JOURNAL OF INDUSTRIAL AND
MANAGEMENT OPTIMIZATION
Volume **7**, Number **1**, February **2011**

doi:10.3934/jimo.2011.7.67

pp. **67–78**

GLOBAL OPTIMALITY CONDITIONS FOR SOME CLASSES OF POLYNOMIAL INTEGER PROGRAMMING PROBLEMS

Jing Quan

Department of Mathematics, College of Sciences
Shanghai University, Shanghai 200444, China

Zhiyou Wu

School of Information Technology and Mathematical Sciences
University of Ballarat, VIC 3353, Australia

Guoquan Li

Department of Mathematics, College of Sciences
Shanghai University, Shanghai 200444, China

(Communicated by David Y. Gao)

Abstract. In this paper, some verifiable necessary global optimality conditions and sufficient global optimality conditions for some classes of polynomial integer programming problems are established. The relationships between these necessary global optimality conditions and these sufficient global optimality conditions are also discussed. The main theoretical tool for establishing these optimality conditions is abstract convexity.

1. **Introduction.** Consider the following classes of polynomial integer programming problems:

$$(POP)_I \qquad \min \quad f(x) = \sum_{i=1}^{n}\sum_{k=3}^{m} b_i^{(k)} x_i^k + \frac{1}{2}x^T A x + a^T x$$

$$s.t. \quad x \in U_I = \{(x_1, \dots, x_n)^T \mid x_i \in \{0, 1, \dots, J\}, i = 1, 2, \cdots, n\}.$$

where $a \in R^n, A \in S^n$ and S^n is the set of all symmetric $n \times n$ matrices, $k \geq 3$ is a positive integer, J is an positive integer.

Many combinatorial optimization problems can be modeled as polynomial programming problems with a polynomial scalar objective function in integer variables of the form $(POP)_I$. Finding the global optimal solution and how to characterize it for general nonlinear integer programming problems are very difficult tasks except for some special problems.

For polynomial programming problem whose objective and constraints are given by multivariate polynomials, one popular method for obtaining the global optimal solutions is to use the SDP-relaxations skill, see [9, 15, 12]. Some analytical approach methods are also applied, such as, [7] and [6]. Horst and Tuy [7] proposed outer approximation techniques for solving a polynomial programming problem with

2000 *Mathematics Subject Classification.* Primary: 90C10, 90C46, 90C26; Secondary: 26C05.
Key words and phrases. Global optimality conditions, polynomial integer programming problem, abstract convexity.

Lipschitzian objective function and constraint functions. Hansen, Jaumard and Lu [6] developed interval analysis based sufficient conditions for convergence, and provided ways to eliminate variables and reduce the ranges of variables. In addition, Sherali and Tuncbilek [14], Adams and Sherali [13] and [5] derived some reformulation linearization technique (RLT) to solve some kinds of polynomial programming problems.

Recently much attention has been focused on characterizing global minimizers of minimization problems (see for example [2, 1, 10] and references therein.) Also some global optimality conditions for some special kinds of nonconvex optimization problems have been studied by many researchers. When all $b_i^{(k)} = 0$, problem $(POP)_I$ is quadratic integer programming problem, its global optimality conditions have just been studied in [4, 16]. When all $b_i^{(k)} = 0$ and $J = 1$, problem $(POP)_I$ is called quadratic $\{0, 1\}$ programming problem, its optimality conditions have been studied in [3, 8], etc.

The purpose of this paper is to establish some verifiable global optimality conditions for polynomial integer programming problem $(POP)_I$ by using the abstract convexity as a tool. First we investigate some sufficient global optimality conditions which ensure a feasible point is a global minimizer of problem $(POP)_I$. Then some necessary global optimality conditions for problem $(POP)_I$ are presented. The relationships between these necessary global optimality conditions and sufficient global optimality conditions for problem $(POP)_I$ are also be discussed in this paper. Some examples to illustrate the optimality conditions given in this paper are verifiable and valuable.

The lay-out of the paper is as follows. Section 2 presents preliminaries from abstract convexity. Section 3 provides some sufficient global optimality conditions and some necessary global optimality conditions, as well as their relationships for problem $(POP)_I$. Several numerical examples are given in Section 4 to illustrate how to use the global optimality conditions to check a given point is or is not a global minimizer.

2. **Preliminary.**

We begin this section by presenting basic definitions and preliminary results that will be used throughout the paper. The real line is denoted by R and the n-dimensional Euclidean space is denoted by R^n. For vectors $x, y \in R^n$, $x \geq y$ means that $x_i \geq y_i$, for $i = 1, \ldots, n$. The notation $A \succeq B$ means $A - B$ is a positive semidefinite and $A \preceq 0$ means $-A \succeq 0$. A diagonal matrix with diagonal elements $\alpha_1, \ldots, \alpha_n$ is denoted by $\mathrm{diag}(\alpha_1, \ldots, \alpha_n)$.. Let L be a set of real-valued functions defined on R^n.

$L-$**Subdifferentials** (see [11]). Let $f : R^n \to R$ and $x_0 \in \mathrm{dom}\, f$. An element $l \in L$ is called an L-subgradient of f at a point $x_0 \in R^n$ if

$$f(x) \geq f(x_0) + l(x) - l(x_0), \ \forall x \in R^n.$$

The set $\partial_L f(x)$ of all L-subgradients of f at x_0 is referred to as $L-$subdifferential of f at x_0.

Note that here L can be any set of real-valued functions defined on R^n. If L is the set of all linear functions defined on R^n, then for any proper lower semicontinuous convex function f defined on R^n, $\partial_L f(x) = \partial f(x)$, where $\partial f(x)$ is the convex subdifferential in the sense of convex analysis.

L-normal Cones. For a set $D \subset R^n$ and $x_0 \in D$, the *normal cone* of D at x_0 with respect to L, called as L-normal cone, is given by

$$N_{L,D}(x_0) := \{l \in L : l(x) - l(x_0) \le 0 \text{ for each } x \in D\}.$$

Observe that if L is the set of all linear functions defined on R^n, then $N_{L,D}(x_0)$ coincides with the normal cone $N_D(x_0)$ in the sense of convex analysis.

