&\quad + \sum_{i=1}^{l}\psi_{\frac{3r^2}{4}}\Big(h_j^2(x) - \frac{r^2}{4}\Big)\Bigg), \qquad (3.1)
\end{aligned}
$$

where $r > 0$ is a parameter, ϕ and ψ_r are defined by (2.4) and (2.3), respectively. Here $\psi_{\frac{r}{2}}\Big(f(x) - f(x_r^*) + \frac{r}{2}\Big)$ is used to penalize the points who satisfy that $f(x) - f(x_r^*) > \frac{r}{2}$; $\sum_{i=1}^{l}\psi_{\frac{3r^2}{4}}\Big(h_j^2(x) - \frac{r^2}{4}\Big)$ is used to penalize the points who satisfy that $|h_j(x)| > \frac{r}{2}, j = 1, \ldots, l$; $\sum_{i=1}^{m}\psi_{\frac{r}{2}}\Big(g_i(x) - \frac{r}{2}\Big)$ is used to penalize the points who satisfy that $g_i(x) > \frac{r}{2}, i = 1, \ldots, m$.

Consider the following problem:

$$
\begin{array}{lll}
(B)_r & \min & F_{r,x_r^*}(x) \\
 & \text{s.t.} & x \in X,
\end{array}
$$

where X is the open box or the closed box given in problem (P). Then, we have the following properties:

Theorem 3.1. *For any $r > 0$, x_r^* is a strict local maximizer of problem $(B)_r$.*

Proof. Since x_r^* is a local minimizer of problem $(P)_r$, there exists $\delta_0 > 0$, such that $f(x) \geq f(x_r^*)$ for any $x \in S_r \cap N(x_r^*, \delta_0)$, where $N(x_r^*, \delta_0)$ denotes the neighborhood of x_r^* with radius δ_0, i.e., $N(x_r^*, \delta_0) = \{x \in R^n \mid \|x - x_r^*\| < \delta_0\}$.

Then, for any $x \in X \cap N(x_r^*, \delta_0)$, we have $f(x) \geq f(x_r^*)$ or $x \in X \setminus S_r$, where $x \in X \setminus S_r$ implies that

$\exists\, i_1 \in \{1, \ldots, m\}$ such that $g_{i_1}(x) > r$ or $\exists\, j_1 \in \{1, \ldots, l\}$ such that $h_{j_1}^2(x) > r^2$,

i.e.,

$$g_{i_1}(x) - \frac{r}{2} > \frac{r}{2} \text{ or } h_{j_1}^2(x) - \frac{r^2}{4} > \frac{3r^2}{4}.$$

Therefore, for any $x \in X \cap N(x_r^*, \delta_0)$, we have

$$\psi_{\frac{r}{2}}\Big(f(x) - f(x_r^*) + \frac{r}{2}\Big) + \sum_{i=1}^{m} \psi_{\frac{r}{2}}\Big(g_i(x) - \frac{r}{2}\Big) + \sum_{i=1}^{l} \psi_{\frac{3r^2}{4}}\Big(h_j^2(x) - \frac{r^2}{4}\Big) \geq 1,$$

which implies that

$$\phi\Bigg(\psi_{\frac{r}{2}}\Big(f(x) - f(x_r^*) + \frac{r}{2}\Big) + \sum_{i=1}^{m} \psi_{\frac{r}{2}}\Big(g_i(x) - \frac{r}{2}\Big) + \sum_{i=1}^{l} \psi_{\frac{3r^2}{4}}\Big(h_j^2(x) - \frac{r^2}{4}\Big)\Bigg) = 1,$$

i.e.,

$$F_{r,x_r^*}(x) = \frac{1}{\|x - x^*\|^2 + 1} < 1 = F_{r,x_r^*}(x_r^*)$$

for any $x \in X \cap (N(x_r^*, \delta_0) \setminus \{x_r^*\})$. Thus, x_r^* is a strict local maximizer of problem $(B)_r$. □

Theorem 3.2. *For any $r > 0$,*

a. if X is an open box, then any local minimizer $\bar{x}_r \in X$ of problem $(B)_r$ satisfies that $\bar{x}_r \in S_r$ and $f(\bar{x}_r) < f(x_r^*)$;

b. if X is a closed box, let $\bar{x}_r$ be a local minimizer of problem $(A)_r$, then one of the following results holds:

(1). $\bar{x}_r \in S_r$ and $f(\bar{x}_r) < f(x_r^*)$;

(2). $\bar{x}_r \in V(X)$, where $V(X)$ is the set of vertex point of X.

