G-coupling Functions and Properties of Strongly Star-Shaped cones

Daniel Mauricio Morales Silva

This thesis is submitted in total for the fulfilment of the PhD degree.

Graduate School of Information Technology and Mathematical Sciences
University of Ballarat
PO Box 663 University Drive,
Victoria 3353, Australia

December, 2009

Abstract

The main part of this thesis presents a new approach to the topic of conjugation, with applications to various optimization problems. It does so by introducing (what we call) G-coupling functions.

The G-coupling function is our generalization of the concept of gap functions. It is significant that these coupling functions are non-negative and they do not necessarily attain the zero value.

After giving some examples and properties of these functions, we use them, initially, to generate a dual scheme for the minimization problem. This is done by using the generalized conjugation theory expounded by Rubinov [34] and Singer [38]. In a very natural way, this duality scheme induces Lagrangian functions.

Let us emphasize an important point. The usual approach in the literature is to fix the coupling function and consider a family of objective functions. This allows seeing which properties the coupling function induces to the general conjugate of the objective function. In this work, we present an unusual type of duality. We will fix the objective function f, and then define a special subfamily of the G-coupling functions depending on f. Any coupling function in this subfamily will generate a duality scheme with many interesting properties.

We study as well Strongly Star- Shaped cones. In Vector Optimization problems, there is a partial order which tells us which object is more preferable than others. If the partial order is induced by a solid convex cone, the theory for facing this situation is already developed (Luc [28]). Nonetheless, in real problems, often the partial order is induced by different kind of cones. In [35], Rubinov and Gasimov studied the partial orders induced by strongly star-shaped cones and introduced a scalarization technique for a certain sub-family of these cones. In this work, we put in evidence that the results obtained in [35] can be extended to the whole family of proper closed strongly star-shaped cones. We present a characterization for a sub-family of these cones. To do so, we extend to arbitrary Banach spaces many results from [34] originally established only for finite dimensional vector spaces.

Statement of Authorship

Except where explicit reference is made in the text of the thesis, this thesis contains no material published elsewhere or extracted in whole or in part from a thesis by which I have qualified for or been awarded another degree or diploma. No other person's work has been relied upon or used without due acknowledgment in the main text and bibliography of the thesis.

Daniel M. Morales Silva
PhD Candidate

Acknowledgements

Since before my arrival, my first principal supervisor professor Alex Rubinov, god rests his soul, saw in me a lot of potential. I hope in this work not only reflect that but much more. I will always be grateful to him for bringing me here, guiding me in my first year and giving me a high standard of being a mathematician and a gentleman.

Upon my arrival, Zari and Maxine gave me the kind of support that only mothers do.

Deep thanks to Dr Regina Burachik and Dr David Yost, who took good care of my guidance in the last two years, right after Alex's passing. Their patience and input through the entire process has been extremely valuable.

I want to thank as well the Centre of Informatics and Applied Optimization (CIAO) and its director, Professor John Yearwood, for the financial support through my entire candidature.

There are many people who supported me a lot even though they just barely know me. I will always be in debt to Nadya, Jade, Julien, Guillermo, David and Moses.

Last, but certainly not least, I want to thank Maria, Angela, Delia, Julio, Mauricio, Juan Carlos, Oscar and Chris, family and friends, I will always appreciate the love and caring they gave me.

To my parents
Julio and Delia.

Contents

List of Symbols (in order of appearance)

$\mathbb{R}^n$	The n-th dimensional Euclidian vector space (page xi).
$\\|\cdot\\|$	Norm in a normed space (page xi).
$\text{int}A$	Interior of the set A (page 1).
$\text{bd}A$	Boundary of the set A (page 1).
$\text{cl}A$ or $\overline{A}$	Closure of the set A (page 1).
A^c	Complement of the set A with respect to the entire space it belongs (page 1).
$B_\varepsilon(x_0)$	Ball of radius ε with center in x_0 (page 1).
$\mathbb{R}_{+\infty}$	The set of Real numbers with $+\infty$ (page 1).
$\mathbb{R}_{-\infty}$	The set of Real numbers with $-\infty$ (page 1).
$dom(f)$	Effective domain of the function f (page 3).
$epi(f)$	Epigraph of the function f (page 3).
$hyp(f)$	Hypograph of the function f (page 3).
$S_\lambda(f)$	Level set λ of f (page 3).
f^*	The (Fenchel) conjugate of the function f (page5).
$\overline{\mathbb{R}}$	The set of Real numbers with $+\infty$ and $-\infty$ (page 6).
f^{**}	The (Fenchel) bi-conjugate of the function f (page 6).
g_*	The (Fenchel) concave-conjugate of the function g (page 11).
$P(f,g)$	The minimization problem with f as the objective function and g the constraints (page 12).
$M(f,g)$	The optimal value of $P(f,g)$ (page 12).
$f_k \xrightarrow{e} f$	The sequence $\{f_k\}_{k\in\mathbb{N}}$ converges epigraphically to f (page 16).
A^∞	Recession cone of the set A (page 18).
f^φ	Abstract-convex conjugate (or φ-conjugate) of the function f (page 20).

g_φ	Abstract-concave conjugate of the function g (page 20).
$f^{\varphi\varphi}$	Abstract-convex biconjugate (or φ-biconjugate) of the function f (page 20).
$supp_l(f, H)$	Lower support set of the function f with respect to H (page 21).
$supp^u(f, H)$	Upper support set of the function f with respect to H (page 21).
$co_H f$	H-convex hull of the function f (page 22).
$cv_H g$	H-concave hull of the function g (page 22).
2^X	Power set of X (page 23).
R_x	Ray starting at zero and going through x (page 27).
$p_{u,K}$	Positively homogeneous function depending on the vector u and the cone K (page 29).
μ_U	Minkowski gauge of the set U (page 31).
$\mathcal{F}^{A,B}$	Set of all G-coupling functions with domain $A \times B$ (page 36).
$\mathcal{F}^A$	Set of all proper functions bounded from below and $dom(f) \subset A$ (page 41).
$\mathcal{F}_f^{A,B}$	Set of all G-coupling functions related to f (page 41).
$f_k \xrightarrow{p} f$	The sequence $\{f_k\}_{k\in I\!N}$ converges punctually to f (page 47).
$\mathcal{H}^{C,B}$	Set of all G-coupling functions related to the constraints (page 58).

List of Figures (in order of appearance)

Preface

The basic tool in many fields of Optimization is Convex Analysis.

Suppose that we want to calculate the distance of a point $x_0 \in I\!R^n$ to a convex set $C \subset I\!R^n$. We can formulate this basic problem in the following way:

$$\min_{x \in C} \|x - x_0\|,$$

where

$$\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$$

is the euclidean norm in $I\!R^n$. Furthermore, we can describe the set C as follows

$$C := \{x \in I\!R^n : h(x) \leq 0\},$$

where $h : I\!R^n \to I\!R \cup \{+\infty\}$ is defined by

$$h(x) := \begin{cases} 0 & x \in C \\ +\infty & \text{otherwise.} \end{cases}$$

From this example, we can see immediately that the objective function $f : I\!R^n \to I\!R$ defined by $f(x) = \|x - x_0\|$ and the constraint function h are convex. In fact, this is what it is called a Convex Minimization Problem with a single constraint.

We are going to see how Convex Analysis has already been used to attack Convex Minimization Problems with multiple constraints. The most important results that interest us, are the results concerning convex conjugation. When we want to minimize a function f, in some cases is better to consider the greatest convex lower semi-continuous function majorized by f, denoted by $\overline{co}f$. The main idea of defining a *regularization* of a given function f is to have another function which has better analytical and/or geometrical properties than f. It is known that the set of minimizers of f is always contained in the set of minimizers of $\overline{co}f$. However, the latter set can be much larger and in such a case, this set cannot help. The convex conjugation theory allows us to calculate this regularized function.

The convex conjugate of a function $f : I\!R^n \to I\!R \cup \{+\infty\}$ is defined as follows:

$$f^*(x^*) := \sup[\langle x^*, x\rangle - f(x) : x \in I\!R^n], \ (x^* \in I\!R^n),$$

where $\langle x, y\rangle := \sum_{i=1}^{n} x_i y_i$ is the inner-product in $I\!R^n$. A remarkable well known fact is that the function $f^{**} = (f^*)^*$ is, precisely, the greatest convex lower semi-continuous function majorized by f.

Despite the good structure of f^{**}, in real problems one quite often finds objective functions which have a trivial convex regularization. Let us consider the following example: we want to minimize over $I\!R$ the function f defined by

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0, \\ \sqrt{-x}, & x < 0. \end{cases} \tag{1}$$

We see that f is not differentiable at the origin and $x_0 = 0$ is its only minimizer. If we calculate its second convex conjugate, we would have $f^{**} \equiv 0$. We have lost all information about f. Therefore, a different technique is required. To handle this situation, we find in the literature, the Generalized Conjugation Theory.

The main idea in Generalized Conjugation is to replace the inner-product, in the definition of the convex conjugate, by another *coupling* function, let us say φ. Of course, all the results must be coherent with the ones of convex conjugation if we use the inner-product as a coupling function.

For example, if in (1), instead of the convex conjugate we use the generalized bi-conjugate $f^{\varphi\varphi}$, with $\varphi : I\!R \times I\!R \to I\!R$ defined by

$$\varphi(x, x^*) := \begin{cases} x^2 - x^*, & |x| \leq 1, \\ f(x) - x^* & |x| > 1. \end{cases}$$

We will have the following

$$\begin{aligned} f^{\varphi}(x^*) &= \sup_{x\in I\!R}[\varphi(x,x^*) - f(x)] \\ f^{\varphi}(x^*) &= \max\begin{cases} \sup\limits_{|x|\leq 1}[x^2 - f(x)] - x^* \\ \sup\limits_{|x|>1}[f(x) - f(x)] - x^* \end{cases} \\ f^{\varphi}(x^*) &= -x^*. \end{aligned}$$

And therefore

$$f^{\varphi\varphi}(x) = \begin{cases} x^2, & |x| \leq 1, \\ f(x), & |x| > 1. \end{cases}$$

In this particular example we have that $\arg\min f = \arg\min f^{\varphi\varphi}$ and it is twice differentiable at the origin.

There is yet another useful technique for attacking non-convex minimization problems: Lagrange-type functions. These are used to re-formulate a constrained minimization problem into one without any constraint.

The objective function of the re-formulated problem is a *convolution* of the objective and constraints functions of the original problem. The purpose is to reduce the constrained optimization problem to a sequence of unconstrained problems or, even more, to solve only one unconstrained problem.

Once the basic ideas are stated, our objective is to define a family of coupling functions, the G-coupling functions, in order to study the minimization problem and, moreover, the equilibrium problem.

For the case of strongly star-shaped cones, in [35] the vector optimization problem is studied using a sub-family of this large family of cones. They do it by means of the functions $p_{u,K}$ which are defined as follows: given a set $K \subset X$ and a vector $u \in X$ the function $p_{u,K} : X \to \overline{I\!R}$ is defined as

$$p_{u,K}(x) = \inf\{\lambda \in I\!R : \lambda u - x \in K\}$$

where the infimum over the empty set is equal to $+\infty$. By definition, these functions are positively homogeneous.

If K belongs to a particular sub-family of strongly star-shaped cones and u belongs to a special subset of kern_*K, the function $p_{u,K}$ will enjoy some good properties. These properties are very important for studying the vector optimization problem (see Section 1.7 and [35]).

The main result we present in this work related to this topic is that in fact, this sub-family is actually the whole family of strongly star-shaped cones.

This work is divided in four chapters.

Chapter 1: Overview

We give in this chapter a detailed explanation of Convex Analysis, Lagrange-type functions, Epigraphical Limit, Recession cones, Generalized Conjugation and Abstract Convexity, the Equilibrium Problem and the relationship between Vector Optimization problems and Strongly Star Shaped Cones.

Chapter 2: G-coupling functions

In this Chapter we define our G-coupling functions. By means of the generalized duality theory, we will use these coupling functions to generate dual formulations for the minimization and the equilibrium problems. It is important to emphasize that our dual formulation for the minimization problem is different to others (see [12], [32] and [34]) in the sense that we will not fix the coupling function.

Chapter 3: Strongly star-shaped cones

We show in this Chapter that the results in [35], related to a sub-family of Strongly Star-shaped cones, can be extended in fact to the whole family of proper strongly star-shaped cones. We will introduce also a characterization for a large sub-family of the strongly star-shaped cones.

Chapter 4: Conclusions and Further Research

In this short Chapter, we will summarize our results and present our intended path of future research.

Chapter 1

Overview

In this chapter, we will fix notations and present results and examples found in already published works.

Unless it is otherwise specified, X will denote an arbitrary Banach space and $\|\cdot\|$ its norm. Take $A \subset X$. Denote by $\text{int}A$, $\text{bd}A$, $\text{cl}A$ (or $\overline{A}$), A^c the interior of A, the boundary of A, the closure of A and the complement with respect to X of A ($A^c = X \setminus A$) respectively.

Given $\varepsilon > 0$ and $x_0 \in X$, we denote by $B_\varepsilon(x_0)$ the open ball of radius ε with center in x_0:

$$B_\varepsilon(x_0) := \{x \in X : \|x - x_0\| < \varepsilon\}.$$

Finally, X' will denote the Banach space of all continuous functionals on X (also called the topological dual of X).

In the extended reals, we use as well the following conventions and notations:

$$+\infty + \alpha = \alpha + \infty = +\infty \quad \forall \alpha \in I\!R,$$

$$-\infty + \alpha = \alpha - \infty = -\infty \quad \forall \alpha \in I\!R,$$

$$I\!R_{+\infty} := I\!R \cup \{+\infty\},$$

$$I\!R_{-\infty} := I\!R \cup \{-\infty\}.$$

1.1 Convex Analysis

This section is devoted to giving the basic definitions and results of convex analysis. These are required for understanding the motivation for a generalized conjugation theory (see [7], [17], [18], [21], [23], [14], [31] and [42] for proofs and further details).

1.1.1 Convex Sets

Definition 1.1.1 *A non-empty set set $C \subset X$ will be called convex if for every $x, y \in C$ and every $t \in [0, 1]$, we have that $tx + (1-t)y \in C$.*

Remark: If $C \subset X = I\!R^n$, Definition 1.1.1 geometrically means that for any pair of points x and y in C, the segment $[x, y]$ is entirely contained in C (see Figure 1.1).

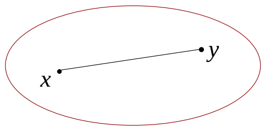

Figure 1.1: Convex Set

Definition 1.1.2 *Given $x \in X$ and $C \subset X$, we say that x is a (finite) convex combination of elements of C, if there exist $p \in I\!N$, $\{t_i\}_{i=1}^{p} \subset [0, 1]$ and $\{x_i\}_{i=1}^{p} \subset C$ such that*

$$x = \sum_{i=1}^{p} t_i x_i \qquad and \qquad \sum_{i=1}^{p} t_i = 1.$$

It is not difficult to verify that the intersection of an arbitrary collection of convex sets is convex. Thus, for every $S \subset X$, the intersection of all the convex subsets of X containing S is the smallest convex subset of X containing S.

Definition 1.1.3 *Given a non-empty set $S \subset X$, we define its convex hull, denoted by $co(S)$, as the smallest convex subset of X containing S.*

Proposition 1.1.1 *The convex hull of a non-empty set $S \subset X$ is the set of all (finite) convex combinations of elements of S.*

1.1.2 Topological Properties

Theorem 1.1.2 *Let $C \subset X$ be a non-empty convex set. Then:*

1. *$intC$ and clC are convex.*

2. *If* $x \in intC$ *and* $y \in clC$*, then for every* $t \in [0,1)$, $x + t(y-x) = z \in intC$.

Proposition 1.1.3 *If* $C \subset X$ *is a non-empty convex set with non-empty interior then:*

1. $intC = int\overline{C}$.

2. $\overline{C} = \overline{intC}$.

Definition 1.1.4 *Given a non-empty set* $S \subset X$*, we define its closed convex hull, denoted by* $\overline{co}(S)$*, as the intersection of all the closed convex sets which contain* S.

Proposition 1.1.4 $\overline{co}(S) = \overline{co(S)}$.

1.1.3 Convex Functions

Given a function $f : S \to I\!R$ with $S \subset X$, we can extend it to a function $g : X \to I\!R_{+\infty}$, defined by:

$$g(x) = \begin{cases} f(x) & \text{if } x \in S \\ +\infty & \text{if } x \notin S. \end{cases}$$

Henceforth, we suppose that the functions are defined on X with values in $I\!R_{+\infty}$.

Definition 1.1.5 *Let be* $f : X \to I\!R_{+\infty}$ *and* $\lambda \in I\!R$*. Let us define,*

1. *The effective domain of* f*:* $dom(f) = \{x \in X : f(x) < +\infty\}$.