Lemma 2.1. *Let L be a set of real-valued functions defined on R^n such that $-l \in L$ for each $l \in L$. Let $\bar{x} \in U_I$. If*

$$-\partial_L f(\bar{x}) \cap N_{L,U_I}(\bar{x}) \neq \emptyset. \tag{1}$$

Then $\bar{x}$ is a global minimizer of $(POP)_I$.

Proof. Let $l \in N_{L,U_I}(\bar{x})$ be a vector such that $-l \in \partial_L f(\bar{x})$. It follows from the definition of $\partial_L f(\bar{x})$, that

$$f(x) - f(\bar{x}) \ge -l(x) + l(\bar{x}), \forall x \in X. \tag{2}$$

The inclusion $l \in N_{L,U_I}(\bar{x})$, implies that

$$l(x) - l(\bar{x}) \le 0, \forall x \in U_I. \tag{3}$$

(2) and (3) imply $f(x) - f(\bar{x}) \ge 0$ for any $x \in U_I$, i.e., $\bar{x}$ is a global minimizer of problem $(POP)_I$. □

3. Global optimality conditions for problem $(POP)_I$.

In the rest of the paper, we take L as the set of the following special polynomial functions.

$$L := \{\sum_{i=1}^{n}\sum_{k=3}^{m} b_i^{(k)} x_i^k + \frac{1}{2}x^T Q x + \beta^T x \mid Q = \text{ diag } (q), q, \beta \in R^n\}. \tag{4}$$

To obtain the sufficient global optimality conditions of problem $(POP)_I$, we first need to calculate the L-subdifferential $\partial_L f(\bar{x})$ and the L-normal cone $N_{L,U_I}(\bar{x})$ at a feasible point $\bar{x} \in U_I$.

Proposition 1. *Let $f(x) := \sum_{i=1}^{n}\sum_{k=3}^{m} b_i^{(k)} x_i^k + \frac{1}{2}x^T A x + x^T a$ and let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T \in R^n$. Then,*

$$\partial_L f(\bar{x}) = \left\{ \sum_{i=1}^{n}\sum_{k=3}^{m} b_i^{(k)} x_i^k + \frac{1}{2}x^T Q x + x^T \beta \;\middle|\; \begin{array}{c} A - Q \succeq 0, Q = \text{diag}(q_1, \ldots, q_n), \\ q_i \in R, \beta = a + (A - Q)\bar{x}, \beta \in R^n \end{array} \right\}.$$

Proof. By definition of L-subdifferential, we have that $l_0 \in \partial_L f(\bar{x})$ if and only if

$$l_0(x) - l_0(\bar{x}) \le f(x) - f(\bar{x}), \ \forall x \in \mathbb{R}^n. \tag{5}$$

Let $l_0(x) = \sum_{i=1}^{n}\sum_{k=3}^{m} b_i^{(k)} x_i^k + \frac{1}{2}x^T Q x + \beta^T x$ and let $\varphi(x) = f(x) - l_0(x)$, then $\varphi(x) = \frac{1}{2}x^T(A - Q)x + (a - \beta)^T x$.. By (5), for each $x \in R^n$, we can get

$$\varphi(x) = \frac{1}{2}x^T(A - Q)x + (a - \beta)^T x \ge f(\bar{x}) - l_0(\bar{x}).$$

Thus φ is bounded below and attains its minimum at $\bar{x}$. We have $A - Q \succeq 0$ and φ is a convex function on R^n. So φ attains its minimum at $\bar{x}$ if and only if $\nabla\varphi(\bar{x}) = 0$. This gives us that

$$(A - Q)\bar{x} + (a - \beta) = 0 \ \text{ and } \ \beta = a + (A - Q)\bar{x}.$$

□

For $\bar{x} \in U_I$, let

$$\begin{aligned} \alpha_{\bar{x}_i} : &= \min\{\frac{(a + A\bar{x})_i + \sum_{k=3}^{m} b_i^{(k)}(x_i^{k-1} + x_i^{k-2}\bar{x}_i + \ldots + x_i\bar{x}_i^{k-2} + \bar{x}_i^{k-1})}{(x_i - \bar{x}_i)}, \\ & x_i \in \{0, 1, \ldots, J\}, x_i \neq \bar{x}_i\} \quad (6) \\ \alpha_{\bar{x}} : &= (\alpha_{\bar{x}_1}, \ldots, \alpha_{\bar{x}_n})^T. \quad (7) \end{aligned}$$

Remark 1. We should note that for a given point $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T \in U_I$, it is very easy to calculate $\alpha_{\bar{x}_i}, \forall i = 1, \ldots, n$. In fact, for a given $i \in \{1, \ldots, n\}$, to calculate $\alpha_{\bar{x}_i}$, we just need first to calculate the following values:

$$g(x_i) := \frac{(a + A\bar{x})_i + \sum_{k=3}^{m} b_i^{(k)}(x_i^{k-1} + x_i^{k-2}\bar{x}_i + \ldots + x_i\bar{x}_i^{k-2} + \bar{x}_i^{k-1})}{(x_i - \bar{x}_i)},$$

where $x_i \in \{0, 1, \ldots, J\} \setminus \{\bar{x}_i\}$. So here only J values need to be calculate and $\alpha_{\bar{x}_i} = \min\{g(x_i), x_i \in \{0, 1, \ldots, J\} \setminus \{\bar{x}_i\}\}$.