Proof. a. Since x_r^* is a strict local maximizer of problem $(B)_r$, then $\bar{x}_r \neq x_r^*$. Moreover, since X is an open box, any local minimizer $\bar{x}_r \in X$ of problem

$(B)_r$ must satisfy that $\nabla F_{r,x_r^*}(\bar{x}_r) = 0$. By (3.1), we have that

$$\begin{aligned}
& \nabla F_{r,x_r^*}(x) \\
= \quad & \frac{-2(x-x_r^*)}{\left(\|x-x_r^*\|^2+1\right)^2}\phi\Bigg(\psi_{\frac{r}{2}}\Big(f(x)-f(x_r^*)+\frac{r}{2}\Big)+\sum_{i=1}^{m}\psi_{\frac{r}{2}}\Big(g_i(x)-\frac{r}{2}\Big) \\
& +\sum_{i=1}^{l}\psi_{\frac{3r^2}{4}}\Big(h_j^2(x)-\frac{r^2}{4}\Big)\Bigg)+\frac{1}{\|x-x_r^*\|^2+1}\cdot\phi'\Bigg(\psi_{\frac{r}{2}}\Big(f(x)-f(x_r^*)+\frac{r}{2}\Big) \\
& +\sum_{i=1}^{m}\psi_{\frac{r}{2}}\Big(g_i(x)-\frac{r}{2}\Big)+\sum_{i=1}^{l}\psi_{\frac{3r^2}{4}}\Big(h_j^2(x)-\frac{r^2}{4}\Big)\Bigg)\cdot\Bigg(\psi'_{\frac{r}{2}}\Big(f(x)-f(x_r^*)+\frac{r}{2}\Big)\nabla f(x) \\
& +\sum_{i=1}^{m}\psi'_{\frac{r}{2}}\Big(g_i(x)-\frac{r}{2}\Big)\nabla g_i(x)+\sum_{i=1}^{l}\psi'_{\frac{3r^2}{4}}\Big(h_j^2(x)-\frac{r^2}{4}\Big)2h_j(x)\nabla h_j(x)\Bigg).
\end{aligned}$$

Suppose that $f(\bar{x}_r) \geq f(x_r^*)$ or $\bar{x}_r \notin S_r$. By $f(\bar{x}_r) \geq f(x_r^*)$, we have that

$$f(\bar{x}_r) - f(x_r^*) + \frac{r}{2} \geq \frac{r}{2}.$$

By $\bar{x}_r \notin S_r$, there exists $i_r \in \{1, \dots, m\}$ such that

$$g_{i_r}(\bar{x}_r) > r$$

or there exists $j_r \in \{1, \dots, l\}$ such that

$$h_{j_r}^2(\bar{x}_r) > r^2,$$

i.e.,

$$g_{i_r}(\bar{x}_r) - \frac{r}{2} > \frac{r}{2} \text{ or } h_{j_r}^2(\bar{x}_r) - \frac{r^2}{4} > \frac{3r^2}{4}.$$

Hence,

$$\psi_{\frac{r}{2}}\Big(f(\bar{x}_r)-f(x_r^*)+\frac{r}{2}\Big)+\sum_{i=1}^{m}\psi_{\frac{r}{2}}\Big(g_i(\bar{x}_r)-\frac{r}{2}\Big)+\sum_{i=1}^{l}\psi_{\frac{3r^2}{4}}\Big(h_j^2(\bar{x}_r)-\frac{r^2}{4}\Big)\geq 1,$$

i.e.,

$$\phi\Bigg(\psi_{\frac{r}{2}}\Big(f(\bar{x}_r)-f(x_r^*)+\frac{r}{2}\Big)+\sum_{i=1}^{m}\psi_{\frac{r}{2}}\Big(g_i(\bar{x}_r)-\frac{r}{2}\Big)+\sum_{i=1}^{l}\psi_{\frac{3r^2}{4}}\Big(h_j^2(\bar{x}_r)-\frac{r^2}{4}\Big)\Bigg)=1$$

and

$$\phi'\Bigg(\psi_{\frac{r}{2}}\Big(f(\bar{x}_r)-f(x_r^*)+\frac{r}{2}\Big)+\sum_{i=1}^{m}\psi_{\frac{r}{2}}\Big(g_i(\bar{x}_r)-\frac{r}{2}\Big)+\sum_{i=1}^{l}\psi_{\frac{3r^2}{4}}\Big(h_j^2(\bar{x}_r)-\frac{r^2}{4}\Big)\Bigg)=0.$$