2. *The epigraph of* f*:* $epi(f) = \{(x,\lambda) \in X \times I\!R : f(x) \leq \lambda\}$.

3. *The hypograph of* f*:* $hyp(f) = \{(x,\lambda) \in X \times I\!R : f(x) \geq \lambda\}$.

4. *The level-set* λ *of* f*:* $S_\lambda(f) = \{x \in X : f(x) \leq \lambda\}$.

5. *If* $dom(f) \neq \emptyset$ *then we say that* f *is proper.*

Let us recall that:

- A function f is called lower semi-continuous (l.s.c.) at x_0 if for every $\lambda < f(x_0)$ there exists a neighborhood V of x_0 such that for all $x \in V$ we have that $f(x) > \lambda$.

- A function $f : X \to I\!R_{-\infty}$ is called upper semi-continuous (u.s.c.) at x_0 if $-f$ is l.s.c. at x_0.

- A function f is called l.s.c. (respectively, u.s.c.) if it is l.s.c. (respectively, u.s.c.) at every $x \in X$.

In a well-known way, $X \times I\!R$ can be made a Banach space (which can be denoted by $X \bigoplus I\!R$) if we endow it with the product topology, which is in turn the norm topology of one of the (equivalent) norms $(x, \lambda) \mapsto \|x\| + |\lambda|$ and $(x, \lambda) \mapsto \max(\|x\|, |\lambda|)$.

Theorem 1.1.5 *The following are equivalent:*

1. *f is l.s.c.*

2. *$epi(f)$ is closed (as a subset of $X \times I\!R$).*

3. *$S_\lambda(f)$ is closed for all $\lambda \in I\!R$.*

Proposition 1.1.6 *Let be $f, g : X \to I\!R$, functions l.s.c at x_0. Then $(f+g)$, $\min(f, g)$, kf for all $k > 0$, are l.s.c. functions at x_0. If $\{f_i\}_{i \in I}$ is a family of function l.s.c. at x_0 (I is an arbitrary index set), then the function $f = \sup_{i \in I} f_i$ is l.s.c. at x_0.*

Definition 1.1.6 *Given a function $f : X \to I\!R_{+\infty}$, we define the closure of f, denoted by $\overline{f}$ (or by clf), as the greatest l.s.c. function majorized by f. If $\overline{f} = f$ then we say that f is closed.*

Proposition 1.1.7 *Consider $f : X \to I\!R_{+\infty}$.*

1. *$epi(\overline{f}) = \overline{epi(f)}$.*

2. *f is l.s.c. at x_0 if and only if $f(x_0) = \overline{f}(x_0)$.*

3. *f is l.s.c. if and only if $f = \overline{f}$.*

Definition 1.1.6 together with Proposition 1.1.7 item 1. imply that the closure of a function always exists.

Definition 1.1.7 *A proper function $f : X \to I\!R_{+\infty}$ is said to be convex if its epigraph is a convex set of $X \times I\!R$.*

Proposition 1.1.8 *Consider $f, g, f_i : X \to I\!R_{+\infty}$ proper functions with $i \in I$ (I is an arbitrary index set). The following are true:*

1. *f is convex if and only if for every $x, y \in I\!R^n$ and $t \in (0, 1)$ we have that*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

2. *If f is convex, then $\overline{f}$ is convex ($epi(\overline{f}) = \overline{epi(f)}$).*

3. *If f is convex and if $\lambda > 0$ then λf is convex.*

4. *If f and g are convex then $f + g$ is convex.*

5. *If $\{f_i\}_{i\in I}$ is a family of convex functions, then*

$$f = \sup_{i\in I} f_i \text{ is a convex function and } epi(f) = \bigcap_{i\in I} epi(f_i).$$

Theorem 1.1.9 *Let $f : X \to I\!R_{+\infty}$ be convex and $a \in X$. If there exists a neighborhood of a on which f is bounded from above then:*

1. *f is a proper convex function and $int(dom(f)) \neq \emptyset$.*

2. *At each point of $int(dom(f))$ there exists a neighborhood on which f is bounded from above.*

3. *f is continuous on $int(dom(f))$.*

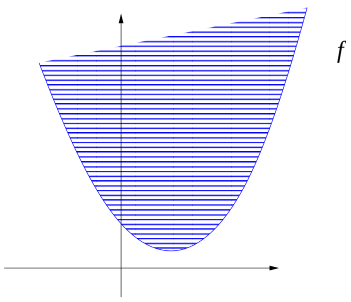

Figure 1.2: Epigraph of a convex function

1.1.4 Conjugation

Definition 1.1.8 *Let be $f : X \to I\!R_{+\infty}$, we define the conjugate function (in the sense of Fenchel) to be the following function:*

$$f^*(x^*) = \sup_{x\in X}[x^*(x) - f(x)],\ \forall x^* \in X'.$$

Definition 1.1.9 *Given $g : X' \to I\!R_{+\infty}$, we define the conjugate function (in the sense of Fenchel) to be the following function:*

$$g^*(x) = \sup_{x^*\in X'}[x^*(x) - g(x^*)],\ \forall x \in X.$$

Remark: It is immediate to see that $f^*(x^*) = \sup_{x \in dom(f)} [x^*(x) - f(x)]$ and since the functions $x^* \mapsto x^*(x) - f(x)$ are l.s.c. and convex for every x then f^* is always a l.s.c. convex function (f^* is the supremum of l.s.c. convex functions).

Example: The following figure shows geometrically what $f^*(v)$ represents when $X = X' = I\!R$ and f is given by its graph:

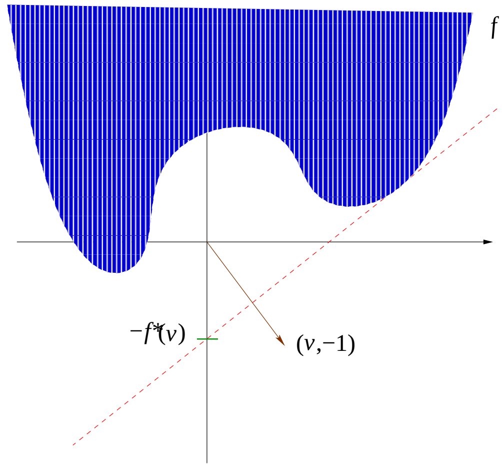

Figure 1.3: Fenchel conjugate

For the following proposition, $\overline{I\!R}$ denotes the set $I\!R \cup \{-\infty, +\infty\}$.

Proposition 1.1.10 *Let be $f, g : X \to \overline{I\!R}$. The following are true:*

1. *If there exists $x_0 \in X$ such that $f(x_0) = -\infty$, then for every x^* we have that*

$$f^*(x^*) \geq x^*(x_0) - f(x_0) = +\infty.$$

Therefore $f^*(x^*) = +\infty, \ \forall x^* \in X'$.

2. *If $f(x) = +\infty$, for every $x \in X$ then $f^*(x^*) = -\infty$, for each $x^* \in X'$.*

3. *If $f(x) \leq g(x)$, for every $x \in X$ then $f^*(x^*) \geq g^*(x^*)$, for each $x^* \in X'$.*

4. *It is always satisfied:*

$$f^*(x^*) + f(x) \geq x^*(x), \ \forall x \in X, \ x^* \in X'.$$

This is called Fenchel-Young Inequality.

5. *The following is always true:*

$$f^{**} \leq f.$$

Theorem 1.1.11 *Let be $f : X \to I\!R_{+\infty}$. Then:*

1. *f^{**} is the pointwise supremum of the collection of all continuous affine minorants of f.*
2. *f^{**} is a l.s.c. convex function on E.*
3. *$f^{***} = f^*$.*

Theorem 1.1.12 *Let be $f : X \to \overline{I\!R}$. The following conditions are equivalent:*

1. *There exists a family of functions H such that $f(x) = \sup_{h \in H} h(x)$ and every $h \in H$ is of the form $h(x) = x^*(x) + \alpha$ for some $x^* \in X'$ and $\alpha \in I\!R$.*
2. *$f = f^{**}$.*
3. *f is a lower semi-continuous proper convex function or f is one of the constant functions $-\infty$ and $+\infty$.*

1.1.5 Convex Duality in finite dimensions

The convex conjugation theory has been used to formulate a dual problem for the minimization problem. Furthermore, with this dual formulation in the case of the constrained minimization problem, we can recover the classical Lagrangian duality.

General Scheme

The constrained minimization problem can be stated in the following way. Given $\tilde{f} : X \to I\!R_{+\infty}$ we define

$$(\tilde{P}) : \ \alpha = \inf[\tilde{f}(x) : \ x \in C], \text{ where } C \subset I\!R^n.$$

If we consider $f : I\!R^n \to I\!R_{+\infty}$ defined by:

$$f(x) = \begin{cases} \tilde{f}(x) & x \in C \\ +\infty & x \notin C \end{cases}$$

we obtain an equivalent problem where there are no constraints

$$(P) : \ \alpha = \inf[f(x) : \ x \in I\!R^n].$$

We say that (P) is the primal problem.

We are going to use now the notion of *perturbation function* ([32]). Given $p \in I\!N$ and $\varphi : I\!R^n \times I\!R^p \to \overline{I\!R}$ we say that φ is a perturbation function for f if it satisfies the following property

$$\varphi(x, 0) = f(x), \ \forall x \in I\!R^n.$$

Its second variable $u \in I\!R^p$ will be called the perturbation variable.

Taking an arbitrary perturbation function for f, namely φ, we define the function $h : I\!R^p \to \overline{I\!R}$ as follows:

$$h(u) = \inf_{x \in I\!R^n} \varphi(x, u).$$

This function h is called marginal function. It is clear that

$$\alpha = h(0) = \inf_{x \in I\!R^n} \varphi(x, 0) = \inf_{x \in I\!R^n} f(x).$$

Consider now h^{**}, the bi-conjugate of the function h. We have that:

$$h^{**}(0) \leq h(0) = \alpha$$

where

$$h^{**}(0) = \sup_{u^* \in I\!R^n} [\langle 0, u^* \rangle - h^*(u^*)].$$

Then, making $-\beta = h^{**}(0)$:

$$(Q) : \ \beta = \inf[h^*(u^*) : \ u^* \in I\!R^n].$$

(Q) is called the dual problem of (P) and, in general, $-\beta \leq \alpha$. The interval $[-\beta, \alpha]$ is called *duality gap*. There is no duality gap if $h^{**}(0) = h(0)$.

On the other hand, we know that:

$$h^*(u^*) = \sup_{u \in I\!R^n} [\langle u^*, u \rangle - h(u)] = \sup_{u \in I\!R^n} \left[\langle u^*, u \rangle - \inf_{x \in I\!R^n} \varphi(x, u) \right]$$

$$h^*(u^*) = \sup_{(x,u) \in I\!R^n \times I\!R^n} [\langle 0, x \rangle + \langle u^*, u \rangle - \varphi(x, u)]$$

$$h^*(u^*) = \varphi^*(0, u^*).$$

Denote by k the following function

$$k(x^*) = \inf_{u^* \in I\!R^n} \varphi^*(x^*, u^*).$$

We have that $\beta = k(0)$.

Now, observe the symmetry between the two problems:

$$\begin{array}{rclcrcl}
\alpha & = & \inf f(x) \qquad (P) & & \beta & = & \inf h^*(u^*) \qquad (Q) \\
\varphi(x,0) & = & f(x),\ \forall x \in I\!R^n & & \varphi^*(0,u^*) & = & h^*(u^*),\ \forall u^* \in I\!R^n \\
h(u) & = & \inf_x \varphi(x,u) & & k(x^*) & = & \inf_{u^*} \varphi^*(x^*,u^*) \\
\alpha & = & h(0) & & \beta & = & k(0)
\end{array}$$

$$-\beta \leq \alpha.$$

By construction h^* is a l.s.c. convex function, thus, (Q) is always a convex minimization problem. If φ is convex over $I\!R^n \times I\!R^p$, then h is convex over $I\!R^p$. If h is proper and convex, there is no duality gap if and only if h is l.s.c. at 0. In general the lower semi-continuity of the function φ does not imply that h is l.s.c.

If h is convex and l.s.c. at 0 and, moreover, if $\alpha = h(0) < +\infty$, then necessarily we have that $h(0) = h^{**}(0)$. In this case there is no duality gap.

Particular case: Lagrangian Duality

Let us consider the following problem:

$$\alpha = \inf[\tilde{f}(x) :\ g_i(x) \leq 0,\ i = 1, \dots, p\ ],$$

with the following hypothesis:

(H_0) $\tilde{f},\ g_i : I\!R^n \to I\!R_{+\infty}$ are proper l.s.c. and convex functions (for every $i = 1, \dots, p$).

(H_1) There exists $\overline{x}$ such that $\tilde{f}(\overline{x}) < +\infty$ and $g_i(\overline{x}) < 0,\ \forall i = 1, \dots, p$ (this condition is known as Slater condition).

(H_2) $\alpha > -\infty$.

From (H_1) and (H_2) we have that $|\alpha| < +\infty$. Define:

$$f(x) = \begin{cases} \tilde{f}(x) & \text{if } g_i(x) \leq 0,\ \forall i = 1, \dots, p \\ +\infty & \text{otherwise.} \end{cases}$$

$$\varphi(x,u) = \begin{cases} f(x) & \text{if } g_i(x) + u_i \leq 0,\ \forall i = 1, \dots, p \\ +\infty & \text{otherwise.} \end{cases}$$

It follows that φ is a proper l.s.c. convex function thanks to (H_0) and (H_1). Therefore $\varphi = \varphi^{**}$. On the other hand,

$$h(u) = \inf_x \varphi(x, u),$$

is convex (since φ is convex). Furthermore if we take $u_i \leq -g_i(\overline{x})$ for all $i = 1, \ldots, p$, we have that $h(u) \leq f(\overline{x})$. Then

$$V = \prod_{i=1}^{p} (-\infty, -g_i(\overline{x})) \subset dom(h).$$

Thus, V is a neighborhood of 0 and $0 \in int(dom(h))$. In consequence, $h(0) = h^{**}(0)$ (in fact h is continuous at 0) which implies that there is no duality gap ($-\alpha = \beta$). Let us calculate h^*:

$$h^*(u^*) = \sup_{x,u} [\langle u^*, u\rangle - \varphi(x, u)]$$

$$h^*(u^*) = \sup_{x,u} [\langle u^*, u\rangle - f(x) : \ g_i(x) + u_i \leq 0].$$

- If u^* has a component lesser than zero, let us say the component i_0, we can take $x = \overline{x},\ u_i = -g_i(\overline{x})$ if $i \neq i_0$, and make u_{i_0} go to $-\infty$, then $h^*(u^*) = +\infty$.
- If $u^* \geq 0$,
$$h^*(u^*) = \sup_{x \in dom(f) \cap (\bigcap dom(g_i))} \left[-f(x) - \sum_{i=1}^{p} u_i^* g_i(x) \right].$$

Therefore

$$h^*(u^*) = \begin{cases} \sup_{x \in dom(f) \cap (\bigcap dom(g_i))} \left[-f(x) - \sum_{i=1}^{p} u_i^* g_i(x) \right], & u^* \geq 0 \\ +\infty, & u^* \ngeq 0. \end{cases}$$

Thus we recover the classical lagrangian duality. Finally, because $\alpha = h(0) = h^{**}(0)$ we can write

$$-\beta = \sup_{u^*} [-h^*(u^*)] = \alpha.$$

Hence,

$$\alpha = \sup_{u^* \geq 0} \inf_x \left[f(x) + \sum_{i=1}^{p} u_i^* g_i(x) \right].$$

1.1.6 Fenchel's duality theorem

In [18], conjugation techniques were introduced to minimize the difference of two functions.

In this section, we consider $X = X' = I\!R^n$. We say that a function $g : X \to I\!R_{-\infty}$ is *concave* if $-g$ is convex.

For a concave function g, we say that g is closed (respectively, proper) if $-g$ is closed (respectively, proper).

Definition 1.1.10 *Let $g : X \to I\!R_{-\infty}$ be a concave function. We define its concave-conjugate, g_*, as follows:*

$$g_*(x^*) := \inf_{x \in X} [x^*(x) - g(x)].$$

Definition 1.1.11 *Let $S \subset X$ be a non-empty set. Take $a \in S$.*

1. *The affine hull of S, denoted by aff(S), is defined by:*

$$aff(S) := \left\{ a + \sum_{i=1}^{p} t_i(b_i - a) : p \in I\!N, \{t_i\}_{i=1}^{p} \subset I\!R, \{b_i\}_{i=1}^{p} \subset S \right\}.$$

2. *The relative interior of S, denoted by ri(S), is the interior of S with respect to its affine hull aff(S).*

Theorem 1.1.13 *Let $f : X \to I\!R_{+\infty}$ be proper and convex and $g : X \to I\!R_{-\infty}$ be proper and concave. One has*

$$\inf_x [f(x) - g(x)] = \sup_{x^*} [g_*(x^*) - f^*(x^*)]$$

if either of the following conditions is satisfied:

1. $ri(dom(f)) \cap ri(dom(g)) \neq \emptyset$;

2. *f and g are closed, and $ri(dom(g_*)) \cap ri(dom(f^*)) \neq \emptyset$.*

Under 1. the supremum is attained at some x^*, while under 2. the infimum is attained at some x; if 1. and 2. both hold, the infimum and supremum are necessarily finite.