Proposition 2. *Let $\bar{x} \in U_I$ and let $\beta = a + (A - Q)\bar{x}$. Then $-\sum_{i=1}^{n}\sum_{k=3}^{m} b_i^{(k)} x_i^k - \frac{1}{2}x^T Q x - \beta^T x \in N_{L,U_I}(\bar{x})$ if and only if*

$$-\operatorname{diag}(\alpha_{\bar{x}}) \preceq \frac{Q}{2}. \quad (8)$$

Proof. By definition, $l = -\sum_{i=1}^{n}\sum_{k=3}^{m} b_i^{(k)} x_i^k - \frac{1}{2}x^T Q x - \beta^T x \in N_{L,U_I}(\bar{x})$ if and only if

$$\sum_{i=1}^{n}\sum_{k=3}^{m} b_i^{(k)}(x_i^k - \bar{x}_i^k) + \sum_{i=1}^{n}[\frac{1}{2}q_i(x_i - \bar{x}_i)^2 + (\beta_i + q_i\bar{x}_i)(x_i - \bar{x}_i)] \geq 0, \forall x \in U_I. \quad (9)$$

By $\beta = a + (A - Q)\bar{x}$, (9) is equivalent to

$$\sum_{i=1}^{n}\sum_{k=3}^{m} b_i^{(k)}(x_i^k - \bar{x}_i^k) + \sum_{i=1}^{n}[\frac{1}{2}q_i(x_i - \bar{x}_i)^2 + (a + A\bar{x})_i(x_i - \bar{x}_i)] \geq 0, \forall x \in U_I. \quad (10)$$

Thus, $l \in N_{L,U_I}(\bar{x})$ if and only if for any $i = 1, \ldots, n$,

$$\sum_{k=3}^{m} b_i^{(k)}(x_i^k - \bar{x}_i^k) + \frac{1}{2}q_i(x_i - \bar{x}_i)^2 + (a + A\bar{x})_i(x_i - \bar{x}_i) \geq 0, \forall x_i \in \{0, 1, \ldots, J\}. \quad (11)$$

In fact, if there exist a $i_0 \in \{1, \ldots, n\}$ and a $y_{i_0} \in \{0, 1, \ldots, J\}$ such that

$$\sum_{k=3}^{m} b_{i_0}^{(k)}(y_{i_0}^k - \bar{x}_{i_0}^k) + \frac{1}{2}q_{i_0}(y_{i_0} - \bar{x}_{i_0})^2 + (a + A\bar{x})_{i_0}(y_{i_0} - \bar{x}_{i_0}) < 0.$$

We let $x_{i_0} = y_{i_0}$ and $x_i = \bar{x}_i, i = 1, \ldots, n, i \neq i_0$, then $x = (x_1, \ldots, x_n)^T \in U_I$ and we have that

$$\begin{aligned} & \sum_{i=1}^{n}\sum_{k=3}^{m} b_i^{(k)}(x_i^k - \bar{x}_i^k) + \sum_{i=1}^{n}[\frac{1}{2}q_i(x_i - \bar{x}_i)^2 + (a + A\bar{x})_i(x_i - \bar{x}_i)] \\ = & \sum_{k=3}^{m} b_{i_0}^{(k)}(y_{i_0}^k - \bar{x}_{i_0}^k) + \frac{1}{2}q_{i_0}(y_{i_0} - \bar{x}_{i_0})^2 + (a + A\bar{x})_{i_0}(y_{i_0} - \bar{x}_{i_0}) \ < \ 0, \end{aligned}$$

which contradicts (10).

Moreover, by $x_i^k - \bar{x}_i^k = (x_i - \bar{x}_i)(x_i^{k-1} + x_i^{k-2}\bar{x}_i + \ldots + x_i\bar{x}_i^{k-2} + \bar{x}_i^{k-1})$, we know that (11) is equivalent to

$$\begin{aligned} & \sum_{k=3}^{m} b_i^{(k)}(x_i - \bar{x}_i)(x_i^{k-1} + x_i^{k-2}\bar{x}_i + \ldots + x_i\bar{x}_i^{k-2} + \bar{x}_i^{k-1}) + \frac{1}{2}q_i(x_i - \bar{x}_i)^2 \\ + \quad & (a + A\bar{x})_i(x_i - \bar{x}_i) \geq 0, \text{ for any } x_i \in \{0, 1, , \ldots, J\}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} & -\frac{(a + A\bar{x})_i + \sum_{k=3}^{m} b_i^{(k)}(x_i^{k-1} + x_i^{k-2}\bar{x}_i + \ldots + x_i\bar{x}_i^{k-2} + \bar{x}_i^{k-1})}{(x_i - \bar{x}_i)} \leq \frac{1}{2}q_i, \\ & \text{for any } x_i \in \{0, 1, , \ldots, J\}, x_i \neq \bar{x}_i. \end{aligned} \tag{12}$$

Hence (12) is equivalent to

$$-\alpha_{\bar{x}_i} \leq \frac{q_i}{2}, \text{ for any } i = 1, \ldots, n,$$

i.e., (8) holds. □

Theorem 3.1 (Sufficient Global Optimality Condition for $(POP)_I$). *Let* $\bar{x} \in U_I, J \geq 1$. *If*

$$[SC1] \qquad -\operatorname{diag}(\alpha_{\bar{x}}) \preceq \frac{1}{2}A,$$

then $\bar{x}$ *is a global minimizer of problem* $(POP)_I$.

Proof. By Lemma 2.1, we know that if $\partial_L f(\bar{x}) \cap (-N_{L,S_I}(\bar{x})) \neq \emptyset$, then $\bar{x}$ is a global minimizer of problem $(POP)_I$. In the following, we can prove that $\partial_L f(\bar{x}) \cap (-N_{L,S_I}(\bar{x})) \neq \emptyset$ if and only if $[SC1]$ holds.