Thus, we have that

$$\nabla F_{r,x_r^*}(\bar{x}_r) = -\frac{2(\bar{x}_r - x_r^*)}{(\|\bar{x}_r - x_r^*\|^2 + 1)^2} \neq 0,$$

which contradicts $\nabla F_{r,x_r^*}(\bar{x}_r) = 0$. Hence, any local minimizer $\bar{x}_r$ of problem $(B)_r$ satisfies that $\bar{x}_r \in S_r$ and $f(\bar{x}_r) < f(x_r^*)$.

b. Let $X = \prod_{i=1}^n [c_i, d_i]$ and $\bar{x}_r$ be a local minimizer of $F_{r,x_r^*}(x)$ on X. If $\bar{x}_r$ is neither a point satisfying

$$f(\bar{x}_r) < f(x_r^*) \text{ and } \bar{x}_r \in S_r$$

nor a vertex of X, then there exist two different points $z_1, z_2 \in X$ and a positive number α with $0 < \alpha < 1$ such that $\bar{x}_r = \alpha z_1 + (1-\alpha) z_2$ and $F_{r,x_r^*}(\bar{x}_r) = \dfrac{1}{\|\bar{x}_r - x_r^*\|^2 + 1}$.

Since X is a box, we have that $[z_1, z_2] := \{\lambda z_1 + (1-\lambda) z_2 \mid 0 \leq \lambda \leq 1\} \subset X$. Let $d_0 = z_2 - z_1$, $\alpha_0 = \min\{\alpha, 1-\alpha\}$. Then, we can easily verify that for any s with $|s| \leq \alpha_0$, it holds

$$\bar{x}_r + s d_0 \in [z_1, z_2] \subset X.$$

Let λ_0 be a positive number such that $0 < \lambda_0 \leq \alpha_0$. Let $z_{1,\lambda_0} = \bar{x} + \lambda_0 d_0$ and $z_{2,\lambda_0} = \bar{x} - \lambda_0 d_0$. Then $z_{1,\lambda_0}, z_{2,\lambda_0} \in X$ and

$$\begin{aligned}
F_{r,x_r^*}(z_{1,\lambda_0}) &\leq \frac{1}{\|z_{1,\lambda_0} - x_r^*\|^2 + 1} \\
&= \frac{1}{\|(\bar{x}_r - x_r^*) + \lambda_0 d_0\|^2 + 1}, \\
F_{r,x_r^*}(z_{2,\lambda_0}) &\leq \frac{1}{\|z_{2,\lambda_0} - x_r^*\|^2 + 1} \\
&= \frac{1}{\|(\bar{x}_r - x_r^*) - \lambda_0 d_0\|^2 + 1}.
\end{aligned}$$

By

$$\begin{aligned}
& \|(\bar{x}_r - x_r^*) + \lambda_0 d_0\|^2 + \|(\bar{x}_r - x_r^*) - \lambda_0 d_0\|^2 \\
= \; & \|\bar{x}_r - x_r^*\|^2 + \lambda^2 \|d_0\|^2 + 2\lambda_0 \langle d_0, \bar{x}_r - x_r^* \rangle + \|\bar{x}_r - x_r^*\|^2 + \lambda_0^2 \|d_0\|^2 \\
& -2\lambda_0 \langle d_0, \bar{x}_r - x_r^* \rangle \\
= \; & 2\|\bar{x}_r - x_r^*\|^2 + 2\lambda_0^2 \|d_0\|^2 \\
> \; & 2\|\bar{x}_r - x_r^*\|^2,
\end{aligned}$$

we have that one of $\|(\bar{x}_r - x_r^*) + \lambda_0 d_0\|^2$ and $\|(\bar{x}_r - x_r^*) - \lambda_0 d_0\|^2$ is larger than $\|\bar{x}_r - x_r^*\|^2$. Thus one of $F_{r,x_r^*}(z_{1,\lambda_0})$ and $F_{r,x_r^*}(z_{2,\lambda_0})$ is less than $F_{r,x_r^*}(\bar{x}_r)$. Since