1.2 Lagrange type-functions

In [36] can be found the notion of *Lagrange type-functions*. In the next chapter we will present, in a different way, Lagrange type-functions which satisfy most of the results we will present now in this Section.

See [36] and references therein for more details and proofs of the results presented here.

Consider the following problem $P(f,g)$

$$\min f(x) \text{ subject to } x \in I\!R^n,\ g(x) \le 0,$$

where $f : I\!R^n \to I\!R$ and $g : I\!R^n \to I\!R^m$ (notice that f and g are not assumed to be continuous). The unconstrained optimization method for solving the constrained minimization problem $P(f,g)$ is to reduce it to a sequence of unconstrained minimization problems of the form

$$\min_x F(x,\omega),\ x \in I\!R^n,$$

where $F : I\!R^n \times \Omega \to I\!R$ and Ω is a set of parameters.

We denote by X_0 the set of feasible elements of $P(f,g)$:

$$X_0 = \{x \in I\!R^n : g(x) \le 0\};$$

by $M(f,g)$ the optimal value of $P(f,g)$: $M(f,g) = \{\inf f(x) : x \in X_0\}$ and by $\arg\min P(f,g)$ the set of optimal solutions of this problem:

$$\arg\min P(f,g) = \{x \in X_0 : f(x) = M(f,g)\}.$$

We make the following assumption:

(A): The set of feasible elements X_0 of problem $P(f,g)$ is non-empty and the objective function f is bounded from below on this set.

It follows from this assumption that $-\infty < M(f,g) < +\infty$.

We are interested in finding a sequence $\{\omega_t\}$ such that

$$\inf_x F(x,\omega_t) \to M(f,g),$$

as $t \to +\infty$. Such a sequence exists if

$$\sup_{\omega \in \Omega} \inf_{x} F(x, \omega) = M(f, g). \tag{1.1}$$

If the function F satisfies (1.1) and it appears as a convolution of the objective and the constraint functions, we shall call it *Lagrange-type function.*

The simplest example is given by the classic Lagrange function:

$$F(x, \omega) = f(x) + \sum_{i=1}^{m} \omega_i g_i(x), \ x \in I\!R^n, \omega \in \Omega,$$

where $f, g_i : I\!R^n \to I\!R$ are convex and lower semi-continuous, and $\Omega = I\!R^m_+$.

Non-linear analogues of Lagrange functions can arise if we try to give a convenient presentation of sufficient conditions for a minimum. They are expressed in terms of the empty intersection of certain sets. The following subsections are meant to put in evidence this affirmation. In later chapters, when we introduce our Lagrange-type functions, we will see that they satisfy these conditions as well.

1.2.1 Saddle-points of Lagrange-type functions

Let us consider the following Lagrange-type functions:

$$L(x, \omega) = f(x) + \chi(g(x), \omega), \ x \in I\!R^n, \omega \in \Omega,$$

where f and g are as in $P(f, g)$.

The linear case can be seen as one of these functions if we take $\Omega = I\!R^m_+$ and $\chi \equiv \langle \cdot, \cdot \rangle$ the inner-product in $I\!R^m$.

Let **K** be a set of functions $\chi : I\!R^m \times \Omega \to I\!R$ with the following two properties:

1) $\chi(\cdot, \omega)$ is lower semi-continuous for all $\omega \in \Omega$.

2) $\sup_{\omega \in \Omega} \chi(v, \omega) = 0$ for all $v \in I\!R^m_-$.

Proposition 1.2.1 *Let $\chi \in$ **K** and L be the Lagrange-type function induced by χ. Then*

$$\sup_{\omega \in \Omega} \inf_{x} L(x, \omega) \le M(f, g).$$

Consider a point $(x_*, \omega_*) \in I\!R^n \times \Omega$ such that

$$L(x_*, \omega_*) = \min_x L(x, \omega_*), \tag{1.2}$$

and

$$\chi(g(x_*), \omega_*) = 0. \tag{1.3}$$

Proposition 1.2.2 *Let $\chi \in \mathbf{K}$. If* (1.2) *and* (1.3) *hold for $x_* \in X_0$ and $\omega_* \in \Omega$, then x_* is a solution of $P(f, g)$.*

Recall that a point $(x_*, \omega_*) \in I\!R^n \times \Omega$ is called a *saddle point* of the function L on the set $I\!R^n \times \Omega$ if

$$L(x_*, \omega) \leq L(x_*, \omega_*) \leq L(x, \omega_*), \text{ for all } x \in I\!R^n \text{ and } \omega \in \Omega. \tag{1.4}$$

Proposition 1.2.3 *A point $(x_*, \omega_*) \in X_0 \times \Omega$ is a saddle point of L (with $\chi \in \mathbf{K}$) on $I\!R^n \times \Omega$ if and only if* (1.2) *and* (1.3) *hold.*

1.2.2 Saddle points and separation

Let $\chi \in \mathbf{K}$, $\omega \in \Omega$ and $\eta \in I\!R$. Consider the following sets:

$$\mathcal{L}^+_\chi(\omega) = \{(u, v) \in I\!R^{1+m} : u + \chi(v, \omega) \geq 0\}, \tag{1.5}$$

$$\mathcal{T}_\eta = \{(f(x) - \eta, g(x)) : x \in I\!R^n\}. \tag{1.6}$$

Proposition 1.2.4 *Let $\chi \in \mathbf{K}$. Then $(x_*, \omega_*) \in X_0 \times \Omega$ is a saddle point of L on $I\!R^n \times \Omega$ if and only if*

$$\mathcal{T}_\eta \subset \mathcal{L}^+_\chi(\omega_*), \text{ where } \eta = f(x_*).$$

It follows from Proposition 1.2.4 that the function $h((u, v), \omega_*) = u + \chi(v, \omega_*)$ separates the sets $I\!R^{1+m}_-$ and $\mathcal{T}_\eta$.

1.2.3 A non-linear example of a Lagrange-type function

Consider the function

$$\chi(v, \omega) = \max\{\langle \omega_0, v\rangle, \ldots, \langle \omega_p, v\rangle\}, \tag{1.7}$$

where $p \geq 1$, $\omega = (\omega_0, \omega_1, \ldots, \omega_p)$, $\omega_i \in I\!R^m_+$ (that is, $\Omega = (I\!R^m_+)^{1+p}$). It is easy to check that $\chi \in \mathbf{K}$. The Lagrange-type function L_χ, which corresponds to χ, has the form

$$L_\chi(x, \omega) = f(x) + \max\{\langle \omega_0, g(x) \rangle, \ldots, \langle \omega_p, g(x) \rangle\}. \tag{1.8}$$

We say that the function L_χ, defined by (1.8), has a saddle point on $I\!R^n \times \Omega$ if there exists a point $(x_*, \omega_*) \in X_0 \times \Omega$ such that (1.4) holds. Thus we are imposing that x_* belongs to X_0 in this definition of saddle point.

For the following Propositions and Theorem, we are considering χ and L_χ as in (1.7) and (1.8) respectively.

Proposition 1.2.5 *The function L_χ has a saddle point on $I\!R^n \times \Omega$ if and only if there exists a vector $\omega_* \in \Omega$ such that*

$$\mathcal{T}_{M(f,g)} \subset \bigcup_{i=0,\ldots,p} \mathcal{L}_i^+(\omega_*),$$

where $\mathcal{L}_i^+(\omega_*) = \{(u, v) \in I\!R^{1+m} : \langle (u, v), (1, (\omega_*)_i) \rangle \geq 0\}$ is the half-space of the space $I\!R^{1+m}$ associated with $(\omega_*)_i$.

Let $\mathcal{L}_\chi^+(\omega)$ be the set defined by (1.5). It easy to check that

$$\mathcal{L}_\chi^+(\omega) = \bigcup_{i=0,\ldots,p} \mathcal{L}_i^+(\omega).$$

Consider now the complement $\mathcal{L}_\chi^-(\omega)$ to the set $\mathcal{L}_\chi^+(\omega)$. Clearly,

$$\mathcal{L}_\chi^-(\omega) = \{(u, v) : u + \langle \omega_i, v \rangle < 0, i = 0, \ldots, p\}$$

is an open convex cone. Define the set

$$\mathcal{H}^- = \{(u, v) \in I\!R^{1+m} : u < 0, v \leq 0\}.$$

This set $\mathcal{H}^-$ is neither closed nor open. Obviously, $\mathrm{cl}\mathcal{H}^- = I\!R^{1+m}_-$. Let $(u, v) \in \mathcal{H}^-$, that is, $u < 0$, $v \leq 0$. Since $\omega_i \geq 0$, it follows that $u + \langle \omega_i, v \rangle < 0$, so $\mathcal{H}^- \subset \mathcal{L}_\chi^-(\omega)$. The previous proposition can be re-formulated as follows:

Proposition 1.2.6 *The function L_χ has a saddle point on the set $I\!R^n \times \Omega$ if and only if there exists a vector $\omega_* = ((\omega_*)_0, \ldots, (\omega_*)_p)$ such that the open convex cone $\mathcal{L}_\chi^-(\omega_*)$ separates the sets $\mathcal{T}_{M(f,g)}$ and $\mathcal{H}^-$ in the following sense:*

$$\mathcal{H}^- \subset \mathcal{L}_\chi^-(\omega_*) \text{ and } \mathcal{T}_{M(f,g)} \cap \mathcal{L}_\chi^-(\omega_*) = \emptyset.$$

Remark: We can express the separation stated in the previous proposition in terms of the function $h((u,v),\omega) := u + \chi(v,\omega)$:

$$h((u,v),\omega_*) < 0, \text{ for all } (u,v) \in \mathcal{H}^-;$$

$$h((u,v),\omega) \geq 0, \text{ for all } (u,v) \in \mathcal{T}_{M(f,g)}.$$

There is a better result if $p = m$:

Theorem 1.2.7 *Consider $P(f,g)$. The Lagrange-type function L_χ has a saddle point on the set $I\!R^n \times \Omega$ if and only if there exists an open convex cone $K \subset I\!R^{1+m}$ which separates $\mathcal{H}^-$ and $\mathcal{T}_{M(f,g)}$, that is $\mathcal{H}^- \subset K$ and $\mathcal{T}_{M(f,g)} \cap K = \emptyset$.*

1.3 Epigraphical Limit

The following definitions and results can be found in [33].

Definition 1.3.1 *For a sequence $\{C^\nu\}_{\nu \in I\!N}$ of subsets of $I\!R^n$, the outer limit is the set*

$$\limsup_{\nu \to +\infty} C^\nu := \{x : \text{ exists an infinite set of indexes } \{\nu_k\} \subset I\!N \text{ and } x_{\nu_k} \in C^{\nu_k} \text{ with } x_{\nu_k} \to x\},$$

while the inner limit is the set

$$\liminf_{\nu \to +\infty} C^\nu := \{x : \exists N_0 \in I\!N, x_\nu \in C^\nu \text{ for each } \nu \geq N_0 \text{ with } x_{\nu_k} \to x\}.$$

The limit of the sequence exists if the outer and inner limit sets are equal:

$$\lim_{\nu \to \infty} C^\nu := \limsup_{\nu \to \infty} C^\nu = \liminf_{\nu \to \infty} C^\nu,$$

in this event the sets C^ν are said to converge to C ($C := \limsup_{\nu \to \infty} C^\nu = \liminf_{\nu \to \infty} C^\nu$) and it is symbolized by $C^\nu \to C$.

Definition 1.3.2 *For any sequence $\{f_k\}_{k \in I\!N}$ of functions on $I\!R^n$ ($f_k : I\!R^n \to \overline{I\!R}$), the functions f_k are said to epi-converge to a function $f : I\!R^n \to \overline{I\!R}$, a condition symbolized by $f_k \xrightarrow{e} f$ if:*

$$epi(f_k) \to epi(f).$$

Example: Consider $f_k : [-1,1] \to I\!R$ given by

$$f_k(x) := \min[1 - x, 1, 2k|x + 1/k| - 1].$$

Then it is not difficult to prove that the epigraphical limit of f_k is given by

$$f(x) := \begin{cases} 1, & x \in [-1,0) \\ -1, & x = 0 \\ 1-x, & x \in (0,1]. \end{cases}$$

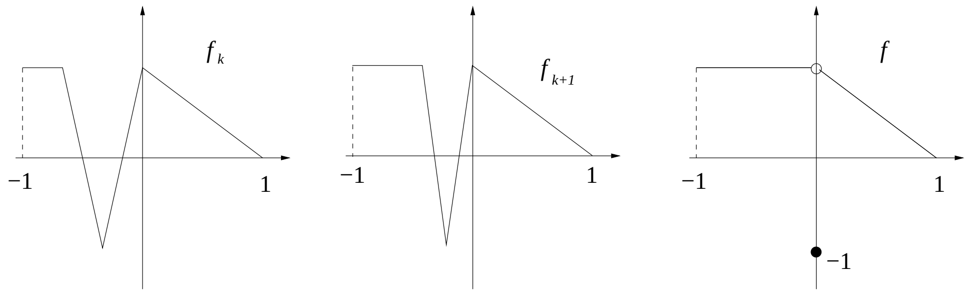

Figure 1.4: Epigraphical limit

Proposition 1.3.1 *Let $\{f_k\}_{k\in I\!N}$ be a sequence of functions on $I\!R^n$. If the sequence is non-decreasing ($f_k \le f_{k+1}$), then $f_k \xrightarrow{e} \sup_k[\overline{f}_k]$.*

Theorem 1.3.2 *Suppose $f_k \xrightarrow{e} f$ with $-\infty < \inf f < +\infty$. We have that $\inf f_k \to \inf f$ if and only if there exists for every $\varepsilon > 0$ a compact set $B \subset I\!R^n$ and $N_0 \in I\!N$ such that*

$$\inf_{x\in B} f_k(x) \le \inf f_k + \varepsilon \text{ for all } k \ge N_0.$$

Definition 1.3.3 *A sequence of functions $\{h_k\}_{k\in I\!N}$ on $I\!R^n$ converges continuously to a function h on $I\!R^n$ if and only if for every $x \in I\!R^n$ and any sequence $\{x_k\}_{k\in I\!N} \subset I\!R^n$ such that $x_k \to x$ then $h_k(x_k) \to h(x)$.*

Theorem 1.3.3 *Let $\{f_k^1\}_{k\in I\!N}$, $\{f_k^2\}_{k\in I\!N}$ be two sequences of functions. Consider f^1 and f^2 functions on $I\!R^n$ such that $f_k^1 \xrightarrow{e} f^1$ and $f_k^2 \xrightarrow{e} f^2$. If both sequences of functions $\{f_k^1\}_{k\in I\!N}$ and $\{f_k^2\}_{k\in I\!N}$ converge pointwise to f^1 and f^2 respectively, then $f_k^1 + f_k^2 \xrightarrow{e} f^1 + f^2$.*

1.4 Cones and Recession Cones

Definition 1.4.1 *Let $K \subset X$ be a non-empty set. We say that K is a cone if $\lambda x \in K$ whenever $x \in K$ and $\lambda > 0$.*

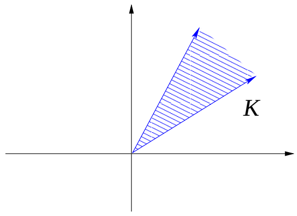

Figure 1.5: Example of a cone in $I\!R^2$

Definition 1.4.2 *Let $K \subset X$ be a non-empty set. We define the sets K^+ and K^{++} as follows:*

$$K^+ := \{x^* \in X' : x^*(x) \geq 0, \ \forall x \in K\},$$

$$K^{++} := \{x \in X : x^*(x) \geq 0, \ \forall x^* \in K^+\}.$$

Theorem 1.4.1 *Take $K \subset X$. $K = K^{++}$ if and only if K is a non-empty closed convex cone.*

Let be $A \subset X$. We define its recession cone as follows (see [28]):

$$A^\infty := \{d \in X : \ \exists \{x_k\}_{k \in I\!N} \subset A, \{t_k\}_{k \in I\!N} \downarrow 0 : \ t_k x_k \to d\} \tag{1.9}$$

with the convention, $\emptyset^\infty = \{0\}$. The following properties are satisfied:

Lemma 1.4.2 *Let it be $A, B \subset X$. The following are true:*

i) *A^∞ is a closed cone.*

ii) *If A is bounded then $A^\infty = \{0\}$. If X is finite dimensional, then the converse is true.*

iii) *If $A \subset B$ then $A^\infty \subset B^\infty$.*

iv) *$A^\infty = (\overline{A})^\infty$.*

v) *If A is a cone, then $A^\infty = \overline{A}$.*

vi) *If $(A_i)_{i \in I}$ is a family of sets in $I\!R^p$, we have that*

$$\left(\bigcap_{i \in I} A_i\right)^\infty \subset \bigcap_{i \in I} A_i^\infty.$$

Remark: The converse of item ii) does not hold for infinite dimensional Banach spaces. For example, consider X a real Hilbert space with a countable ortonormal base $\{e_\lambda\}_{\lambda\in I\!N} \subset X$ (see [27] chapter 3 for definitions and the properties used here). Take

$$A := \{ke_k\}_{k\in I\!N}.$$

Clearly A is unbounded. We will prove that $A^\infty = \{0\}$. Since A^∞ is a closed cone, then $\{0\} \subset A^\infty$. For the converse, take $d \neq 0$ and assume it belongs to A^∞. Then, there must exist $\{t_k\}_{k\in I\!N}$ a sequence of non-negative numbers converging to zero and $\{x_k\}_{k\in I\!N}$ a sequence of elements in A which are of the form $x_k = n_k e_{n_k}$ such that:

$$d = \lim_{k\to+\infty} t_k x_k = \lim_{k\to+\infty} \left[(t_k n_k)\, e_{n_k}\right]. \tag{1.10}$$

From (1.10) we have:

$$1 = \frac{\|d\|}{\|d\|} = \frac{\left\|\lim\limits_{k\to+\infty} t_k n_k e_{n_k}\right\|}{\|d\|} = \frac{\lim\limits_{k\to+\infty} t_k n_k}{\|d\|},$$

therefore

$$t_k n_k \to \|d\|. \tag{1.11}$$

On the other hand, from (1.10) we also have:

$$\|d\|^2 = \langle d, d\rangle = \left\langle \lim_{k\to+\infty} \left[(t_k n_k)\, e_{n_k}\right], d \right\rangle = \lim_{k\to+\infty} t_k n_k \langle e_{n_k}, d\rangle.$$

But since $\{e_\lambda\}_{\lambda\in I} \subset X$ is an ortonormal base of X we have that

$$\sum_{k=1}^{+\infty} |\langle e_{n_k}, d\rangle|^2 \leq \sum_{\lambda\in I} |\langle e_\lambda, d\rangle|^2 = \|d\|^2,$$

therefore, we must have that $\langle e_{n_k}, d\rangle \to 0$. This together with (1.11) we must have that

$$\lim_{k\to+\infty} t_k n_k \langle e_{n_k}, d\rangle = 0 \neq \|d\|^2,$$

which is a contradiction. Therefore $A^\infty = \{0\}$.

1.5 Generalized Conjugation and Abstract Convexity

As the title of this Section suggests, we will present definitions and results which generalize those presented in Section 1.1.4.

Consider two arbitrary sets A and B. Any function $\varphi : A \times B \to \mathbb{R}$ is called *coupling function*. For example, if $A = B = \mathbb{R}^n$ then the inner product, $\varphi(a, b) = \langle a, b\rangle$, is a coupling function.

The notion of coupling function allows us to consider an element $b \in B$ as a function defined on A, namely $b(a) = \varphi(a, b)$ for all $a \in A$ and, similarly, an element $a \in A$ as a function on B.

In this general setting, we can find the following results and definitions (see [34], [39] and references therein):

Definition 1.5.1 *Given functions $f : A \to \overline{\mathbb{R}}$ and $g : B \to \overline{\mathbb{R}}$, we define:*

i) The abstract-convex conjugate of f (denoted by f^φ) as:

$$f^\varphi(b) := \sup_{a\in A}[\varphi(a, b) - f(a)], \ \forall b \in B.$$

The abstract-convex conjugate of g (denoted by g^φ) as:

$$g^\varphi(a) := \sup_{b\in B}[\varphi(a, b) - g(b)], \ \forall a \in A.$$

ii) The abstract-concave conjugate of f (denoted by f_φ) as:

$$f_\varphi(b) := \inf_{a\in A}[\varphi(a, b) - f(a)], \ \forall b \in B.$$

The abstract-concave conjugate of g (denoted by g_φ) as:

$$g_\varphi(a) := \inf_{b\in B}[\varphi(a, b) - g(b)], \ \forall a \in A.$$

The following results are related only to the *abstract-convex* case. We can state analogous facts regarding the *abstract-concave* case.

Lemma 1.5.1 *If $f : A \to \mathbb{R}_{+\infty}$ $(g : B \to \mathbb{R}_{+\infty})$ is proper, then*

$$f^\varphi : B \to \mathbb{R}_{+\infty} \ (g^\varphi : A \to \mathbb{R}_{+\infty}).$$

Let be $f : A \to \mathbb{R}_{+\infty}$, a proper function. We denote by $f^{\varphi\varphi}$ the $\varphi-$biconjugate of f. That is $f^{\varphi\varphi} = (f^\varphi)^\varphi$.

Proposition 1.5.2 *Let be $f : A \to \mathbb{R}_{+\infty}$ any function. The following properties are satisfied:*

1. *For each* $h : A \to I\!R_{+\infty}$ *such that* $f \le h$, *we have that* $h^\varphi \le f^\varphi$. *And if* $h^\varphi : B \to I\!R_{+\infty}$, *then we have that* $f^{\varphi\varphi} \le h^{\varphi\varphi}$.

2. *If* $f : A \to I\!R_{+\infty}$ *and* $f^\varphi : B \to I\!R_{+\infty}$ *are proper, then* $f(a)+f^\varphi(b) \ge \varphi(a,b)$ *and* $f^\varphi(b) + f^{\varphi\varphi}(a) \ge \varphi(a,b)$ *for every* $(a,b) \in A \times B$. *This is called the Generalized Fenchel-Young's inequality.*

3. *If* $f^\varphi : B \to I\!R_{+\infty}$ *then* $f^{\varphi\varphi} \le f$.

4. *If* $f^\varphi : B \to I\!R_{+\infty}$ *and* $a \in A$ *we have that*

$$f^{\varphi\varphi}(a) = \sup\{\vartheta(a) : \vartheta \in \Theta,\ \vartheta \le f\},$$

where $\Theta := \{\vartheta : X \to I\!R,\ \vartheta = \varphi(\cdot, b) - r,\ b \in B,\ r \in I\!R\}$.

5. $f^{\varphi\varphi\varphi} \equiv f^\varphi$.

Example: Let $A = X$, $B = X'$ and $\varphi : X \times X' \to I\!R$ be the evaluation function: $(x, x^*) \mapsto x^*(x)$. We are in the case summarized in Section 1.1.

Definition 1.5.2 *Let* H *be a non-empty set of functions* $h : A \to I\!R$ *and functions* $f, g : A \to I\!R_{+\infty}$.

1. f *is called abstract-convex with respect to* H *(*H*-convex) if there exists a set* $U \subset H$ *such that* f *is the upper envelope of this set:*

$$f(a) = \sup\{h(a) : h \in U\}.$$

g *is called abstract-concave with respect to* H *(*H*-concave) if there exists a set* $U \subset H$ *such that* g *is the lower envelope of this set:*

$$g(a) = \inf\{h(a) : h \in U\}.$$

2. *The set*

$$supp_l(f, H) = \{h \in H, h \le f\}$$

of all H*-minorants of* f *is called the lower support set of the function* f *with respect to the set* H.

The set

$$supp^u(g, H) = \{h \in H, h \ge f\}$$

of all H*-majorants of* g *is called the upper support set of the function* g *with respect to the set* H.

3. *The function $co_H f$ defined by*

$$co_H f(x) = \sup\{h(x) : h \in supp_l(f, H)\}, \ x \in X$$

is called the H-convex hull of the function f. Clearly, f is H-convex if and only if $f = co_H f$.

The function $cv_H g$ defined by

$$cv_H g(x) = \inf\{h(x) : h \in supp^u(g, H)\}, \ x \in X$$

is called the H-concave hull of the function g. Clearly, g is H-concave if and only if $g = cv_H g$.

Proposition 1.5.3 *Given a coupling function $\varphi : A \times B \to I\!R$ define H as*

$$H := \{h : A \to I\!R, \ h(a) = \varphi(a, b) - c, \ b \in B, c \in I\!R\}.$$

Then

$$supp_l(f, H) = epi(f^g) \text{ and } supp^u(f, H) = hyp(f_g).$$

Remark: If we consider H as above then Proposition 1.5.2 item 4 affirms what is known as the Fenchel-Moreau theorem

$$f^{\varphi\varphi} = co_H f.$$

1.6 The Equilibrium Problem

In [24], [25] and [39], the Equilibrium Problem is studied:

$$(EP) \text{ Find } x \in K, \text{ such that } f(x, y) \geq 0, \ \forall y \in K,$$

where $K \subset X$ is a non-empty closed convex set and $f : K \times K \to I\!R$ is a function that satisfies:

i) $f(x, x) = 0$, for all $x \in K$.

ii) $f(x, \cdot) : K \to I\!R$ is convex and l.s.c. for all $x \in K$.

iii) $f(\cdot, y) : K \to I\!R$ is u.s.c. for all $y \in K$.

This formulation includes, as particular cases, minimization problems, Nash equilibria problems, complementary problems, fixed point problems, variational inequality problems and vector optimization problems. Before pointing out some of these problems, we need the following definitions (see [1] and [30]):

Definition 1.6.1 *Let $T : X \to 2^{X'}$ a point-to-set mapping.*

i) We say that $x \in X$ belongs to the domain of T, denoted as $D(T)$ if and only if $T(x) \neq \emptyset$.

ii) We say that T is upper semi-continuous at $x_0 \in D(T)$ if, for all $\varepsilon > 0$, there exists V, a neighborhood of x_0, such that $T(x) \subset B_\varepsilon(T(x_0))$ for every $x \in V$. T is an upper semi-continuous mapping if it is upper semi-continuous at every x of X.

For the following, see [24] and references therein:

(a) *Convex minimization problem (CMP)*: Let $h : X \to I\!R_{+\infty}$ be a convex, l.s.c. and proper function. The convex minimization problem is defined as

$$\text{Find } x \in K \text{ such that } h(x) \leq h(y),\ \forall y \in K.$$

If we take $K := \{x \in X : h(x) < +\infty\}$ and $f(x, y) := h(y) - h(x)$ for all $x, y \in K$, then x is a solution of (CMP) if and only if x is a solution of (EP).

(b) *Complementarity problem (CP)*: Let K be a closed convex cone, and let $T : K \to X'$ be a continuous operator. The complementarity problem is defined as

$$\text{Find } x \in K \text{ such that } T(x) \in K^+,\ \langle T(x), x\rangle = 0.$$

If we take $f(x, y) := \langle T(x), y - x\rangle$ for all $x, y \in K$, then x is a solution of (CP) if and only if x is a solution (EP).

(c) *Vector minimization problem (VMP)*: Let $C \subset I\!R^m$ be a closed convex cone, such that both C and its polar cone C^+ have nonempty interior. Consider the partial order in $I\!R^m$ given by

$$x \leq y \text{ if and only if } y - x \in C,$$

$$x < y \text{ if and only if } y - x \in \text{int}(C).$$

A function $F : K \subset I\!R^n \to I\!R^m$ is said to be $C-$convex if and only if K is convex and $F(tx + (1-t)y) \leq tF(x) + (1-t)F(y)$ for all $t \in (0, 1)$. Let $C \subset I\!R^m$ a closed convex cone with nonempty interior, C^+ its polar cone and $F : K \to I\!R^m$ a $C-$convex function (K is closed) such that $\langle z, F(\cdot)\rangle$ is l.s.c. for every $z \in C^+$. The vector minimization problem is defined as

$$\text{Find } \overline{x} \in K \text{ such that } F(x) \not< F(\overline{x}),\ \forall x \in K.$$

If we take $f(x, y) := \max_{\|z\|=1, z\in C^+} \langle z, F(y) - F(x)\rangle$, then x is a solution of (VMP) if and only if x is a solution of (EP).

(d) *Fixed point problem (FPP)*: Assume that X is a Hilbert space with inner product $\langle \cdot, \cdot\rangle$. Let $T : X \to 2^X$ be an upper semi-continuous point-to-set mapping such that $T(x)$ is a nonempty, closed, bounded, convex subset of K for each $x \in K$, where $K \subset X$ is nonempty, convex and closed. The fixed point problem is defined as

$$\text{Find } x \in K \text{ such that } x \in T(x).$$

If we take $f(x, y) := \max_{\varphi\in T(x)} \langle x - \varphi, y - x\rangle$ for all $x, y \in K$, then x is a solution of (FPP) if and only if x is a solution of (EP).

(e) *Variational inequality problem (VIP)*: Let X be a Banach space, $K \subset X$ a nonempty, closed and convex set and $T : K \to 2^{X'}$ a point-to-set mapping such that $T(x)$ is compact for every $x \in K$. The variational inequality problem is defined as

$$\text{Find } x \in K,\ u \in T(x) \text{ such that } \langle u, y - x\rangle \geq 0,\ \forall y \in K.$$

If we take $f(x, y) := \sup_{u\in T(x)} \langle u, y - x\rangle$ for all $x, y \in K$, then x is a solution of (VIP) if and only if x is a solution of (EP).

It is clear now, why it is so important to study this problem.

1.7 Vector optimization and Strongly Star-Shaped cones

In Problems of Vector Optimization, in order to find an equivalent minimization problem, the strongly star-shaped cones play a key role. In this section we will present results and definitions mostly from [35] (unless stated otherwise).

Definition 1.7.1 *[28] Take a binary relation B on X (i.e. $B \subset X \times X$). Denote every $(x, y) \in B$ by $x \succeq y$. We say that $\succeq$ is a pre-order if it satisfies:*

i) (Reflexivity) For every $x \in X$, $x \succeq x$.

ii) (Transitivity) For every $x, y, z \in X$ *such that* $x \succeq y$ *and* $y \succeq z$ *then* $x \succeq z$.

Problems of vector (multi-criteria) optimization arise when there are some different criteria for the choice of a preferable object. When an object x is more preferable than an object y it is often denoted by $x \succeq y$. As a rule it is assumed that the totality of these criteria ($\succeq$) forms a pre-order relation.

Let us present an example (see [1] and references therein).

A non co-operative two persons game: Cournot's Duopoly.

The two players are each manufacturers of the same commodity. In this case, the objective functions are cost functions which depend on the production of the two players. This means that each player will produce an amount of units which depend on the amount of units of the other player. The first player will produce x units while the second one will produce y units (both are non-negative numbers). The problem is to find a vector $(\overline{x}, \overline{y})$ such that there can not exist another vector (x, y) such that $f_1(x, y) < f_1(\overline{x}, \overline{y})$ and $f_2(x, y) < f_2(\overline{x}, \overline{y})$ where f_i is the cost function of player i $(i = 1, 2)$. This means that after both players share information, there should not exist a better pair of numbers (x, y) such that reduces the cost of both of them. Let us consider the case when

$$f_1(x, y) := x(x + y - u), \ f_2(x, y) := y(x + y - u), \ (x, y) \in [0, u] \times [0, u].$$

In this case, we have that the square $[0, u]^2$ is mapped by $f := (f_1, f_2)$ into:

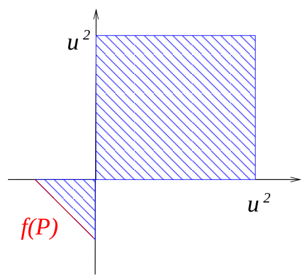

Figure 1.6: The set $f([0, u]^2)$.

From this, it can be seen that any vector of the set

$$P := \left\{(x, y) \in [0, u]^2 : x + y = \frac{u}{2}\right\}$$

is a solution of our problem. If we choose the *strategy* pair

$$x_P := \frac{u}{4}, \qquad y_P := \frac{u}{4},$$

we have that each player will have a loss of $-\frac{u^2}{8}$. Thus, if the manufacturers agree to co-operate, this is a reasonable compromise.

In the previous example, a strategy pair (x_1, y_1) is *more preferable* to the strategy pair (x_2, y_2), $(x_1, y_1) \succeq (x_2, y_2)$, if and only if $f_i(x_1, y_1) \leq f_i(x_2, y_2)$ for $i = 1, 2$. This order relation $\succeq$ is a pre-order relation.