By Proposition 1, we know that $l = \sum_{i=1}^{n}\sum_{k=3}^{m} b_i^{(k)} x_i^k + \frac{1}{2}x^TQx + \beta^T x \in \partial_L f(\bar{x})$ in and only if $\beta = a + (A - Q)\bar{x}$ and $Q \preceq A$. By Proposition 2, we know that if $\beta = a + (A - Q)\bar{x}$, then $-l = -\sum_{i=1}^{n}\sum_{k=3}^{m} b_i^{(k)} x_i^k - \frac{1}{2}x^TQx - \beta^T x \in N_{L,U_I}(\bar{x})$ if and only if

$$-\operatorname{diag}(\alpha_{\bar{x}}) \preceq \frac{Q}{2}.$$

Hence, $l \in \partial_L f(\bar{x}) \cap (-N_{L,S_I}(\bar{x}))$ implies that condition $[SC1]$ holds.

Conversely, if $[SC1]$ holds, then take $Q = -2\operatorname{diag}(\alpha_{\bar{x}})$, we have that

$$Q \preceq A \text{ and } -\operatorname{diag}(\alpha_{\bar{x}}) = \frac{Q}{2}.$$

Take $\beta = a + (A - Q)\bar{x}$, then we have that $l \in \partial_L f(\bar{x}) \cap (-N_{L,S_I}(\bar{x}))$. □

Let

$$\begin{aligned} \tilde{\bar{x}}_i : &= \begin{cases} -1, & \text{if } \bar{x}_i = 0 \\ 1, & \text{if } \bar{x}_i = J \\ \operatorname{sign}(a + A\bar{x})_i, & \text{if } 0 < \bar{x}_i < J \end{cases}, \\ \tilde{\bar{X}} &= \operatorname{diag}(\tilde{\bar{\mathrm{x}}}_1, \ldots, \tilde{\bar{\mathrm{x}}}_\mathrm{n}), \end{aligned}$$

where $\operatorname{sign}(a + A\bar{x})_i = \begin{cases} -1, & (a + A\bar{x})_i < 0 \\ 0, & (a + A\bar{x})_i = 0 \\ 1, & (a + A\bar{x})_i > 0 \end{cases}$, and let

$$\hat{\bar{x}}_i : = \max\{\tilde{\bar{x}}_i(a + A\bar{x})_i, \frac{\tilde{\bar{x}}_i(a + A\bar{x})_i}{J}\}, i = 1, \ldots, n \tag{13}$$

$$\hat{\bar{x}} : = (\hat{\bar{x}}_1, \ldots, \hat{\bar{x}}_n)^T. \tag{14}$$

Corollary 1. *Let* $\bar{x} \in U_I$*, if* $b_i^{(k)} = 0, i = 1, \ldots, n, k = 3, \ldots, m$ *and if*

$$[SC2] \qquad \operatorname{diag}(\widehat{\bar{x}}) \preceq \frac{1}{2}A,$$

then $\bar{x}$ *is a global minimizer of problem* $(POP)_I$*.*

Proof. We can easily verify that if $b_i^k = 0, i = 1, \ldots, n, k = 3, \ldots, m$, then $\alpha_{\bar{x}} = -\widehat{\bar{x}}$. Hence, if $[SC2]$ holds, then $\bar{x}$ is a global minimizer of problem $(POP)_I$. □

Remark 2. Condition $[SC2]$ gives a verifiable sufficient global optimality condition for integer quadratic programming problems. Note that for a give point $\bar{x}$, it is very easy to verify whether condition $[SC2]$ holds since here we just need to calculate two values for any $i = 1, \ldots, n$. Moreover, condition $[SC2]$ extends the results given by Theorem 5 in reference [4], where a sufficient condition for integer quadratic programming problems is given as follows when A is positive semidefinite matrix:

$$[SC2]' \qquad \begin{cases} -(a + A\bar{x})_i \leq 0, & \bar{x}_i = 0 \\ (a + A\bar{x})_i \leq 0, & \bar{x}_i = J \\ (a + A\bar{x})_i = 0, & 0 < \bar{x}_i < J \end{cases}.$$

We can easily verify that if A is positive semidefinite matrix, $[SC2]'$ implies condition $[SC2]$.

Corollary 2. *Let* $\bar{x} \in U_I$*, if* $J = 1$ *and if*

$$[SC3] \qquad \operatorname{diag}\Big(\widetilde{\bar{X}}(a + A\bar{x} + \sum_{k=3}^{m} b^{(k)})\Big) \preceq \frac{A}{2},$$

then $\bar{x}$ *is a global minimizer of problem* $(POP)_I$*, where* $b^{(k)} = (b_1^{(k)}, \ldots, b_n^{(k)})^T$*.*

Proof. We can easily verify that if $J = 1$, then $-\alpha_{\bar{x}_i} = \widetilde{\bar{x}}_i[(a + A\bar{x})_i + \sum_{k=3}^{m} b_i^{(k)}]$. Hence, if $J = 1$ and condition $[SC3]$ holds, then $\bar{x}$ is a global minimizer of problem $(POP)_I$. □

Corollary 3. *Let* $\bar{x} \in U_I$*, if* $b_i^{(k)} = 0, i = 1, \ldots, n, k = 3, \ldots, m,$ $J = 1$ *and if*

$$[SC4] \qquad \operatorname{diag}\Big(\widetilde{\bar{X}}(a + A\bar{x})\Big) \preceq \frac{A}{2},$$

then $\bar{x}$ *is a global minimizer of problem* $(POP)_I$*.*

Proof. It can be obtained from Corollary 2. □

Note that the condition $[SC4]$ is just the condition given in reference [8]. Hence sufficient global optimality condition $[SC1]$ extends the results given in reference [4, 8].

In the following, we will discuss the necessary conditions for problem $(POP)_I$.