λ_0 can approach 0 to any extent, we obtain that $\bar{x}_r$ is not a local minimizer of $F_{r,x_r^*}(x)$ on X. This is a contradiction. Therefore, if $\bar{x}_r$ is a local minimizer of $F_{r,x_r^*}(x)$ on X, then we have $f(\bar{x}_r) < f(x_r^*)$ and $\bar{x}_r \in S_{\frac{r}{2}}$, or $\bar{x}_r$ is a vertex of X. □

Remark 3.1. *Note that from the proof of Theorem 3.2, we know that if $\bar{x}_r$ is a stationary point of function $F_{r,x_r^*}(x)$ ($\nabla F_{r,x_r^*}(i.e., \bar{x}_r) = 0$ which is not necessarily local minimizer of problem (B_r)), then $\bar{x}_r$ also satisfies that $\bar{x}_r \in S_r$ and $f(\bar{x}_r) < f(x_r^*)$.*

Theorem 3.3. *Assume that there is at least one $\frac{r}{2}$-approximate global minimizer of problem (P). If x_r^* is not a $\frac{r}{2}$-approximate global minimizer of problem (P), then there exists a $\bar{x} \in S_{\frac{r}{2}}$ such that $\bar{x}$ is a local minimizer of problem $(B)_r$.*

Proof. i) If x_r^* is not a $\frac{r}{2}$-approximate global minimizer of problem (P), then there exists $\bar{x} \in S_{\frac{r}{2}}$ such that $f(\bar{x}) < f(x_r^*) - \frac{r}{2}$. Hence, we have that

$$f(\bar{x})-f(x_r^*)+\frac{r}{2} < 0 \text{ and } g_i(\bar{x})-\frac{r}{2} \leq 0, h_j^2(\bar{x})-\frac{r^2}{4} \leq 0,\ \forall i = 1,\dots,m,\ j = 1,\dots,l.$$

Thus,

$$\psi_{\frac{r}{2}}\Big(f(\bar{x}) - f(x_r^*) + \frac{r}{2}\Big) + \sum_{i=1}^{m} \psi_{\frac{r}{2}}\Big(g_i(\bar{x}) - \frac{r}{2}\Big) + \sum_{i=1}^{l} \psi_{\frac{3r^2}{2}}\Big(h_j^2(\bar{x}) - \frac{r^2}{4}\Big) = 0$$

and $F_{r,x_r^*}(\bar{x}) = 0$. Therefore, $\bar{x}$ is a global (and also local) minimizer of problem $(B)_r$ since $F_{r,x_r^*}(x) \geq 0$ for any $x \in X$. □

Theorem 3.4. *For any $x_1,\ x_2 \in X$ with*
i) *$f(x_1) \geq f(x_r^*)$ or $x_1 \notin S_r$*
and
ii) *$f(x_2) \geq f(x_r^*)$ or $x_2 \notin S_r$, we have that*

$$\|x_2 - x_r^*\| > \|x_1 - x_r^*\| \text{ if and only if } F_{r,x_r^*}(x_2) < F_{r,x_r^*}(x_1).$$

Proof. For such $x_i, i = 1, 2$, we have that

$$F_{r,x_r^*}(x_i) = \frac{1}{\|x_i - x_r^*\|^2 + 1}, i = 1, 2.$$

Thus, $\|x_2 - x_r^*\| > \|x_1 - x_r^*\|$ if and only if $F_{r,x_r^*}(x_2) < F_{r,x_r^*}(x_1)$. □

From the properties discussed above, we know that for a given local minimizer x_r^* of problem (P_r), if we can find a local minimizer $\bar{x}_r$ of problem $(B)_r$, then point $\bar{x}_r$ must be a better point than x_r^* since $\bar{x}_r \in S_r$ and $f(\bar{x}_r) < f(x_r^*)$. And then a better local minimizer for problem $(P)_r$ can be obtained by local search problem $(P)_r$ starting from $\bar{x}_r$ and finally a $\frac{r}{2}$-approximate global minimizer for problem (P) can be obtained. But how to find the first local minimizer x_r^* of problem $(P)_r$ is also an important task, here we can use the method proposed in Section 2 to get a feasible point x_r to problem $(P)_r$ and then we can find the first local minimizer x_r^* of problem $(P)_r$ by local search problem $(P)_r$ starting from the feasible point x_r.