The theory of vector optimization with respect to pre-order relation is well developed [28]. However, often we get preferences that form a relation which is not a pre-order. Let us give some simple examples. Assume that we have $m > 1$ criteria (objective functions) $f_1, \ldots, f_m$ defined on a set X. Each element $x \in X$ can be estimated by the vector $(f_1(x), \ldots, f_m(x))$. Usually it is assumed that x is more preferable than y ($x \succeq y$) if $f_i(x) \geq f_i(y)$ for all $i \in I = \{1, \ldots, m\}$. Clearly $\succeq$ is a pre-order relation. However, sometimes we need different kinds of preferences, which are either weaker or stronger than $\succeq$. For example, let $m > 2$ and $I_1 = \{2, \ldots, m\}$, $I_m = \{1, \ldots, m-1\}$. Consider preferences $\succeq_1$ defined in the following way: $x \succeq_1 y$ if either $f_i(x) \geq f_i(y)$ for $i \in I_1$ or $f_i(x) \geq f_i(y)$ for $i \in I_m$. The preferences $\succeq_1$ are weaker than $\succeq$ (i.e. $x \succeq y$ implies $x \succeq_1 y$) and these preferences are not transitive, so $\succeq_1$ is not a pre-order relation. Consider now another preference $\succeq_2$. We say that $x \succeq_2 y$ if $f_i(x) \geq f_i(y)$ for all $i \in I$ and either $f_1(x) - f_1(y) \geq f_2(x) - f_2(y)$ or $f_2(x) - f_2(y) \geq f_3(x) - f_3(y)$. Clearly $\succeq_2$ is stronger than $\succeq$ and $\succeq_2$ is not a pre-order relation. Both relations $\succeq_i$ ($i = 1, 2$) have the following structure: $x \succeq_i y$ means that the vector $(f_1(x) - f_1(y), \ldots, f_m(x) - f_m(y))$ belongs to a conic set K_i. This set can be represented as the union of two convex cones, however K_i itself is not a convex cone.

In [35] a large class of preferences, namely those that are defined by means of the so-called strongly star-shaped conic sets in a Banach space X, is examined. The simplest example of a strongly star-shaped conic set is the union K of a finite number of closed convex cones K_i ($i \in I$) such that the intersection $\bigcap_{i \in I} \mathrm{int} K_i$ is not empty. Each strongly star-shaped set K determines the relation $\succeq_K$ on X, where $x \succeq_K y$ if and only if $x - y \in K$. If K is not convex, the $\succeq_K$ is not a pre-order relation.

We can also find some approaches to vector optimization duality using a *scalarization* of the relation $\succeq_K$ (in Propositions 5.1 and 5.2 of [35], we can find scalar minimization problems which are equivalent to special cases of vector minimization problems). However for making this scalarization, it is required a function $p_{u,K}$ where u is a particular element of K and K belongs to a sub-family of the strongly star-shaped cones (see Definition 1.7.3 below).

In Chapter 3 we will show that in fact the sub-family used in [35] is actually the whole family of strongly star-shaped cones.

Let us present now the main definitions and results from [35] which are going to be essential for our study.

Let X be a Banach space and $K \subset X$. For each $x \in X$ denote by R_x the ray starting at zero and going through x:

$$R_x := \{\lambda x : \lambda \geq 0\}.$$

Definition 1.7.2 *[34]*

i) K is called star-shaped if there exists a point $u \in K$ such that $\alpha u + (1-\alpha)x \in K$ for all $x \in K$ and $\alpha \in (0,1)$. The set of all points u which possess this property is denoted by $kernK$ (see Figure 1.7).

ii) K is called strongly star-shaped if there exists a point $u \in intK$ such that the ray $u + R_x$ does not intersect the boundary bdK of the set K more than once for each $x \in X$. The set of all points u which enjoy this property is denoted by $kern_* K$ (see Figure 1.8).

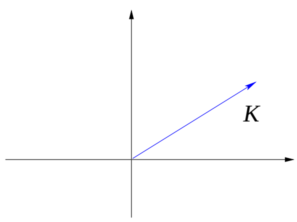

Figure 1.7: Star-shaped set with $\mathrm{kern}K = K$

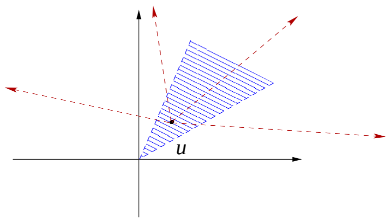

Figure 1.8: Strongly star-shaped set with $\text{kern}_* K = \text{int}K$

Remark: If $u \in \text{kern}_* K$ then for each x, the ray $u + R_x$ either intersects the boundary of K once or does not intersect this boundary. The latter means that $u + R_x \subset \text{int}K$. Every strongly star-shaped set is star-shaped and $\text{kern}_* K \subset \text{kern}K$. Its proof for finite dimensional case can be found for example in [34] (Proposition 5.18), however this proof is valid for an arbitrary Banach space.

Recall that a set $K \subset X$ is called conic if $x \in K$ implies that $(R_x \backslash \{0\}) \subset K$. If K is a conic strongly star-shaped set, it is easy to check that the set $\text{kern}_* K$ is a conic set.

Definition 1.7.3 *Let K be a conic strongly star-shaped set. Denote by $U(K)$ the set of points $u \in K$, which possess the following properties:*

(1) $u \in \text{kern}_* K$;

(2) for each $x \in X$ the line $x + \{\lambda u : \lambda \in I\!R\}$ is not contained in K.

It is easy to check that the set $U(K)$ is conic.

Example 1: If K is a convex cone, $K \neq X$ and $\text{int}K$ is nonempty, then $\text{kern}_* K = U(K) = \text{int}K$.

Example 2: If $K = X$, then $\text{kern}_* K = X$, however $U(K) = \emptyset$.

Example 3: Let us give an example of a strongly star-shaped cone in infinite dimensions. Consider $\mathcal{C}([0,1])$ the set of all continuous functions $f : [0,1] \to I\!R$ with the norm $\|f\| = \sup_x |f(x)|$. In this space, take the set C of all non-negative functions:

$$f \in C \text{ if and only if } f(x) \geq 0, \ \forall x \in [0,1].$$

This set C is a convex cone and therefore its kern$_*$ must be its interior (see Example 1 above). Let us show that its interior is equal to the set of all strictly positive functions:

$$g \in \text{int}C \text{ if and only if } g(x) > 0,\ \forall x \in [0,1].$$

$\Longrightarrow$ Take a strictly positive function f and consider $\varepsilon = \dfrac{\inf f}{2}$. The number ε is not zero since f is a continuous strictly positive function and therefore f attains its minimum in $[0,1]$. We will show that if $g \in \mathcal{C}([0,1])$ is such that $\|f-g\| < \varepsilon$ then g is strictly positive. In fact, $\|f-g\| < \varepsilon$ is equivalent to:

$$|f(x) - g(x)| < \varepsilon,\ \forall x \in [0,1],$$

which in turn is equivalent to

$$f(x) - \varepsilon < g(x) < f(x) + \varepsilon,\ \forall x \in [0,1].$$

Then, taking infimum over x in the left side of the inequality, we have

$$\frac{\inf f}{2} = \inf f - \varepsilon < g(x),\ \forall x \in [0,1].$$

This shows us that $g > \dfrac{\inf f}{2}$ and therefore, g is a strictly positive function.

$\Longleftarrow$ Take a function $f \in$int$C$, so there exists $\varepsilon > 0$ such that $B_\varepsilon(f) \subset C$. Let us assume that there exists $t_0 \in [0,1]$ such that $f(t_0) = 0$. Define the function g as $g := f - \dfrac{\varepsilon}{2}$. This function belongs to $B_\varepsilon(f)$ but is negative on t_0. This contradiction implies that $f > 0$.

Denote by $\mathcal{K}(X)$ the set of all conic closed sets $K \subset X$ with nonempty $U(K)$.

Proposition 1.7.1 *Let $K \in \mathcal{K}(X)$ and $u \in U(K)$. Then*

i) *$\nu x + \lambda u \in K$ for each $x \in K$ and $\lambda > 0, \nu > 0$.*

ii) *For each $x \in X$ there exists $\lambda > 0$ and $\nu \leq 0$ such that $\lambda u - x \in K$ and $x - \nu u \in K$.*

iii) *For each $x \in X$ the set $\Lambda_x = \{\lambda \in I\!R : \lambda u - x \in K\}$ is a closed segment of the form $[\lambda_x, +\infty)$ with $\lambda_x > -\infty$.*

A useful function for determining when a point $u \in X$ belongs to $U(K)$ is the following: let $u \in X$, $K \subset X$ and let $p_{u,K} : X \to \overline{I\!R}$ be defined by

$$p_{u,K}(x) := \inf[\lambda \in I\!R : \lambda u - x \in K], \tag{1.12}$$

where the infimum over the empty set is equal to $+\infty$.

Proposition 1.7.2 *Let $K \in \mathcal{K}(X)$ and $u \in U(K)$. The following are satisfied:*

i) $p_{u,K}(x + \mu u) = p_{u,K}(x) + \mu, \ \forall x \in X, \mu \in I\!R,$

ii) $p_{u,K}(x) \in I\!R$ *for every* $x \in X$ *and*

iii) $p_{u,K}$ *is a continuous positively homogeneous function.*

The converse is also true: consider a continuous positively homogeneous function $p : X \to I\!R$ such that there exists $u \in X$ which satisfies, $p(x + \mu u) = p(x) + \mu$ for every $x \in X$ and $\mu \in I\!R$. Then there exists $K \in \mathcal{K}(X)$ such that $u \in U(K)$ and $p \equiv p_{u,K}$.

Proposition 1.7.3 *Take $u \in X$, $K_i \subset X$ for $i = 1, \ldots, m$. Define $K = \bigcup_{i=1}^{m} K_i$, then*

$$p_{u,K}(x) = \min_i p_{u,K_i}(x), \ \forall x \in X.$$

Remark: It is not difficult to prove that if $K \subset X$ is a cone, so it is K^c. Moreover, the following holds.

Proposition 1.7.4 *Let $K \in \mathcal{K}(X)$ and $u \in U(K)$. Then $p_{-u,CK} = -p_{u,K}$, where $CK := (intK)^c$.*

Proposition 1.7.5 *Let $K_i \in \mathcal{K}(X)$, $i = 1, \ldots, m$ and $\bigcap_{i=1}^{m} U(K_i) \neq \emptyset$. Let $K = \bigcup_{i=1}^{m} K_i$. Then $K \in \mathcal{K}(X)$ and*

$$\bigcap_{i=1}^{m} U(K_i) \subset U(K).$$

(For the proofs of Propositions 1.7.1 - 1.7.4, refer to [35].)

In Chapter 3, we will present a characterization for strongly star-shaped closed conic sets which the interior of their kern$_*$ is nonempty. For doing this, we need to extend to the infinite dimensional setting some results which in [34] are established only for finite dimensional spaces.

We recall below some definitions from [34].

Definition 1.7.4 *A positively homogeneous function $f : I\!R^n \to I\!R$ is called min-sublinear if for each $x \in I\!R^n$ there exists a finite sublinear function p_x such that $p_x(y) \geq f(y)$ for all $y \in I\!R^n$ and $p_x(x) = f(x)$.*

Definition 1.7.5 *Let Q be a conic set in $I\!R^n$. A nonempty set $U \subset Q$ is called a radiant subset of the cone Q if*

$$(x \in U,\ \lambda \in (0,1]) \Longrightarrow \lambda x \in U.$$

Definition 1.7.6 *Let Q be a conic subset of $I\!R^n$ and U a radiant subset of Q. The function $\mu_U : X \to [0, +\infty]$ defined by*

$$\mu_U(x) = \inf\{\lambda > 0 : x \in \lambda U\}$$

is called the Minkowski gauge of the set U (here $\lambda U = \{\lambda x : x \in U\}$).

Remark: It is not difficult to show from this definition that

$$\{x : \mu_U(x) < 1\} \subset U \subset \{x : \mu_U(x) \leq 1\}.$$

Proposition 1.7.6 *A positively homogeneous function $f : I\!R^n \to I\!R$ is min-sublinear if and only if for any $z \in I\!R^n$ there exists a number $k_z > 0$ such that*

$$f(x) - f(z) \leq k_z \|x - z\| \text{ for all } x \in I\!R^n.$$

Proposition 1.7.7 *Let f be a positively homogeneous Lipschitz function defined on $I\!R^n$ with a Lipschitz constant L. Let $S = \{x : \|x\| = 1\}$ be the unit sphere. Then there exists a family $(p_z)_{z \in S}$ of sublinear functions such that:*

1) $f(x) = \min_{z \in S} p_z(x)$ for all $x \in I\!R^n$.

2) $p_z(x) \leq f(z) + L\|x - z\|$ for all $x \in I\!R^n$.

Proposition 1.7.8 *Let U be a radiant subset of $I\!R^n$ and $0 \in int\ kernU$. Then there exist $\varepsilon > 0$ and a family of convex closed sets $(K_u)_{u \in U}$ such that:*

1) $u \in K_u$ and $B(0, \varepsilon) \subset K_u$ for all $u \in U$.

2) $U = \bigcup_{u \in U} K_u$.

Proposition 1.7.9 *Let U be a closed radiant subset of $I\!R^n$. The Minkowski gauge μ_U of the set U is Lipschitz if and only if there exists $\varepsilon > 0$, a set of indices T and a family $(U_t)_{t \in T}$ of convex sets containing the ball $B(0, \varepsilon) = \{x \in I\!R^n : \|x\| \leq \varepsilon\}$ such that*

$$U = cl \bigcup_{t \in T} U_t.$$

Proposition 1.7.10 *Let $U \subset I\!R^n$ be a closed radiant set. Then the Minkowski gauge μ_U of the set U is Lipschitz if and only if $0 \in int\ kernU$.*

(See [34], Propositions 5.13, 5.14, 5.17 and Theorems 5.2 and 5.3 for proofs.)

Chapter 2

G-coupling functions

This chapter is devoted to introducing the G-coupling functions and to presenting many properties which are satisfied when these functions are applied in various optimization problems.

Unless stated otherwise, the definitions, lemmas, theorems and propositions stated in this Chapter are new.

Thanks to Fenchel and Rockafellar, duality schemes for convex minimization problems are very well explained (see [17], [18] and [31]) by means of the *Fenchel conjugate*. As we have seen in Section 1.1.5, the Dual problem is a convex problem.

For the nonconvex case, an interesting theory which can be used is the generalized conjugation (see [34] and [38]).

Our G-coupling functions are motivated by GAP functions. In the examples we are about to consider, GAP functions have many similar properties. We are going to put in evidence these properties.

After introducing the definition of G-coupling functions, we will induce naturally a duality scheme for the minimization problem. This duality scheme will induce a Lagrangian function which can be compared to some Lagrangians stated in Section 1.2.

At the end of this Chapter, we will see that even for the Equilibrium Problem we can use G-coupling functions. Our point of view will generalize the duality scheme found in [25].

2.1 Motivation

In several works already published, there can be found definitions of GAP functions for particular problems. Now we present 3 concrete examples.

- In [8], the Variational Inequality Problem is studied. Let H be a Hilbert space, $T : H \to 2^H$ a point-to-set mapping and $C \subset H$ a nonempty closed convex set:

$$(VIP) \text{ Find } x_0 \in C, \text{ such that, } \exists y^* \in T(x_0) \text{ with } \langle y^*, x - x_0\rangle \geq 0 \ \forall x \in C,$$

  where T is a maximal monotone point-to-set mapping which is defined as follows: it will be said that T is a maximal monotone correspondence if it satisfies that $\langle u - v, x - y\rangle \geq 0$ for every $u \in T(x),\ v \in T(y)$ with $x, y \ \in H$ and if for some pair $(v, y) \in H \times H$ we have that $\langle u - v, x - y\rangle \geq 0$, for all $x \in C$ and for all $u \in T(x)$, then we must have that $v \in T(y)$. The following is given as a GAP function:

$$h_{T,C}(x) := \sup_{(y,v)\in G_C(T)} \langle v, x - y\rangle,$$

  where $G_C(T) = \{(y, v) : \ y \in C, v \in T(y)\}$. This function happens to be non-negative and convex, and it is equal to zero only in solutions of (VIP).

- In [44], the Extended Pre-Variational Inequality Problem is studied:

$$\begin{aligned}(EPVIP): \quad & \text{Find } x_0 \in I\!R^n, \text{ such that} \\ & \langle F(x_0), \eta(x, x_0)\rangle \geq f(x_0) - f(x),\ \forall x \in I\!R^n.\end{aligned}$$

  Where $F : I\!R^n \to I\!R^n$, $\eta : I\!R^n \times I\!R^n \to I\!R^n$ and $f : I\!R^n \to I\!R_{+\infty}$. In this work, the GAP function is

$$\min_{y\in I\!R^n} [\langle F(x), \eta(y, x)\rangle - f(x) + f(y)],$$

  which is non-positive and it only reaches the value zero in solutions of $(EPVIP)$.