Theorem 3.2. *Let* $\bar{x} \in S_I$*,* $e := (1, \ldots, 1)^T$ *and let* $\operatorname{diag}(A) = \operatorname{diag}(a_{11}, \ldots, a_{nn})$*. If* $\bar{x}$ *is a global minimizer of* $(POP)_I$*, then the following conditions hold:*

$$[NC1] \qquad -\operatorname{diag}(\alpha_{\bar{x}}) \preceq \frac{1}{2}\operatorname{diag}(A).$$

Proof. Let $\bar{x}$ be a global minimizer of problem $(POP)_I$. Then

$$\sum_{i=1}^{n}\sum_{k=3}^{m} b_i^{(k)}(x_i^k - \bar{x}_i^k) + \frac{1}{2}x^T Ax + a^T x - \frac{1}{2}\bar{x}^T A\bar{x} + a^T\bar{x} \geq 0, \forall x \in U_I$$

$$\Leftrightarrow \sum_{i=1}^{n}\sum_{k=3}^{m} b_i^{(k)}(x_i^k - \bar{x}_i^k) + \frac{1}{2}(x-\bar{x})^T A(x-\bar{x}) + (x-\bar{x})^T(a + A\bar{x}) \geq 0, \forall x \in U_I.$$

For any $i = 1, \dots, J$, we let $x := (\bar{x}_1, \dots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \dots, \bar{x}_n)^T$, where $x_i \in \{0, 1, \dots, J\}$. Then

$$\sum_{i=1}^{n}\sum_{k=3}^{m} b_i^{(k)}(x_i^k - \bar{x}_i^k) + \frac{1}{2}x^T Ax + a^T x - \frac{1}{2}\bar{x}^T A\bar{x} + a^T\bar{x} \geq 0$$

$$\Rightarrow \sum_{k=3}^{m} b_i^{(k)}(x_i^k - \bar{x}_i^k) + \frac{1}{2}(x_i - \bar{x}_i)^2 a_{ii} + (x_i - \bar{x}_i)(a + A\bar{x})_i \geq 0, \forall x_i \in \{0, 1, \dots, J\}$$

$$\Leftrightarrow -\frac{\sum_{k=3}^{m} b_i^{(k)}(x_i^{k-1} + x_i^{k-2}\bar{x}_i + \ldots + x_i\bar{x}_i^{k-2} + \bar{x}_i^{k-1}) + (a + A\bar{x})_i}{x_i - \bar{x}_i} \leq \frac{a_{ii}}{2},$$
$$\forall x_i \in \{0, 1, \dots, J\}, x_i \neq \bar{x}_i$$

$$\Leftrightarrow -\alpha_{\bar{x}_i} \leq \frac{a_{ii}}{2}.$$

Hence, if $\bar{x}$ is a global minimizer of $(POP)_I$, then condition $[NC1]$ holds. □

Corollary 4. *Let $\bar{x} \in U_I$, if $b_i^{(k)} = 0, i = 1, \dots, n, k = 3, \dots, m$ and if $\bar{x}$ is a global minimizer of problem $(POP)_I$, then*

$$[NC2] \qquad \mathrm{diag}(\widehat{\bar{x}}) \preceq \frac{1}{2}\mathrm{diag}(A),$$

where $\widehat{\bar{x}}$ is defined by (14).

Proof. We can easily verify that if $b_i^{(k)} = 0, i = 1, \dots, n, k = 3, \dots, m$, then $-\alpha_{\bar{x}} = \widehat{\bar{x}}$. Hence, if $\bar{x}$ is a global minimizer of problem $(POP)_I$, then $[NC2]$ holds. □

Note that here condition $[NC2]$ gives a necessary global optimality condition for integer quadratic programming problem.

Corollary 5. *Let $\bar{x} \in U_I$, if $J = 1$ and if $\bar{x}$ is a global minimizer of problem $(POP)_I$, then*

$$[NC3] \qquad \mathrm{diag}(\widetilde{\bar{X}}(a + A\bar{x} + \sum_{k=3}^{m} b^{(k)})) \preceq \frac{1}{2}\mathrm{diag}(A),$$

where $b^{(k)} = (b_1^{(k)}, \dots, b_n^{(k)})^T$.

Proof. We can easily verify that if $J = 1$, then $-\alpha_{\bar{x}_i} = \widetilde{\bar{x}}_i(a + A\bar{x})_i + \sum_{k=3}^{m} b_i^{(k)}$. Hence, if $J = 1$ and if $\bar{x}$ is a global minimizer of problem $(POP)_I$, then condition $[NC3]$ holds. □

Corollary 6. *Let $\bar{x} \in U_I$, if $b_i^{(k)} = 0, i = 1, \dots, n, k = 3, \dots, m$, $J = 1$ and if $\bar{x}$ is a global minimizer of problem $(POP)_I$, then*

$$[NC4] \qquad \mathrm{diag}\Big(\widetilde{\bar{X}}(a + A\bar{x})\Big) \preceq \frac{A}{2}.$$

Proof. It can be obtained from Corollary 5. □

Note that the condition $[NC4]$ is just the global optimality condition given in reference [3] and [8]. Hence, necessary global optimality condition $[NC1]$ extends the results in references [3] and [8].

Here we will discuss the relationships between the sufficient global optimality condition $[SC1]$ and the necessary global optimality condition $[NC1]$. Obviously, we have that

$$[SC1] \Rightarrow [NC1].$$

But generally, $[NC1]$ can not imply $[SC1]$. If A is a diagonal matrix, then

$$[SC1] \Leftrightarrow [NC1].$$

The following example illustrate that $[NC1]$ can not imply $[SC1]$ when A is not a diagonal matrix.

Example 1. Consider the problem

$$\begin{array}{lll} (EP1) & \min & f(x) := 2x_1^3 - 3x_2^3 + x_3^3 + x_1^4 + 2x_2^4 - 3x_3^4 + \dfrac{1}{2}x^T Ax + a^T x \\ & s.t. \quad x & \in \{0,1,2\}^3 \end{array}$$

Here $A = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & -1 \end{pmatrix}$, $a = (1,-4,1)^T$ and $J = 2$. Let $\bar{x} = (0,1,2)^T$, then $A\bar{x} = (0,6,0)^T$, $a + A\bar{x} = (1,2,1)^T$, $\alpha_{\bar{x}} = (4,3,9.5)^T$, Hence, $\frac{1}{2}A + \mathrm{diag}(\alpha_{\bar{x}}) = \begin{pmatrix} 5.5 & 1 & -0.5 \\ 1 & 4 & 1 \\ -0.5 & 1 & 9 \end{pmatrix} \succeq 0$. So condition $[SC1]$ is satisfied at $\bar{x}$ and $\bar{x}$ is a global minimizer of $(EP1)$.