4 An algorithm for finding an approximate global minimizer for problem (P)

The general idea of this algorithm is as follows: for the given $r > 0$, solving unconstrained problem $(A)_r$ by using some local optimization methods. Let x_r be a local minimizer of problem $(A)_r$. Then $x_r \in S_r$. Solve problem $(P)_r$ by using some local optimization methods starting from x_r and let x_r^* be the obtained local minimizer. Solve unconstrained problem $(B)_r$ by using some local optimization methods starting from the neighboring points of x_r^*. If we can find a local minimizer or a stationary point $\bar{x}_r$ of problem $(B)_r$, then $\bar{x}_r$ is a better point to problem $(P)_r$ than x_r^*. Solve problem $(P)_r$ again starting from $\bar{x}_r$ and then we can obtain a better local minimizer of problem $(P)_r$. Continue the process. If we cannot find any local minimizer of problem $(B)_r$, then x_r^* is a $\frac{r}{2}-$ approximate global minimizer of problem (P). If r is small enough, then we can stop; otherwise, we can reduce r and repeat the above process. Finally, an approximate global minimizer to problem (P) with preset precision can be obtained. The corresponding algorithm is denoted by Algorithm **AFM**(Auxiliary Function Method) and detailed as follows:

Algorithm AFM

Step 0. Choose a small positive number μ(the required precision)(In the examples of Section 5, we take $\mu = 10^{-3}$).

Choose a positive integer number K and directions $e_1, \ldots, e_K$ (In the numerical examples in Section 5, we let $K = 2n$ and let $e_i,\ i = 1, \ldots, K$, be the coordinate directions, where n is the number of dimensions of the variable).

Let $k := 0$ and let $r = \mu$.

Step 1. Choose an initial point $x_1^0 \in X$.

Step 2. Solving the following problem by local optimization methods starting from x_1^0:

$$
\begin{array}{lll}
(A)_r & \min & G_r(x) \\
 & \text{s.t.} & x \in X,
\end{array}
$$

where $G_r(x)$ is defined by (2.5).

Step 3. *a*. If X is an open box and if we can find a local minimizer or a stationary point of problem $(A)_r$, let x_r be the local minimizer or the stationary point and goto *Step 5*. Otherwise, goto Step 4;

b. If X is an closed box and let x_r be the local minimizer or a stationary point of problem $(A)_r$. Check whether $x_r \in S_r$. If $x_r \in S_r$, goto *Step 5*. Otherwise, goto Step 4;

Step 4. If $k < K$, let $k := k + 1$, Let α be small enough such that $x_1^0 + \alpha e_k \in X$ and let $x_1^0 := x_1^0 + \alpha e_k$, goto *Step 2*. Otherwise goto *Step 10*.

Step 5. Solve the following problem $(P)_r$ by local optimization methods starting from x_r.

$$
\begin{array}{lll}
(P)_r & \min & f(x) \\
 & \text{s.t.} & x \in S_r.
\end{array}
$$

Let x_r^* be the obtained local minimizer of problem $(P)_r$ (Assume that we can always find at least one local minimizer here). Let $k := 1$ and go to Step 6.

Step 6. Let α be small so that $x_r^* + \alpha e_k \in X$ and let $x_1^1 := x_r^* + \alpha e_k$, next step.

Step 7. Solve the following problem by local optimization methods starting from x_1^1:

$$
\begin{array}{lll}
(B)_r & \min & F_{r,x_r^*} \\
 & \text{s.t.} & x \in X,
\end{array}
$$

where $F_{r,x_r^*}(x)$ is defined by (3.1).