- In Section 1.6, we have seen the Equilibrium Problem:

$$(EP) \text{ Find } x \in K, \text{ such that } f(x, y) \geq 0,\ \forall y \in K,$$

  where $K \subset I\!R^n$ is a non-empty closed convex set and $f : K \times K \to I\!R$ is a function that satisfies:

i) $f(x,x) = 0$, for all $x \in K$.

ii) $f(x,\cdot) : K \to I\!R$ is convex and l.s.c.

iii) $f(\cdot,y) : K \to I\!R$ is u.s.c.

In [39] a GAP function is defined:

$$g_1(y) := \begin{cases} \sup_{x \in K} f(x,y), & y \in K \\ +\infty, & \text{otherwise.} \end{cases}$$

In this case, the function g_1 is non-negative, convex and l.s.c. and if it vanishes at x_0, then x_0 is a solution of (EP).

These GAP functions are used to transform an Equilibrium Problem into a minimization problem. The main idea for doing this is to take advantage of all the theory and algorithms developed for the minimization problem. Thus, by using a good algorithm, we can find a solution of a particular Equilibrium Problem.

However, it is important to show that there are some other ideas of GAP functions. In the following examples, these functions have two arguments which can belong to different vector spaces. Therefore, we can consider them as coupling functions. Consider the following 2 examples:

- For the minimization problem, we have seen in Section 1.1.5 that the convex conjugation theory allows us to generate a dual problem. In this approach, there is implicit another concept of gap function. Remember that the following scheme is satisfied under suitable assumptions (following the notation of Section 1.1.5):

$$\begin{array}{rclcrcl} \alpha & = & \inf f(x) \quad (P) & & \beta & = & \inf h^*(u^*) \quad (Q) \\ \varphi(x,0) & = & f(x),\ \forall x \in I\!R^n & & \varphi^*(0,u^*) & = & h^*(u^*),\ \forall u^* \in I\!R^p \\ h(u) & = & \inf_x \varphi(x,u) & & k(x^*) & = & \inf_{u^*} \varphi^*(x^*,u^*) \\ \alpha & = & h(0) & & \beta & = & k(0) \end{array}$$

$$-\beta \leq \alpha.$$

Suppose that there is no duality gap $(-\beta = \alpha)$. Then $\overline{u^*}$ is an optimal solution of (Q) and $\overline{x}$ is an optimal solution of (P) if and only if

$$f(\overline{x}) + h^*(\overline{u^*}) = 0.$$

Let us focus our attention to the function $g_2 : I\!R^n \times I\!R^p \to \overline{I\!R}$ defined by:

$$g_2(x, u^*) = f(x) + h^*(u^*).$$

This function can be seen as a GAP function. It vanishes at (x_0, u_0^*) if and only if x_0 solves the primal problem and u_0^* solves the dual one. In addition, this function is non-negative and if the first variable is kept fixed, the function is convex and l.s.c.

- In [25], the (EP) is consider in the following way:

$$(EP) \text{ Find } x \in K, \text{ such that } f(x, y) \geq 0, \ \forall y \in K,$$

where

i) $K \subset I\!R^n$ is a non-empty convex set.

ii) $f : I\!R^n \times I\!R^n \to \overline{I\!R}$ is a function which satisfies the following properties:

a) $f(x, x) = 0$ for each $x \in K$.

b) $f(x, \cdot) : I\!R^n \to I\!R_{+\infty}$ is convex and l.s.c. for all $x \in K$.

The main differences with this formulation and the one given in Section 1.6 are that K is no longer closed, $f(x, \cdot)$ can take the value $+\infty$ and the upper semi-continuity property is no longer needed for $f(\cdot, y)$. In this work yet another GAP function is implicit:

$$g_3(x, x^*) := f_x^*(x^*) - i_K(x^*).$$

Here, $i_K : I\!R^n \to I\!R_{+\infty}$ is defined by

$$i_K(x^*) := \inf_{x \in K} \langle x^*, x \rangle$$

and $f_x^* : I\!R^n \to I\!R_{+\infty}$ is defined by

$$f_x^*(x^*) := \sup_{y \in I\!R^n} [\langle x^*, y \rangle - f(x, y)].$$

This function g_3 is non-negative and if it vanishes in (x_0, x_0^*) with $x_0 \in K$ then x_0 is a solution of (EP).

These GAP functions give us a relation between the primal variable and the dual variable (in [25], a dual formulation for the Equilibrium Problem can be found). The "gap" between these variables is zero, whenever they are solutions of the primal and dual problems.

Since these GAP functions have two arguments, we can consider them as coupling functions and apply the generalized conjugation theory. This theory is becoming more important in solving non-convex optimization problems.

We would like to make a contribution in this field, by introducing the G-coupling functions.

2.2 Definition

Consider two arbitrary Banach spaces, X and Y.

Definition 2.2.1 *Given $A \subset X$ and $B \subset Y$, a non-negative function $g : A \times B \to I\!R$ will be called a G-coupling function if*

(D1) $\inf_{x \in A,\ y \in B} g(x, y) = 0.$

Define $\mathcal{F}^{A,B} := \{g : A \times B \to I\!R : \ g \text{ is a G-coupling function}\}$.

Not every G-coupling function has zeros:

Example: Take $A = B = I\!R$ and define

$$g(x, y) := \exp(x + y).$$

Then $g \in \mathcal{F}^{A,B}$ is a continuous function and the infimum in $(D1)$ is not attained.

For the next Proposition, we will use some well known facts and definitions concerning the weak topology in a reflexive Banach space (see Chapter 3 of [6], Section 2.5.1 of [9], Section 4.8 of [27], Section 3.10 of [41] and references therein). We list them in the next Definition.

Definition 2.2.2 *Let Z be an arbitrary reflexive Banach space. We say that*

i) A sequence $(z_k)_{k \in I\!N}$ converges weakly to z if and only if $f(z_k)$ converges to $f(z)$ for every $f \in X'$ (X' is the set of all continuous functionals of X).

ii) $A \subset Z$ is a weakly closed set if and only if for every weakly convergent sequence $(a_k)_{k \in I\!N} \subset A$ with z_0 its limit, we have that $z_0 \in A$.

iii) $A \subset Z$ *is a weakly compact set if and only if* A *is weakly closed and bounded.*

iv) *A function* $h : Z \to I\!R$ *is weakly l.s.c. if and only if its level sets* $S_\lambda(h)$ *are weakly closed for every* $\lambda \in I\!R$.

Theorem 2.2.1 *(Eberlein) Let* X *be a Banach space. A set* $A \subset X$ *is weakly compact if and only if* A *is weakly closed and every sequence contained in* A *has a weakly convergent subsequence.*

With these, we have the following.

Proposition 2.2.2 *Let* $A \subset X$*,* $B \subset Y$ *weakly closed sets and* $g \in \mathcal{F}^{A,B}$ *with* X *and* Y *reflexive Banach spaces. The following statements hold:*

i) If g *is weakly l.s.c. on* $A \times B$ *and there exists* $M > 0$ *such that* $S_M(g)$ *is bounded, then*

$$\{(x, y) : g(x, y) = 0\} = \bigcap_{\varepsilon>0} S_\varepsilon(g) \neq \emptyset.$$

ii) In particular, if $\lim_{\|(x,y)\| \to +\infty} g(x, y) = +\infty$*, with* $(x, y) \in A \times B$ *and* g *is weakly l.s.c. (on* $A \times B$*) then the conclusion in i) follows.*

Proof:

i) It is clear that for every $0 \leq \lambda \leq M$ we have that $S_\lambda(g) \subset S_M(g)$ which implies that every $S_\lambda(g)$ is bounded as well. Since g is weakly l.s.c. then $S_\lambda(g)$ is weakly closed (for every $\lambda \in I\!R$). Then, we have a nested net of weakly compact sets $\{S_\lambda(g)\}_{\lambda \in [0,M]}$. Take $(\lambda_k)_{k \in I\!N}$ a non-increasing sequence of non-negative numbers such that $\lambda_k < M$ and $\lambda_k \downarrow 0$. For each $k \in I\!N$ take (x_k, y_k) an element of $S_{\lambda_k}(g)$. Since the sequence of sets $S_{\lambda_k}(g)$ is nested, we have that $(x_k, y_k) \in S_{\lambda_1}(g)$ for every $k \in I\!N$. Therefore, we can take (x_{n_k}, y_{n_k}) a weakly convergent sub-sequence. Let $(\overline{x}, \overline{y})$ be such limit. Since g is weakly l.s.c. we have:

$$g(\overline{x}, \overline{y}) \leq \liminf_k g(x_{n_k}, y_{n_k}). \tag{2.1}$$

On the other hand,

$$g(x_{n_k}, y_{n_k}) \leq \lambda_{n_k}, \ \forall k \in I\!N,$$

hence,

$$\liminf_k g(x_{n_k}, y_{n_k}) \leq \lim_k \lambda_{n_k} = 0. \tag{2.2}$$

Finally, (2.1) together with (2.2) imply that

$$g(\overline{x}, \overline{y}) \leq 0,$$

but since g is a G-coupling function we always have that $0 \leq g(x, y)$ for every $(x, y) \in A \times B$, thus $g(\overline{x}, \overline{y}) = 0$.

ii) Let us show that $\lim_{\|(x,y)\| \to +\infty} g(x, y) = +\infty$ implies that all level sets $S_\varepsilon(g)$ are bounded. Assume that there exists an unbounded level set, namely $S_\alpha(g)$. There must exist $(x_n, y_n)_{n \in \mathbb{N}} \subset S_\alpha(g)$ a sequence such that $\lim_n \|(x_n, y_n)\| = +\infty$ and $g(x_n, y_n) \leq \alpha$ for every n. This contradicts the assumption over g. Since every level set of g is bounded, we are now in the conditions of item i) and the statement follows.

Remark: In item i), we have just proved the well known fact that every weakly l.s.c. function attains its minimum in a weakly compact set. We presented the proof here for completeness.

2.3 The Minimization Problem

Let us turn our attention now to how the family of functions $\mathcal{F}^{A,B}$ will allow us to establish duality schemes for the minimization problem. It is important to point out that in the following we consider an unusual type of duality: $f : A \to \mathbb{R}_{+\infty}$ is kept fixed and $g \in \mathcal{F}^{A,B}$ is variable.

Unless stated otherwise, X and Y will denote arbitrary Banach spaces.

2.3.1 The set $\mathcal{F}_f^{A,B}$

Consider the following problem:

$$(P) : \min_{x \in A} f(x),$$

where $f : A \to \mathbb{R}_{+\infty}$ is a proper function and $A \subset X$ is non-empty. For a given $B \subset Y$ take $g \in \mathcal{F}^{A,B}$. Define $f^g : B \to \mathbb{R}_{+\infty}$ and $f^{gg} : A \to \mathbb{R}_{+\infty}$ as follows (see Section 1.5):

$$f^g(y) := \sup_{x \in A} \{g(x, y) - f(x)\} \ \forall y \in B,$$
$$f^{gg}(x) := \sup_{y \in B} \{g(x, y) - f^g(y)\} \ \forall x \in A.$$

In some cases, it would be better to consider a $g \in \mathcal{F}^{A,B}$ which satisfies:

(D2) B is non-empty closed convex and $g(x,\cdot) : B \to I\!R$ is a convex and l.s.c. function for each x in A.

From the previous definitions, we have the following:

Lemma 2.3.1 *Let $f : A \to I\!R_{+\infty}$ be a proper function and given $B \subset Y$ take $g \in \mathcal{F}^{A,B}$. Then*

$$f^g(y) + f(x) \geq g(x,y) \geq 0, \ \forall (x,y) \in A \times B,$$

which implies

$$f(x) \geq -f^g(y), \ \forall (x,y) \in A \times B.$$

Moreover, if g satisfies $(D2)$, then f^g is a convex l.s.c function.

Proof: From Proposition 1.5.2 item 2, we have that

$$f^g(y) + f(x) \geq g(x,y), \ \forall (x,y) \in A \times B.$$

Since g is non-negative by definition

$$f^g(y) + f(x) \geq g(x,y) \geq 0, \ \forall (x,y) \in A \times B.$$

By using the first part and the last part of this chain of inequalities, we conclude that

$$f(x) \geq -f^g(y), \ \forall (x,y) \in A \times B.$$

If g satisfies $(D2)$, then f^g will be a convex l.s.c function thanks to Propositions 1.1.6 and 1.1.8 which ensure that the supremum of a family of convex l.s.c. functions is in turn a convex l.s.c. function.

Unless it is mentioned, not every $g \in \mathcal{F}^{A,B}$ satisfies $(D2)$.

It would be interesting to know which condition either a G-coupling function g or the function f must satisfy in order to guarantee properness of f^g, because this would provide a non-trivial function related to f, namely f^g. Furthermore, any value of f^g will give us a lower estimate of the optimal value of (P), i.e.

$$-f^g(y) \leq \inf f, \ \text{for all } y \in B.$$

The following lemma ensures the existence of such a function $g \in \mathcal{F}^{A,B}$ for any non-empty $B \subset Y$, taking as a starting point a natural condition on f which must be imposed if we want to find a solution of (P).

Lemma 2.3.2 *Let $f : A \to I\!R_{+\infty}$ be a proper function. Then f is bounded from below if and only if, for every non-empty $B \subset Y$, there exists $g \in \mathcal{F}^{A,B}$ such that f^g is proper.*

Proof:

- Suppose that $\inf f > -\infty$, then for a non-empty $B_0 \subset Y$ fixed, take $y_0 \in B_0$ and consider $g \in \mathcal{F}^{A,B_0}$ as follows:

$$g(x,y) = \|y - y_0\|,$$

thus

$$f^g(y) = \|y - y_0\| - \inf f \quad \forall y \in B_0,$$

which is clearly a proper function and since B_0 was fixed arbitrarily, the result is satisfied for every non-empty $B \subset Y$.

- Take a non-empty set $B_0 \in Y$ and $g \in \mathcal{F}^{A,B_0}$ such that f^g is proper. Let us suppose that $\inf f = -\infty$, from Proposition 1.5.2 item 3 we can see that this implies that $\inf f^{gg} = -\infty$. Then:

$$-\infty = \inf_{x\in A} f^{gg}(x) = \inf_{x\in A}\left(\sup_{y\in B_0}[g(x,y) - f^g(y)]\right).$$

It is well known that for any bi-function $F : A \times B_0 \to \overline{I\!R}$ we have that $\inf_{x\in A}\sup_{y\in B_0} F(x,y) \geq \sup_{y\in B_0}\inf_{x\in A} F(x,y)$ (see [3] or [31]). Let us apply this in the previous equation where for us $F(x,y) = g(x,y) - f^g(y)$:

$$-\infty = \inf_{x\in A}\left(\sup_{y\in B_0}[g(x,y) - f^g(y)]\right) \geq \sup_{y\in B_0}\left(\inf_{x\in A}[g(x,y) - f^g(y)]\right).$$

On the other hand since g is a G-coupling function, it is not difficult to see that $g(x,y) - f^g(y) \geq -f^g(y)$ for every $y \in B_0$, then $\inf_{x\in A}[g(x,y) - f^g(y)] \geq -f^g(y)$. With this, we have

$$\sup_{y\in B_0}\left(\inf_{x\in A}[g(x,y) - f^g(y)]\right) \geq \sup_{y\in B_0}(-f^g(y)) = -\inf_{y\in B_0} f^g(y).$$

Altogether $-\infty \geq -\inf_{y\in B_0}(f^g(y))$ and this is equivalent to $\inf_{y\in B_0} f^g(y) = +\infty$. Thus, f^g is not proper and we have a contradiction. Therefore, $\inf f > -\infty$.

Remark: Notice that this proof also states, in particular, that there exists $g \in \mathcal{F}^{A,B}$ for every non-empty $B \subset Y$ which satisfies $(D2)$ and f^g is proper.

Henceforth, we will consider only functions f such that $\inf f > -\infty$ and for some fixed non-empty $B \subset Y$, $g \in \mathcal{F}^{A,B}$ will be such that f^g is proper.

Let it be $\mathcal{F}^A = \{f : A \to I\!\!R_{+\infty},\ f \text{ is proper},\ \inf f > -\infty\}$ and $\gamma_{g,f} : A \times B \to I\!\!R_{+\infty}$ defined by:

$$\gamma_{g,f}(x,y) = f(x) + f^g(y) \tag{2.3}$$

where $B \subset Y$ is non-empty.

Remark: Observe that $\gamma_{g,f}(x,y) \geq g(x,y) \geq 0$ for every $(x,y) \times A \times B$, but it could take the value $+\infty$ and therefore $\gamma_{g,f}$ might not be in $\mathcal{F}^{A,B}$. Nevertheless, if $\inf \gamma_{g,f} = 0$ then we can determine a Primal problem (P) and a Dual problem (D) as we will see now.