Let's consider another feasible point $\bar{y} = (0,0,2)^T$. We can verify that the necessary condition $(NC1)$ is satisfied at $\bar{y}$, but the sufficient condition $(SC1)$ does not hold at $\bar{y}$. In fact, we have that $a + A\bar{y} = (-1,0,-1)^T$, $\alpha_{\bar{y}} = (2,-1,10.5)^T$ and $\mathrm{diag}(A) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Hence, $-\alpha_{\bar{y}} \le \frac{1}{2}\mathrm{diag}(A)e$, i.e., $(NC1)$ is satisfied at $\bar{y}$. But $\frac{1}{2}A + \mathrm{diag}(\alpha_{\bar{y}}) = \begin{pmatrix} 3.5 & 1 & -0.5 \\ 1 & 0 & 1 \\ -0.5 & 1 & 10 \end{pmatrix}$ is not positive semidefinite. Hence $(SC1)$ does not hold at $\bar{y}$.

Here we can notice that the point $\bar{y}$ is also a global minimizer since $f(\bar{x}) = f(\bar{y})$, which means that even at the global minimizer , the sufficient global optimality conditions maybe also do not hold.

4. **Numerical examples.** In this section we give some examples to illustrate how to use the global optimality conditions to check a given point is or is not a global minimizer.

Example 2. Consider the problem

$$\begin{array}{lll} (EP2) & \min & f(x) := 3x_1^3 - x_1^4 - 4x_2^4 + 3x_2^5 + \dfrac{1}{2}x^T Ax + a^T x \\ & s.t. \quad x & \in \{0,1,2,3,4,5,6\}^2. \end{array}$$

Here $A = \begin{pmatrix} 1 & -2 \\ -2 & -4 \end{pmatrix}$, $a = (2, -1)^T$ and $J = 6$. Let $\bar{x} = (6, 1)^T$, then $A\bar{x} = (4, -16)^T$, $a + A\bar{x} = (6, -17)^T$. By (6), we have

$$\alpha_{\bar{x}_1} = \min\{\frac{6}{x_1 - 6} + \frac{3x_1^3 - x_1^4 - (3 \times 6^3 - 6^4)}{(x_1 - 6)^2} \mid x_1 \in \{0, 1, 2, 3, 4, 5\}\};$$

$$\alpha_{\bar{x}_2} = \min\{\frac{-17}{x_2 - 1} + \frac{-4x_2^4 + 3x_2^5 - (-4 \times 1^4 + 3 \times 1^5)}{(x_2 - 1)^2} \mid x_2 \in \{0, 2, 3, 4, 5, 6\}\}.$$

We can easily get $\alpha_{\bar{x}_1} = 17$, $\alpha_{\bar{x}_2} = 16$. Then we have $\alpha_{\bar{x}} = (17, 16)^T$. Hence $\frac{1}{2}A + \mathrm{diag}(\alpha_{\bar{x}}) = \begin{pmatrix} 17.5 & -1 \\ -1 & 14 \end{pmatrix} \succeq 0$. So condition $[SC1]$ is satisfied at $\bar{x}$ and $\bar{x}$ is a global minimizer of $(EP2)$ with $f(\bar{x}) = -634$.

We consider another point $\bar{y} = (5, 1)^T$, we have $A\bar{y} = (3, -14)^T$, $a + A\bar{y} = (5, -15)^T$. By (6), we have

$$\alpha_{\bar{y}_1} = \min\{\frac{5}{y_1 - 5} + \frac{3y_1^3 - y_1^4 - (3 \times 5^3 - 5^4)}{(y_1 - 5)^2} \mid y_1 \in \{0, 1, 2, 3, 4, 6\}\},$$

$$\alpha_{\bar{y}_2} = \min\{\frac{-15}{y_2 - 1} + \frac{-4y_2^4 + 3y_2^5 - (-4 \times 1^4 + 3 \times 1^5)}{(y_2 - 1)^2} \mid y_2 \in \{0, 2, 3, 4, 5, 6\}\}.$$

We can get $\alpha_{\bar{y}_1} = -393$; $\alpha_{\bar{y}_2} = 16$. Then we have $\alpha_{\bar{y}} = (-393, 16)^T$ and $\mathrm{diag}(A) = \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}$. Hence $-\alpha_{\bar{y}} \le \frac{1}{2}\mathrm{diag}(A)e$ can't hold, i.e., $(NC1)$ is not satisfied at $\bar{y}$ and $\bar{y}$ is not the global minimizer of problem $(EP2)$.

Example 3. Consider the problem

$$(EP3) \quad \min \quad f(x) := 6x_1^3 - x_1^4 + x_2^4 - x_2^5 + 3x_3^3 - x_3^4 - 4x_4^3 + x_4^5 + \frac{1}{2}x^TAx + a^Tx$$
$$s.t. \quad x \in \{0, 1, 2, 3, 4, 5, 6\}^4.$$