Step 8. *a*. If X is an open box and if we can find a local minimizer or a stationary point of problem $(B)_r$, let y_r^* be the local minimizer or the stationary point and let $x_r := y_r^*$, goto *Step 5*. Otherwise, goto Step 9;

b. If X is an closed box and let y_r^* be the local minimizer or a stationary point of problem $(B)_r$. Check whether $f(y_r^*) < f(x_r^*)$ and $y_r^* \in S_r$. If $f(y_r^*) < f(x_r^*)$ and $y_r^* \in S_r$, let $x_r := y_r^*$ and goto *Step 5*. Otherwise, goto Step 9;

Step 9. If $k < K$, let $k := k + 1$, goto *Step 6*. Otherwise, stop and x_r^* is the obtained μ-approximate global minimizer of problem (P).

Step 10. Stop and problem (P) has no $\frac{\mu}{2}$-approximate feasible point.

Theorem 4.1. *Let x_r^* be the obtained $\frac{r}{2}$-approximate global minimizer of problem (P). Let x^* be an accumulation point of $\{x_r^*\}(r \to 0)$. Then x^* is a global minimizer of problem (P).*

Proof. Since $\{x_r^*\} \subset X$, then there must exist an accumulation point x^* of $\{x_r^*\}(r \to 0)$. Since x_r^* is the $\frac{r}{2}$-approximate global minimizer of problem (P), then we have that

$$f(x) \geq f(x_r^*) - \frac{r}{2}, \forall x \in S_{\frac{r}{2}}$$

which yields

$$f(x) \geq f(x^*), \forall x \in S.$$

Moreover, since $g_i(x_r^*) \leq \frac{r}{2}, i = 1, \ldots, m, h_j^2(x_r^*) \leq \frac{r^2}{4}, j = 1, \ldots, l$, we have that

$$g_i(x^*) \leq 0, i = 1, \ldots, m, h_j(x^*) = 0, j = 1, \ldots, l.$$

Hence $x^* \in S$. Therefore, x^* is a global minimizer of problem (P). $\square$

5 Numerical examples

The algorithm is coded in Fortran 95, and is successfully used to find the (approximate) global minimizers of the following test problems. In our program, the SQP method is performed on the problem $(P)_r$ to obtain a local minimizer, and the quasi-Newton method is performed on problems $(A)_r$ and $(B)_r$. The numerical results are summarized in the following tables for each example. The symbols used are described as follows:

k:	The numbers of the local minimizers obtained by Algorithm **AFM**,
x_1^0:	The initial point,
x_r:	The r-approximate feasible point obtained by *Step 2* in our algorithm,
$C(x_r)$:	the vector $\begin{pmatrix} h^2(x_r) \\ g(x_r) \end{pmatrix}$,
$C(x_r^*)$:	the vector $\begin{pmatrix} h^2(x_r^*) \\ g(x_r^*) \end{pmatrix}$,
x_r^*:	The local minimizer of problem $(P)_r$ starting from x_r,
$f(x_r)$, $f(x_r^*)$:	The values of objective function f at x_r and x_r^*, respectively.

Example 5.1. *[2]*

$$\begin{aligned}
\min \quad & f(x) = -9x_1 - 15x_2 + 6x_3 + 16x_4 + 10(x_6 + x_7) \\
\text{s.t.} \quad & g_1(x) = -2.5x_1 + 2x_6 + x_5x_8 \le 0, \\
& g_2(x) = -1.5x_2 + 2x_7 + x_5x_9 \le 0, \\
& h_1(x) = -x_3 - x_4 + x_8 + x_9 = 0 \\
& h_2(x) = x_1 - x_6 - x_8 = 0 \\
& h_3(x) = x_2 - x_7 - x_9 = 0 \\
& h_4(x) = x_5x_8 + x_5x_9 - 3x_3 - x_4 = 0 \\
& 0 \le x_1 \le 100 \\
& 0 \le x_2 \le 200 \\
& 0 \le x_3, x_4, x_5, x_6, x_7, x_8, x_9 \le 500
\end{aligned}$$

From [2], we know that $x^* = (0.0, 200.0, 0.0, 100.0, 1.0, 0.0, 100.0, 0.0, 100.0)^T$ is a global minimum of problem Example 5.1 with global optimal value $f^* = -400$. Table 5.1 gives the results obtained by Algorithm **AFM**. From Table 5.1, the obtained approximate global minimizer is

$x^*_\mu := (16.0999999E{-}05, 200.0000, 0.0000000E{+}00, 99.99976, 1.000000, 0.0000000E{+}00, 100.0001, 8.1315163E - 20, 99.99982)^T$ with approximate global optimal value $f^*_\mu := -400.0032$.