Definition 2.3.1 *Define the set $\mathcal{F}_f^{A,B}$ as follows:*

$$\mathcal{F}_f^{A,B} := \{g \in \mathcal{F}^{A,B} / f^g \text{ is proper and } \inf \gamma_{g,f} = 0\}.$$

Lemma 2.3.3 *Let $f \in \mathcal{F}^A$. Then $\mathcal{F}_f^{A,B}$ is non-empty for every non-empty $B \subset Y$.*

Proof: Given a non-empty $B \subset Y$, take $y_0 \in B$ and define $g \in \mathcal{F}^{A,B}$ by:

$$g(x,y) = \|y - y_0\|.$$

It is easy to check that g belongs to $\mathcal{F}_f^{A,B}$ (this example also proves that functions can be found in $\mathcal{F}_f^{A,B}$ which satisfy $(D2)$).

Recall that we are considering

$$(P) : \min_x f(x)$$

with $f \in \mathcal{F}^A$. Taking $g \in \mathcal{F}_f^{A,B}$, define the dual problem related to g:

$$(D_g) : \min_{y \in B} f^g(y).$$

Proposition 2.3.4 *Take $f \in \mathcal{F}^A$ and $g \in \mathcal{F}_f^{A,B}$ as above. The Duality Gap between (P) and (D_g) is zero.*

Proof: From the definition of $\mathcal{F}_f^{A,B}$, we can write

$$\inf_{(x,y)\in A\times B} \gamma_{g,f}(x,y) = \inf_{x\in A} f(x) + \inf_{y\in B} f^g(y) = 0,$$

which implies

$$\inf_{x\in A} f(x) = -\inf_{y\in B} f^g(y) = \sup_{y\in B}[-f^g(y)].$$

Finally, this equation proves the statement.

We present next a "saddle-point" property for our primal and dual problems.

Theorem 2.3.5 *Let $g \in \mathcal{F}_f^{A,B}$. Then, $\overline{y}$ is a solution of (D_g) and $\overline{x}$ is a solution of (P) if and only if $\gamma_{g,f}(\overline{x},\overline{y}) = 0$.*

Proof: $\overline{x}$ and $\overline{y}$ are solutions of (P) and (D_g) respectively if and only if

$$f(\overline{x}) = \inf f = -\inf f^g = -f^g(\overline{y}) \Longleftrightarrow f(\overline{x}) + f^g(\overline{y}) = \gamma_{g,f}(\overline{x},\overline{y}) = 0. \square$$

Examples: In the following examples, $A, B \subset I\!R$, $f : A \to I\!R \cup \{+\infty\}$ and $g \in \mathcal{F}^{A,B}$.

1) Given a function $f \in \mathcal{F}^A$, not every G-coupling function g belongs to $\mathcal{F}_f^{A,B}$. Take $A = I\!R$, $B = [0, +\infty)$,

$$f(x) = \begin{cases} x^2, & x \geq 0 \\ +\infty, & x < 0 \end{cases}$$

and

$$g(x,y) = \begin{cases} \frac{1}{xy+1}, & x, y \geq 0 \\ 0, & x < 0. \end{cases}$$

Let us calculate f^g:

$$f^g(y) = \sup_{x\in I\!R}\{g(x,y) - f(x)\} = \sup_{x\geq 0}\left\{\frac{1}{xy+1} - x^2\right\},\ y \geq 0,$$

since $g(\cdot, y)$ and $-f(\cdot)$ are both decreasing for x, y non-negative, then

$$f^g(y) = 1,\ y \geq 0.$$

Hence

$$\gamma_{g,f}(x,y) = f(x) + f^g(y) = \begin{cases} x^2 + 1, & x, y \geq 0 \\ +\infty, & x < 0 \end{cases}$$

and since the infimum of $\gamma_{g,f}$ is not zero, then $g \notin \mathcal{F}_f^{A,B}$.

2) Consider $A = B = I\!R$, $f(x) = x^2$ and $g(x, x^*) = (xx^*)^2$. We will generate a dual problem for this simple convex function f which is different from the classical conjugation. Calculate f^g:

$$f^g(y) = \sup_{x \in I\!R} \{(xy)^2 - x^2\} = \begin{cases} 0, & |y| \leq 1 \\ +\infty, & |y| > 1. \end{cases}$$

Then $\gamma_{g,f}$ is given by

$$\gamma_{g,f}(x, y) = f(x) + f^g(y) = \begin{cases} x^2, & |y| \leq 1 \\ +\infty, & |y| > 1. \end{cases}$$

It is clear that $g \in \mathcal{F}_f^{I\!R, I\!R}$. In this example, $f^g \neq f^*$ and (D) has infinite solutions.

3) In this example, we consider a function f which is nonconvex. Let $A = (0, +\infty)$, $B = [0, +\infty)$,

$$f(x) = \begin{cases} \ln x, & x \geq 1 \\ 0, & 0 < x < 1 \end{cases}$$

and $g(x, y) = f(x) + y$ for every $(x, y) \in A \times B$. Then $f^g(y) = y$ for all $y \in B$ and $\gamma_{g,f} \equiv g$ therefore, $g \in \mathcal{F}_f^{A,B}$.

The next theorem states that given $A \times B \subset X \times Y$ (non-empty), the correspondence defined by

$$\begin{array}{rcl} \mathbf{F}: \mathcal{F}^A & \rightrightarrows & \mathcal{F}^{A,B} \\ f & \mapsto & \mathbf{F}(f) = \mathcal{F}_f^{A,B}, \end{array}$$

is an *outer semi-continuous* [9] correspondence (it is also called a *closed* correspondence in [4], [16] and [45]).

Theorem 2.3.6 *Take $A \subset X$ a non-empty set and $B \subset X$ non-empty, closed, convex and $f \in \mathcal{F}^A$. Suppose that exist $f_k : dom(f) \to I\!R$, $g_k : A \times B \to I\!R$, sequences of functions ($k \in I\!N$), such that*

i) f_k converges uniformly to f on $dom(f)$, in other words:

$$\forall \varepsilon > 0, \exists k_0 \in I\!N \text{ such that } |f_k(x) - f(x)| \leq \varepsilon,$$

for each $k \geq k_0$ and $x \in dom(f)$,

ii) $g_k \in \mathcal{F}_{f_k}^{A,B}$ satisfies $(D2)$ for every $k \in I\!N$, *iii)* g_k *converges uniformly to a function* g *in* $A \times B$*, which means that:*

$$\forall \varepsilon > 0, \exists k_1 \in I\!N \text{ such that } |g_k(x,y) - g(x,y)| \le \varepsilon,$$

for every $k \ge k_1$ *and* $(x,y) \in A \times B$.

Then $g \in \mathcal{F}_f^{A,B}$ *and it satisfies* $(D2)$.

Proof: Let us prove first that $g \in \mathcal{F}^{A,B}$. Since g_k converges uniformly to g, given $\varepsilon > 0$, there exists $N \in I\!N$ such that if $k \ge N$ then

$$|g_k(x,y) - g(x,y)| < \varepsilon, \qquad \forall (x,y) \in A \times B.$$

$$\text{Hence } g_k(x,y) - \varepsilon < g(x,y) < g_k(x,y) + \varepsilon, \ \forall (x,y) \in A \times B.$$

Taking $\inf\limits_{x,y}$ (remember that $\inf g_k = 0$ for all $k \in I\!N$):

$$-\varepsilon < \inf_{x,y} g(x,y) < \varepsilon.$$

Then $|\inf g| < \varepsilon$. And since $\varepsilon > 0$ is arbitrary, one has that $\inf g = 0$. This proves that $g \in \mathcal{F}^{A,B}$.

Now we prove that g satisfies $(D2)$. We need to prove that $g(x,\cdot) : B \to I\!R$ is convex and l.s.c. for all $x \in A$. Let $x_0 \in A$ be fixed arbitrarily.

- $g(x_0,\cdot)$ is convex: since for all $k \in I\!N$, $g_k(x_0,\cdot)$ is convex, one has that given $y_1, y_2 \in B$ and $t \in [0,1]$:

  $$g_k(x_0, ty_1 + (1-t)y_2) \le tg_k(x_0,y_1) + (1-t)g_k(x_0,y_2).$$

  Making $k \to +\infty$:

  $$g(x_0, ty_1 + (1-t)y_2) \le tg(x_0,y_1) + (1-t)g(x_0,y_2),$$

  which proves that $g(x_0,\cdot)$ is convex. Note that so far we have not used the uniform convergence of the sequence g_k.

- $g(x_0,\cdot)$ is l.s.c.: fix $y_0 \in B$ and take $\lambda < g(x_0,y_0)$. Since g_k converges uniformly to g in $A \times B$, there exists $N \in I\!N$ such that

  $$|g_N(x,y) - g(x,y)| < \varepsilon, \ \forall (x,y) \in A \times B,$$

  where $\varepsilon = \dfrac{g(x_0,y_0) - \lambda}{2}$. Hence,

  $$\lambda < \lambda + \varepsilon = g(x_0,y_0) - \varepsilon < g_N(x_0,y_0).$$

Since $g_N(x_0, \cdot)$ is l.s.c., then there exists $V(y_0) \subset B$, a neighborhood of y_0, such that if $y \in V(y_0)$ then

$$\lambda + \varepsilon < g_N(x_0, y).$$

Reducing $g(x_0, y)$:

$$\lambda + \varepsilon - g(x_0, y) < g_N(x_0, y) - g(x_0, y) < \varepsilon.$$

Therefore, if $y \in V(y_0)$, then $\lambda < g(x_0, y)$. Thus $g(x_0, \cdot)$ is l.s.c. in $y_0 \in B$, and since y_0 was fixed arbitrarily then $g(x_0, \cdot)$ is a l.s.c. function.

We have proved that for a fixed $x_0 \in A$, $g(x_0, \cdot)$ is a convex l.s.c. function, and since x_0 was fixed arbitrarily we have proved in fact that $g \in \mathcal{F}^{A,B}$ satisfies $(D2)$.

It remains to prove that $g \in \mathcal{F}_f^{A,B}$. For doing this, let us show that $(f_k^{g_k})_{k \in I\!N}$ converges uniformly to f^g (in B).

Let $\varepsilon > 0$ and $N \in I\!N$ such that if $k \geq N$ then

$$|g_k(x, y) - g(x, y)| < \frac{\varepsilon}{4}, \ \forall (x, y) \in A \times B$$

and

$$|f_k(x) - f(x)| < \frac{\varepsilon}{4}, \ \forall x \in dom(f).$$

Fix $k \geq N$ and take $y \in B$ arbitrarily, then

$$f_k^{g_k}(y) - \frac{\varepsilon}{2} < g_k(x', y) - f_k(x'), \text{ for some } x' \in dom(f).$$

Hence

$$f_k^{g_k}(y) - \varepsilon < g_k(x', y) - f_k(x') - \frac{\varepsilon}{2} < g(x', y) - f(x') \leq f^g(y),$$

and so

$$f_k^{g_k}(y) - \varepsilon < f^g(y). \tag{2.4}$$

This proves that $f_k^{g_k}(y) - f^g(y) < \varepsilon$. On the other hand:

$$f^g(y) - \frac{\varepsilon}{2} < g(x'', y) - f(x''), \text{ for some } x'' \in dom(f),$$

whence

$$f^g(y) - \varepsilon < g(x'', y) - f(x'') - \frac{\varepsilon}{2} < g_k(x'', y) - f_k(x'') \leq f_k^{g_k}(y),$$

and so

$$f^g(y) - \varepsilon < f_k^{g_k}(y).$$

This shows that

$$-\varepsilon < f_k^{g_k}(y) - f^g(y). \tag{2.5}$$

Since $y \in B$ was arbitrary, thanks to (2.4) and (2.5) we have that

$$-\varepsilon < f_k^{g_k}(y) - f^g(y) < \varepsilon, \text{ for every } y \in B.$$

This proves that $(f_k^{g_k})_{k \in I\!N}$ converges uniformly to f^g (in B), and it is immediate to see that f^g is proper and

$$0 \leq f(x) + f^g(y) \leq f_k(x) + f_k^{g_k}(y) + \varepsilon, \; \forall (x,y) \in dom(f) \times B,$$

where $\varepsilon > 0$ is arbitrary and k is large enough. Taking $\inf\limits_{(x,y) \in A \times B}$ one has:

$$0 \leq \inf_{(x,y) \in A \times B} (f(x) + f^g(y)) \leq \varepsilon.$$

Therefore $\inf\limits_{(x,y) \in A \times B} (f(x) + f^g(y)) = 0$ and $g \in \mathcal{F}_f^{n,m}$.□

Remark: It is important to note that the parts of the proof marked with • show that if the functions g_k satisfy $(D2)$, so does g. These parts of the proof can be omitted if we are in the situation that the functions g_k converge uniformly to a function g in $A \times B$ but they do not satisfy $(D2)$. We will still conclude that g belongs to $\mathcal{F}_f^{A,B}$ but g will not satisfy $(D2)$ in general.

The following example shows that the assumption of uniform convergence of the functions g_k is necessary in theorem 2.3.6.

Example: In this example we will show a sequence of functions g_k which converges pointwise but not uniformly to a function g and $g \notin \mathcal{F}_f^{A,B}$.
Consider $A = B = Y = X$, $f_k \equiv f$ where $f : X \to I\!R$, $f(x) = \|x\|^2$ and $g_k \in \mathcal{F}^{X,X}$ is defined by

$$g_k(x,y) := \|x\|^2 \|y - ku\|^2$$

where $u \in X \setminus \{0\}$ is fixed arbitrarily.

Let us calculate the function f^{g_k}:

$$f^{g_k}(y) = \sup_x [g_k(x,y) - f(x)] = \sup_x [\|x\|^2 \|y - ku\|^2 - \|x\|^2]$$

$$= \sup_x [\|x\|^2 (\|y - ku\|^2 - 1)].$$

From this we have

$$f^{g_k}(y) = \begin{cases} 0, & \|y - ku\| \leq 1 \\ +\infty, & \|y - ku\| > 1. \end{cases}$$

Therefore

$$\inf_{(x,y) \in X \times X} [f(x) + f^{g_k}(y)] = 0$$

which implies that $g_k \in \mathcal{F}_f^{X,X}$. However, when $k \to +\infty$ the sequence $\{g_k\}_{k \in I\!N}$ converges pointwise to the function

$$g(x,y) = \begin{cases} 0, & x = 0 \\ +\infty, & x \neq 0. \end{cases}$$

Thus $f^g \equiv +\infty$ and $g \notin \mathcal{F}_f^{X,X}$.

Now we will use the epigraphical limit notion, for the case when $A \subset X = I\!R^n$ and $B \subset Y = I\!R^m$. In the following $\stackrel{p}{\longrightarrow}$ will denote pointwise convergence:

$$f_k \stackrel{p}{\longrightarrow} f \text{ if and only if } \lim_{k \to +\infty} f_k(x) = f(x), \ \forall x.$$

Theorem 2.3.7 *Let $\{f_k\}_{k \in I\!N} \subset \mathcal{F}^A$ and $\{g_k\}_{k \in I\!N} \subset \mathcal{F}^{A,B}$ be sequences of functions such that:*

i) $f_{k+1} \leq f_k$ for every $k \in I\!N$, $f_k \stackrel{p}{\longrightarrow} f$ and $f_k \stackrel{e}{\longrightarrow} f$.

ii) $g_{k+1} \geq g_k$ for every $k \in I\!N$, $g_k \stackrel{p}{\longrightarrow} g$ and $g_k \stackrel{e}{\longrightarrow} g$.

iii) $g_k \in \mathcal{F}_{f_k}^{A,B}$ and $g_k(x,\cdot) : B \to I\!R$ is l.s.c. for all $x \in A$ and $k \in I\!N$.

iv) For every $\varepsilon > 0$ there exists $A_1 \subset A$ compact set and $k_1 \in I\!N$ such that

$$\inf_{x \in A_1} f_k(x) \leq \inf_{x \in A} f_k(x) + \varepsilon, \ \forall k \geq k_1.$$

v) There exists $B_1 \subset B$ compact set such that for every $x \in A$ and $k \in I\!N$, there exists $y_k \in B_1$ such that $g_k(x, y_k) = 0$.

Then $g \in \mathcal{F}_f^{A,B}$.