Here $A = \begin{pmatrix} 1 & -2 & 3 & -4 \\ -2 & 5 & -6 & 7 \\ 3 & -6 & 8 & -1 \\ -4 & 7 & -1 & 9 \end{pmatrix}$, $a = (2, -1, 3, -4)^T$ and $J = 6$. Let $\bar{x} = (0, 6, 6, 0)^T$, then $A\bar{x} = (6, -6, 12, 36)^T$, $a + A\bar{x} = (8, -7, 15, 32)^T$. By (6), we have

$$\alpha_{\bar{x}_1} = \min\{\frac{8}{x_1 - 0} + \frac{6x_1^3 - x_1^4 - (6 \times 0^3 - 1 \times 0^4)}{(x_1 - 0)^2} \mid x_1 \in \{1, 2, 3, 4, 5, 6\}\},$$

$$\alpha_{\bar{x}_2} = \min\{\frac{-7}{x_2 - 6} + \frac{x_2^4 - x_2^5 - (1 \times 6^4 - 1 \times 6^5)}{(x_2 - 6)^2} \mid x_2 \in \{0, 1, 2, 3, 4, 5\}\},$$

$$\alpha_{\bar{x}_3} = \min\{\frac{15}{x_3 - 6} + \frac{3x_3^3 - x_3^4 - (3 \times 6^3 - 1 \times 6^4)}{(x_3 - 6)^2} \mid x_3 \in \{0, 1, 2, 3, 4, 5\}\},$$

$$\alpha_{\bar{x}_4} = \min\{\frac{32}{x_4 - 0} + \frac{-4x_4^3 + x_4^5 - (-4 \times 0^4 + 1 \times 0^5)}{(x_4 - 0)^2} \mid x_4 \in \{1, 2, 3, 4, 5, 6\}\}.$$

We can get $\alpha_{\bar{x}_1} = \frac{4}{3}$; $\alpha_{\bar{x}_2} = 181\frac{1}{6}$; $\alpha_{\bar{x}_3} = 15.5$; $\alpha_{\bar{x}_4} = 16$. Then we have $\alpha_{\bar{x}} = (\frac{4}{3}, 181\frac{1}{6}, 15.5, 16)^T$. And $\frac{1}{2}A + \mathrm{diag}(\alpha_{\bar{x}}) = \begin{pmatrix} \frac{11}{6} & -1 & 1.5 & -2 \\ -1 & 183\frac{5}{6} & -3 & 3.5 \\ 1.5 & -3 & 19.5 & -0.5 \\ -2 & 3.5 & -0.5 & 20.5 \end{pmatrix}$ is positive. So condition $[SC1]$ is satisfied at $\bar{x}$ and $\bar{x}$ is a global minimizer of problem $(EP3)$ with $f(\bar{x}) = -7098$.

We consider another point $\bar{y} = (0, 1, 6, 2)^T$, then $A\bar{y} = (8, -17, 40, 19)^T$, $a + A\bar{y} = (10, -18, 43, 15)^T$. By (6), we have

$$\begin{aligned}
\alpha_{\bar{y}_1} &= \min\{\frac{10}{y_1 - 0} + \frac{6y_1^3 - y_1^4 - (6 \times 0^3 - 0^4)}{(y_1 - 0)^2} \mid y_1 \in \{1, 2, 3, 4, 5, 6\}\},\\
\alpha_{\bar{y}_2} &= \min\{\frac{-18}{y_2 - 1} + \frac{y_2^4 - y_2^5 - (1 \times 1^4 - 1 \times 1^5)}{(y_2 - 1)^2} \mid y_2 \in \{0, 2, 3, 4, 5, 6\}\},\\
\alpha_{\bar{y}_3} &= \min\{\frac{45}{y_3 - 6} + \frac{3y_3^3 - y_3^4 - (3 \times 6^3 - 1 \times 6^4)}{(y_3 - 6)^2} \mid y_3 \in \{0, 1, 2, 3, 4, 5\}\},\\
\alpha_{\bar{y}_4} &= \min\{\frac{15}{y_4 - 2} + \frac{-4y_4^3 + y_4^5 - (-4 \times 2^3 + 1 \times 2^5)}{(y_4 - 2)^2} \mid y_4 \in \{0, 1, 3, 4, 5, 6\}\}.
\end{aligned}$$

We can get $\alpha_{\bar{y}_1} = \frac{5}{3}$; $\alpha_{\bar{y}_2} = -262.8$; $\alpha_{\bar{y}_3} = 10.5$; $\alpha_{\bar{y}_4} = -18$. Obviously $-\alpha_{\bar{y}} \le \frac{1}{2}\mathrm{diag}(A)e$ can't hold, where $\mathrm{diag}(A) = \mathrm{diag}(1, 5, 8, 9)$. Hence $[NC1]$ is not satisfied at the point $\bar{y}$, so $\bar{y}$ is not a global minimizer of problem $(EP3)$.

Example 4.

$$[EP4] \qquad \min f(x) := \frac{1}{2}x^T Ax + a^T x \qquad s.t. \quad x \in \{0, 1, \cdots, 10\}^8,$$

where $A = \begin{pmatrix} 4 & -2 & -3 & 0 & 1 & 4 & 5 & -2\\ -2 & -4 & 0 & 0 & 2 & 2 & 0 & 0\\ -3 & 0 & 8 & -2 & 0 & 3 & 4 & 0\\ 0 & 0 & -2 & -4 & 4 & 4 & 0 & 1\\ 1 & 2 & 0 & 4 & 100 & 2 & 0 & -2\\ 4 & 2 & 3 & 4 & 2 & 100 & 1 & 0\\ 5 & 0 & 4 & 0 & 0 & 1 & 200 & 4\\ -2 & 0 & 0 & 1 & -2 & 0 & 4 & 10 \end{pmatrix}$ and $a = (-4, 1, -8, 3, -100, -10, -20, 0)^T$.