Table 5.1: Numerical results for Example 5.1

μ	x_0	x_r obtained by solving problem $(A)_r$	$f(x_r)$	$C(x_r)$
1e-3	$\begin{pmatrix} 43 \\ 148 \\ 248 \\ 358 \\ 445 \\ 446 \\ 446 \\ 258 \\ 159 \end{pmatrix}$	$\begin{pmatrix} 99.96474 \\ 0.0000000 \\ 0.0000000 \\ 0.0000000 \\ 149.4920 \\ 99.96474 \\ 0.0000000 \\ 0.000000 \\ 0.0000000 \end{pmatrix}$	99.96474	$\begin{pmatrix} 0.0000000 \\ 0.0000000 \\ 0.0000000 \\ 0.0000000 \\ -49.98237 \\ 2.3925282E-16 \end{pmatrix}$
k	$\begin{pmatrix} x_r^* \\ f(x_r^*) \end{pmatrix}$	$C(x_r^*)$	$\begin{pmatrix} x_r \\ f(x_r) \end{pmatrix}$	$C(x_r)$
1	$\begin{pmatrix} 0.0000000 \\ 0.2362123 \\ 0.0000000 \\ 0.1181056 \\ 1.000166 \\ 0.0000000 \\ 0.1181055 \\ 0.0000000 \\ 0.1181056 \\ -0.4724404 \end{pmatrix}$	$\begin{pmatrix} 0.0000000E+00 \\ 1.0000912E-18 \\ 1.4748758E-12 \\ 3.8464765E-10 \\ -2.5001139E-09 \\ 1.7787044E-05 \end{pmatrix}$	$\begin{pmatrix} 3.1622779E-02 \\ 7.052086 \\ 2.0319681E-19 \\ 3.366936 \\ 0.9813899 \\ 2.6563243E-18 \\ 3.621908 \\ 0.0000000E+00 \\ 3.398560 \\ -15.97583 \end{pmatrix}$	$\begin{pmatrix} 9.9890167E-07 \\ 9.9890167E-07 \\ 1.0000003E-06 \\ 9.9089220E-07 \\ 9.9925033E-04 \\ 9.9999993E-04 \end{pmatrix}$
2	$\begin{pmatrix} 100.0000 \\ 6.0000005E-03 \\ 50.00400 \\ 3.8370194E-26 \\ 2.999960 \\ 49.99400 \\ 5.0000004E-03 \\ 50.00500 \\ 0.0000000E+00 \\ -100.0760 \end{pmatrix}$	$\begin{pmatrix} 9.9890167E-07 \\ 9.9890167E-07 \\ 1.0000003E-06 \\ 9.9089220E-07 \\ 9.9925033E-04 \\ 9.9999993E-04 \end{pmatrix}$	$\begin{pmatrix} 2.8499999E-04 \\ 64.00041 \\ 3.2399999E-04 \\ 32.00001 \\ 0.9999900 \\ 3.3499999E-04 \\ 32.00001 \\ 3.4200001E-04 \\ 32.00023 \\ -128.0032 \end{pmatrix}$	$\begin{pmatrix} 4.6987672E-09 \\ 7.0896400E-09 \\ 2.8172508E-08 \\ 1.0753374E-06 \\ -8.3003333E-06 \\ -6.8283314E-04 \end{pmatrix}$
3	$\begin{pmatrix} 16.0999999E-05 \\ 200.0000 \\ 0.0000000E+00 \\ 99.99976 \\ 1.000000 \\ 0.0000000E+00 \\ 100.0001 \\ 8.1315163E-20 \\ 99.99982 \\ -400.0032 \end{pmatrix}$	$\begin{pmatrix} 3.7252903E-09 \\ 3.7210000E-09 \\ 3.7252903E-09 \\ 3.7252903E-09 \\ -1.5250000E-04 \\ 6.1035156E-05 \end{pmatrix}$		