Let $\bar{x} = (10, 10, 7, 10, 0, 0, 0, 1)^T$, then we have $a + A\bar{x} = (-7, -59, -2, -50, -32, 111, 62, 0)^T$, $\widetilde{\widetilde{X}}(a + A\bar{x}) = (-7, -59, 2, -50, 32, -111, -62, 0)^T$, $\widehat{\widehat{x}} = (-0.7, -5.9, 2.0, -5.0, 32, -11.1, -6.2, 0)^T$,

$$\frac{A}{2} - \mathrm{diag}(\widehat{\widehat{x}}) = \begin{pmatrix} 2.7 & -1 & -1.5 & 0 & 0.5 & 2 & 2.5 & -1\\ -1 & 3.9 & 0 & 0 & 1 & 1 & 0 & 0\\ -1.5 & 0 & 2 & -1 & 0 & 1.5 & 2 & 0\\ 0 & 0 & -1 & 3 & 2 & 2 & 0 & 0.5\\ 0.5 & 1 & 0 & 2 & 18 & 1 & 0 & -1\\ 2 & 1 & 1.5 & 2 & 1 & 61.1 & 0.5 & 0\\ 2.5 & 0 & 2 & 0 & 0 & 0.5 & 106.2 & 2\\ -1 & 0 & 0 & 0.5 & -1 & 0 & 2 & 5 \end{pmatrix} \succeq 0,$$

i.e. $[SC2]$ holds at $\bar{x}$. Hence $\bar{x} = (10, 10, 7, 10, 0, 0, 0, 1)^T$ is a global minimizer of problem $(EP4)$ with $f(\bar{x}) = -615$.

Example 5. Consider the problem

$$(EP5) \qquad \min \quad f(x) := -2x_1^3 + x_1^4 + 3x_2^4 - x_2^5 - 2x_3^3 + x_3^4 + \frac{1}{2}x^T Ax + a^T x$$
$$s.t. \qquad x \quad \in \{0,1\}^3$$

Here $A = \begin{pmatrix} -4 & 2 & -3 \\ 2 & 3 & 7 \\ -3 & 7 & -5 \end{pmatrix}$, $a = (1,-2,3)^T$, $b^{(3)} = (-2,0,-2)^T$, $b^{(4)} = (1,3,1)^T$, $b^{(5)} = (0,-1,0)^T$ and $J = 1$. Let $\bar{x} = (1,0,1)^T$, then $A\bar{x} = (-7,9,-8)^T$, $a + A\bar{x} = (-6,7,-5)^T$. We can have $a + A\bar{x} + \sum_{k=3}^m b^{(k)} = (-7,9,-6)^T$, $\tilde{\bar{X}}(a + A\bar{x} + \sum_{k=3}^m b^{(k)}) = (-7,-9,-6)^T$ and

$$\frac{A}{2} - \mathrm{diag}\Big(\tilde{\bar{X}}(a + A\bar{x} + \sum_{k=3}^m b^{(k)})\Big) = \begin{pmatrix} 5 & 1 & -1.5 \\ 1 & 10.5 & 3.5 \\ -1.5 & 3.5 & 3.5 \end{pmatrix} \succeq 0,$$

i.e., $\mathrm{diag}\Big(\tilde{\bar{X}}(a + A\bar{x} + \sum_{k=3}^m b^{(k)})\Big) \preceq \frac{A}{2}$. Hence sufficient condition $[SC3]$ holds. Therefore, $\bar{x} = (1,0,1)^T$ is a global minimizer of problem $(EP5)$ with $f(\bar{x}) = -5.5$.

Acknowledgments. This research was partially supported by the Australian Research Council under Grant DP0771709 and by National Natural Science Foundation of China 10971241.

REFERENCES

[1] J. Baptiste and H. Urruty, *Conditions for global optimality 2*, Journal of Global Optimization, **13** (1998), 349–367.

[2] J. Baptiste and H. Urruty, *Global optimality conditions in maximizing a convex quadratic function under convex quadratic constraints*, Journal of Global Optimization, **21** (2001), 445–455.

[3] A. Beck and M. Teboulle, *Global optimality conditions for quadratic optimization problems with binary constraints*, SIAM Journal on Optimization, **11** (2000), 179–188.

[4] W. Chen and L. S. Zhang, *Global optimality conditions for quadratic integer problems*, Proceedings of Eighth Symposium on Operations Research Society of China, (2006), 206–211.

[5] D. S. Hanif and H. T. CIHAN, *A global optimization algorithm for polynomial programming problems using a reformulation-linearization technique*, Journal of Global Optimization, **2** (1992), 101–112.

[6] P. Hansen, B. Jaumard and S. J. Lu, *An analytical approach to global optimization*, Mathematical Programming, **52** (1991), 227–254.

[7] R. Horst and H. Tuy, "Global Optimization: Deterministic Approaches," Springer-Verlag, Berlin, 1990.

[8] V. Jeyakumar, A. M. Rubinov and Z. Y. Wu, *Sufficient global optimality conditions for non-convex quadratic minimization problems with box constraints*, Journal of Global Optimization, **36** (2006), 471–481.

[9] J. B. Lasserre, *Global optimization with polynomials and the problem of moments*, SIAM Journal on Optimization, **11** (2001), 796–817.

[10] M. C. Pinar, *Sufficient global optimality conditions for bivalent quadratic optimization*, Journal of Optimization Theory and Applications, **122** (2004), 433–440.

[11] A. M. Rubinov, "Abstract Convexity and Global Optimization," Kluwer, 2000.

[12] M. Schweighofer, *Optimization of polynomials on compact semialgebraic sets*, SIAM Journal on Optimization, **15** (2005), 805–825.

[13] H. D. Sherali and W. P. Adams, *A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems*, SIAM Journal on Discrete Mathematics, **3** (1990), 411–430.

[14] H. Sherali and C. Tuncbilek, *A global optimization algorithm for polynomial programming problems using a reformulation-linearization technique*, Journal of Global Optimization, **2** (1992), 101–112.

[15] H. Waki, S. Kim, M. Kojima and M. Maramatsu, *Sums of squares and semidefinite programming relaxations for polynomial optimization problems with structured sparsity*, Dept. of Mathematical and Computing Sciences, Tokyo Institute of Technology, Tokyo, (2004).

[16] Z. B. Wang, S. C. Fang, D. Y. Gao and W. X. Xing, *Global extremal conditions for multi-integer quadratic programming*, Journal of Industrial and Management Optimization, **4** (2008), 213–226.

Received November 2008; 1st revision March 2010; final revision September 2010.

E-mail address: jquan@shu.edu.cn
E-mail address: z.wu@ballarat.edu.au
E-mail address: gqli@shu.edu.cn