Example 5.2. *[2]*

$$\begin{aligned} \min \quad & f(x) = (x_1-1)^2 + (x_1-x_2)^2 + (x_2-x_3)^3 \\ & +(x_3-x_4)^4 + (x_4-x_5)^4 \\ \text{s.t.} \quad & h_1(x) = x_1 + x_2^2 + x_3^3 - 3\sqrt[2]{2.0} - 2 = 0 \\ & h_2(x) = x_2 - x_3^2 + x_4 - 2\sqrt[2]{2.0} + 2 = 0 \\ & h_3(x) = x_1 x_5 - 2 = 0 \\ & -5 \leq x_1, x_2, x_3, x_4, x_5 \leq 5 \end{aligned}$$

From [2], we know that $x^* = (1.1166, 1.2204, 1.5378, 1.9728, 1.7911)^T$ is a global minimum of Example 5.2 with global optimal value $f^* = 0.029290$. Table 5.2 gives the results obtained by Algorithm **AFM**. From Table 5.2, the obtained approximate global minimizer is

$x^*_\mu := (1.116679, 1.220987, 1.537592, 1.971631, 1.790981)^T$ with approximate global optimal value $f^*_\mu := 2.9313827E-02$.

Table 5.2: Numerical results for Example 5.2

μ	x_1^0	x_r obtained by solving problem $(A)_r$	$f(x_r)$	$C(x_r)$
1e-4	$\begin{pmatrix} -3.000000 \\ -3.000000 \\ -3.000000 \\ -3.000000 \\ -3.000000 \end{pmatrix}$	$\begin{pmatrix} -0.9015743 \\ 2.277812 \\ 1.250575 \\ 0.1145480 \\ -2.218336 \end{pmatrix}$	46.09313	$\begin{pmatrix} 5.4752699E-11 \\ 6.6264987E-14 \\ 9.4995713E-12 \end{pmatrix}$
k	$\begin{pmatrix} x_r^* \\ f(x_r^*) \end{pmatrix}$	$C(x_r^*)$	$\begin{pmatrix} x_r \\ f(x_r) \end{pmatrix}$	$C(x_r)$
1	$\begin{pmatrix} 2.410253 \\ 1.195114 \\ -0.1535293 \\ -1.570289 \\ 27.87190 \end{pmatrix}$	$\begin{pmatrix} 5.4752699E-11 \\ 6.6264987E-14 \\ 9.4995713E-12 \end{pmatrix}$	$\begin{pmatrix} 1.019089 \\ 1.237455 \\ 1.545632 \\ 1.979924 \\ 1.962466 \\ 5.4353226E-02 \end{pmatrix}$	$\begin{pmatrix} 5.1341001E-08 \\ 7.2976991E-10 \\ 5.3104343E-09 \end{pmatrix}$
2	$\begin{pmatrix} 1.115719 \\ 1.214690 \\ 1.539896 \\ 1.985005 \\ 1.791729 \\ 2.9440390E-02 \end{pmatrix}$	$\begin{pmatrix} 6.0969199E-09 \\ 1.6063050E-10 \\ 8.7312367E-07 \end{pmatrix}$	$\begin{pmatrix} 1.117583 \\ 1.221715 \\ 1.537239 \\ 1.969797 \\ 1.789373 \\ 2.9325772E-02 \end{pmatrix}$	$\begin{pmatrix} 3.4904694E-08 \\ 3.4979766E-10 \\ 5.1888584E-08 \end{pmatrix}$
3	$\begin{pmatrix} 1.116679 \\ 1.220987 \\ 1.537592 \\ 1.971631 \\ 1.790981 \\ 2.9313827E-02 \end{pmatrix}$	$\begin{pmatrix} 8.0145592E-11 \\ 1.5321868E-12 \\ 2.5110729E-09 \end{pmatrix}$		

6 Conclusion

This paper proposed two methods. One is a method to obtain an approximate feasible point for general constrained global optimization problems (with both inequality and equality constraints). The other one is an auxiliary function method to obtain a global minimizer or an approximate global minimizer with a required precision by locally solving some unconstrained programming problems for the general constrained global optimization problems. Some numerical examples were also reported to demonstrate the efficiency of these two methods.

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