Proof: We divide the proof in five parts.

a) $g \in \mathcal{F}^{A,B}$: Thanks to v), the sequence g_k satisfy Theorem 1.3.2 (page 17), in fact, take $\varepsilon > 0$ and $k \in I\!N$

$$\inf_{(x,y)\in A\times B_1} g_k(x,y) \leq g_k(x', y_k) = 0 \leq \inf_{(x,y)\in A\times B} g_k(x,y) + \varepsilon = \varepsilon,$$

for any $x' \in A$. Therefore $0 = \inf g_k \to \inf g$ which implies that $g \in \mathcal{F}^{A,B}$.

b) $f_k^{g_k} \xrightarrow{p} f^g$: take $y \in B$ then it is clear that

$$f_k^{g_k}(y) \geq g_k(x,y) - f_k(x),\ \forall x \in A,\ k \in I\!N$$

taking $\liminf_k$:

$$\liminf_k f_k^{g_k}(y) \geq g(x,y) - f(x).$$

Taking now $\sup_x$ we have

$$\liminf_k f_k^{g_k}(y) \geq f^g(y). \tag{2.6}$$

On the other hand, take $k \in I\!N$ and $\varepsilon > 0$. There exists x_k^ε such that

$$f_k^{g_k}(y) - \varepsilon < g_k(x_k^\varepsilon, y) - f_k(x_k^\varepsilon).$$

Since $g_k - f_k$ is an increasing sequence we have that $g_k - f_k \leq g - f$ and hence

$$f_k^{g_k}(y) - \varepsilon < g_k(x_k^\varepsilon, y) - f_k(x_k^\varepsilon) < g(x_k^\varepsilon, y) - f(x_k^\varepsilon),$$

which implies that

$$f_k^{g_k}(y) - \varepsilon < f^g(y)$$

and therefore

$$\limsup_k f_k^{g^k}(y) - \varepsilon \leq f^g(y).$$

Since this is valid for any $\varepsilon > 0$, we can conclude that

$$\limsup_k f_k^{g^k}(y) \leq f^g(y). \tag{2.7}$$

From (2.6) and (2.7) we have that $f_k^{g_k} \xrightarrow{p} f^g$.

c) $f_k^{g_k} \xrightarrow{e} f^g$: As we pointed out in b) $g_k - f_k$ is a non-decreasing sequence. Then

$$g_{k+1}(x,y) - f_{k+1}(x) \geq g_k(x,y) - f_k(x),\ \forall k \in I\!N,\ x \in A.$$

Whence

$$f_{k+1}^{g_{k+1}}(y) \geq g_k(x,y) - f_k(x), \ \forall k \in \mathbb{N}, \ x \in A.$$

Taking $\sup\limits_{x\in A}$

$$f_{k+1}^{g_{k+1}}(y) \geq f_k^{g_k}(y), \ \forall k \in \mathbb{N}.$$

This implies that $\{f_k^{g_k}\}$ is a non-decreasing sequence as well and thanks to Proposition 1.3.1 the epigraphical limit exists and it must be equal to $\sup\limits_k[\overline{f_k^{g_k}}]$, but from item iii) the family of functions $g_k(x,\cdot) - f_k(x)$ is l.s.c. for every k and x, hence $f_k^{g_k}$ are l.s.c. for all $k \in \mathbb{N}$. Then

$$f_k^{g_k} \xrightarrow{e} \sup_k f_k^{g_k} = \lim_k f_k^{g_k}.$$

d) $(f_k^{g_k} + f_k) \xrightarrow{e} (f^g + f)$: Thanks to b) and c) and condition i), we can use Theorem 1.3.3 (page 17) to conclude that the affirmation is true.

e) For every $\varepsilon > 0$, there exist $A_1 \subset A$ and $B_1 \subset B$ compact sets such that

$$\inf_{(x,y)\in A_1\times B_1} \gamma_k(x,y) \leq \inf_{(x,y)\in A\times B} \gamma_k(x,y) + \varepsilon,$$

where $\gamma_k := f_k^{g_k} + f_k$. In fact, thanks to iii), $\inf \gamma_k = 0$ for every $k \in \mathbb{N}$, then $\inf f_k^{g_k} = -\inf f_k$. On the other hand, from v) we have that for any $k \in \mathbb{N}$

$$f_k^{g_k}(y_k) \geq \inf_{y\in B_1} f_k^{g_k}(y) \geq \inf_{y\in B} f_k^{g_k}(y) = -\inf_x f_k(x) = f_k^{g_k}(y_k),$$

hence

$$\inf_{y\in B_1} f_k^{g_k}(y) = f_k^{g_k}(y_k) = \inf_{y\in B} f_k^{g_k}(y). \tag{2.8}$$

Thus, thanks to iv), v) and Equation (2.8), we can see that for a given $\varepsilon > 0$ there exist $A_1 \subset A$ compact set and $k_1 \in \mathbb{N}$ such that

$$\inf_{(x,y)\in A_1\times B_1} \gamma_k(x,y) = \inf_{x\in A_1} f_k(x) + \inf_{y\in B_1} f_k^{g_k}(y) \leq \inf_{x\in A} f_k(x) + \inf_y f_k^{g_k}(y) + \varepsilon,$$

whence

$$\inf_{(x,y)\in A_1\times B_1} \gamma_k(x,y) \leq \inf_{(x,y)\in A\times B} \gamma_k(x,y) + \varepsilon,$$

for every $k \geq k_1$. Therefore, the statement follows.

Thanks to e) and Theorem 1.3.2, we can conclude that

$$\inf(f + f^g) = \lim_{k\to+\infty}(\inf(f_k + f_k^{g_k})) = 0. \tag{2.9}$$

Finally, from a) and (2.9) we have that $g \in \mathcal{F}_f^{A,B}$.□

Remark: In [33], Proposition 7.4 items c and d, we have that since the sequence of functions f_k is non-increasing, their epigraphical limit exists and it is equal to $\mathrm{cl}\left[\inf_k f_k\right]$. By making $f = \mathrm{cl}\left[\inf_k f_k\right]$, condition i) can be re-written as follows:

$$f_{k+1} \leq f_k \text{ for every } k \in I\!N \text{ and } f_k \xrightarrow{p} cl[\inf_k f_k].$$

This omission was made to improve readability. The analogous can be said about the sequence g_k in condition ii).

Consider now the case when $A = I\!R^n$ and $B \subset I\!R^m$ is a closed set (non-empty). Define $m(\gamma_{g,f}) := \{(x,y) \in A \times B : \gamma_{g,f}(x,y) = 0\}$, where γ is given by (2.3). From Theorem 2.3.5, (x_0, y_0) belongs to $m(\gamma_{g,f})$ if and only if x_0 is a solution of (P) and y_0 is a solution of (D_g). Take $f \in \mathcal{F}^A$ such that it is a l.s.c. function and $g \in \mathcal{F}_f^{A,B}$. Define the set $R(\gamma_{g,f})$, as follows:

$$R(\gamma_{g,f}) := \bigcap_{(x,y) \in A \times B} \left(S_{\gamma_{g,f}(x,y)}(\gamma_{g,f})\right)^{\infty},$$

where for any set $K \subset A \times B$, K^{∞} is given by (1.9).

Lemma 2.3.8

i) $(m(\gamma_{g,f}))^{\infty} \subset R(\gamma_{g,f})$.

ii) If $m(\gamma_{g,f}) \neq \emptyset$ *then* $R(\gamma_{g,f}) = (m(\gamma_{g,f}))^{\infty}$.

Proof:

i) Since $m(\gamma_{g,f}) = \bigcap_{(x',y') \in A \times B} \left(S_{\gamma_{g,f}(x',y')}(\gamma_{g,f})\right) \subset S_{\gamma_{g,f}(x,y)}(\gamma_{g,f})$ for every $(x,y) \in A \times B$, then by lemma 1.4.2 iii) we have that $(m(\gamma_{g,f}))^{\infty} \subset (S_{\gamma_{g,f}(x,y)}(\gamma_{g,f}))^{\infty}$ for every $(x,y) \in A \times B$ and so $(m(\gamma_{g,f}))^{\infty} \subset R(\gamma_{g,f})$.

ii) If $m(\gamma_{g,f}) \neq \emptyset$ then $S_0(\gamma_{g,f}) = m(\gamma_{g,f}) \neq \emptyset$. So, we have that $R(\gamma_{g,f}) \subset (S_0(\gamma_{g,f}))^{\infty} = (m(\gamma_{g,f}))^{\infty}$. The statement follows from i).

Lemma 2.3.9 *Let* $\{\lambda_k\}_{k \in I\!N}$ *be a sequence such that* $\lim_{k \to +\infty} \lambda_k = 0$ *and* $\lambda_k > 0$ *for each* $k \in I\!N$. *If* $m(\gamma_{g,f}) = \emptyset$, *then* $R(\gamma_{g,f}) = \bigcap_{k \in I\!N} (S_{\lambda_k}(\gamma_{g,f}))^{\infty}$.

Proof: We only need to show that $\bigcap_{k\in I\!N} (S_{\lambda_k}(\gamma_{g,f}))^\infty \subset R(\gamma_{g,f})$. Indeed, take $u \in \bigcap_{k\in I\!N} (S_{\lambda_k}(\gamma_{g,f}))^\infty$, then $u \in (S_{\lambda_k}(\gamma_{g,f}))^\infty$ for all $k \in I\!N$. For every $(x, y) \in A \times B$ (arbitrarily fixed), we have that $0 < \gamma_{g,f}(x, y)$ (because $m(\gamma_{g,f}) = \emptyset$). Since $\lim_{n\to+\infty} \lambda_n = 0$, we have that there exists $q \in I\!N$ such that $\lambda_q \le \gamma_{g,f}(x, y)$. So, $S_{\lambda_q}(\gamma_{g,f}) \subset S_{\gamma_{g,f}(x,y)}(\gamma_{g,f})$. It implies that $(S_{\lambda_q}(\gamma_{g,f}))^\infty \subset (S_{\gamma_{g,f}(x,y)}(\gamma_{g,f}))^\infty$ and so $u \in (S_{\gamma_{g,f}(x,y)}(\gamma_{g,f}))^\infty$. The statement follows.□

Remark: In general, the converse of this lemma is not true. For example, take $A = B = I\!R$, $f(x) = x^2$ and $g(x, y) = x^2 + y^2$. It is not difficult to see that $g \in \mathcal{F}_f^{A,B}$, $f^g(y) = y^2$ and $\gamma_{g,f} \equiv g$. The level sets $S_\lambda(\gamma_{g,f})$ are compact for every $\lambda \in I\!R$, therefore

$$R(\gamma_{g,f}) = \bigcap_{k\in I\!N} (S_{\lambda_k}(\gamma_{g,f}))^\infty = \{(0, 0)\}$$

for every sequence $\{\lambda_k\}_{k\in I\!N}$ such that $\lambda_k \downarrow 0$ and $m(\gamma_{g,f}) = \{(0, 0)\} \neq \emptyset$.

Lemma 2.3.10 *Take $g \in \mathcal{F}_f^{A,B}$ such that $g(x, \cdot)$ is l.s.c. for all $x \in A$. Then $m(\gamma_{g,f}) = \emptyset$ if and only if $(m(\gamma_{g,f}))^\infty \neq R(\gamma_{g,f})$.*

Proof: We will prove that while $(m(\gamma_{g,f}))^\infty$ consists only of the vector zero, the set $R(\gamma_{g,f})$ does not. We divide the proof in two parts:

- If $m(\gamma_{g,f}) = \emptyset$, then $(m(\gamma_{g,f}))^\infty = \{0\}$. Since f and $g(x, \cdot)$ are l.s.c for all $x \in A$ then $S_{\gamma_{g,f}(x,y)}(\gamma_{g,f})$ are closed for all $(x, y) \in A \times B$. If there exists $(x_0, y_0) \in A \times B$ such that $S_{\gamma_{g,f}(x_0,y_0)}(\gamma_{g,f})$ is bounded, then by the Finite Intersection Property we would have $m(\gamma_{g,f}) \neq \emptyset$, a contradiction. Therefore $S_{\gamma_{g,f}(x,y)}(\gamma_{g,f})$ are unbounded for every $(x, y) \in A \times B$. Now, consider $\{\lambda_k\}_{k\in I\!N}$ such that $\lim_{k\to+\infty} \lambda_k = 0$ and $\lambda_k > \lambda_{k+1}$ with $k \in I\!N$. For each $k \in I\!N$ take $u^k \in (S_{\lambda_k}(\gamma_{g,f}))^\infty$ with $\|u^k\| = 1$. Since $\lambda_q \le \lambda_k$ for every $k, q \in I\!N$ such that $q \ge k$, we have that $(S_{\lambda_q}(\gamma_{g,f}))^\infty \subset (S_{\lambda_k}(\gamma_{g,f}))^\infty$, therefore $\{u^q\}_{q\ge k} \subset (S_{\lambda_k}(\gamma_{g,f}))^\infty$ for every $k \in I\!N$ and so any cluster point of $\{u^q\}_{q\in I\!N}$ belongs to $(S_{\lambda_k}(\gamma_{g,f}))^\infty$ for any $k \in I\!N$. Thus, from Lemma 2.3.9 we have that any cluster point of $\{u^k\}_{k\in I\!N}$ belongs to $R(\gamma_{g,f})$. So, the statement follows.

- Item ii) of Lemma 2.3.8 is equivalent to affirm that if $(m(\gamma_{g,f}))^\infty \neq R(\gamma_{g,f})$ then $m(\gamma_{g,f}) = \emptyset$ which is what we wanted to prove.

Theorem 2.3.11 *Take $g \in \mathcal{F}_f^{A,B}$ such that $g(x, \cdot)$ is l.s.c. for all $x \in A$. Then $R(\gamma_{g,f}) = \{0\}$ if and only if $m(\gamma_{g,f})$ is non-empty and compact.*

Proof:

- If $R(\gamma_{g,f}) = \{0\}$ then $m(\gamma_{g,f})$ is non-empty and compact. Indeed, since f and $g(x,\cdot)$ are l.s.c for every $x \in A$, we have that $\gamma_{g,f}$ is l.s.c. on $A \times B$. Since $R(\gamma_{g,f}) = \{0\}$, then from Lemma 2.3.8 item i) $(m(\gamma_{g,f}))^\infty = \{0\}$. So, from Lemma 2.3.10 we have $m(\gamma_{g,f}) \neq \emptyset$, here $m(\gamma_{g,f})$ is closed (thanks to the lower semi-continuity of $\gamma_{g,f}$). The statement follows applying Lemma 1.4.2 item ii).

- If $m(\gamma_g, f)$ is non-empty and compact, we are in the conditions of item ii) of Lemma 2.3.8 and the conclusion follows using the latter result and the fact that $m(\gamma_{g,f})^\infty = \{(0,0)\} \neq \emptyset$.

In the literature, for the minimization problem, it is commonly found that for a point x_0 such that $f(x_0) = \inf f$, this point is called *optimal point*, while the value $\inf f$ if it is not $-\infty$ is called *optimal value*. At this point a natural question arises, if $A \times B \subset X \times Y$ is non-empty and $g \in \mathcal{F}_f^{A,B}$, for a given $f \in \mathcal{F}^A$, would be there any kind of relation between the optimal points and the optimal values of f and f^{gg}? The next lemma answers this.

Lemma 2.3.12 *For a fixed $A \times B \subset X \times Y$, $f \in \mathcal{F}^A$ and every $g \in \mathcal{F}_f^{A,B}$, the following are satisfied:*

i) $\inf f = \inf f^{gg}$,

ii) if x_0 is a global minimum of f, then x_0 is a global minimum of f^{gg}.

Proof: Remember that f^{gg} is defined by:

$$f^{gg}(x) = \sup_{y \in B}\{g(x,y) - f^g(y)\}.$$

i) $\inf f^{gg} \leq \inf f$ is always true. On the other hand

$$f^g(y) + f^{gg}(x) \geq g(x,y) \geq 0,\ \forall (x,y) \in A \times B,$$

which implies that

$$\inf_{x \in A} f^{gg}(x) \geq -\inf_{y \in B} f^g(y).$$

Since $g \in \mathcal{F}_f^{A,B}$ one has that

$$\inf_{x \in A} f(x) = -\inf_{y \in B} f^g(y),$$

which means

$$\inf f \leq \inf f^{gg} \leq \inf f.$$

Therefore $\inf f = \inf f^{gg}$.

ii) $f^{gg}(x_0) \leq f(x_0) = \inf f = \inf f^{gg} \leq f^{gg}(x_0)$, then $f^{gg}(x_0) = \inf f^{gg}$.

Remark: In [34], Proposition 7.14, states the same result, but assuming that the coupling function has a zero. In this lemma we do not need that assumption and because of this, the proof of item i) is different that the one in [34].

2.4 Generalized Lagrangians

We have seen that the functions in $\mathcal{F}_f^{A,B}$, for a given function $f \in \mathcal{F}^A$ can give us an interesting duality scheme. In this section, we will show that in this scheme there is implicit the notion of a Lagrangian function.

Take $f \in \mathcal{F}^A$ and $g \in \mathcal{F}_f^{A,B}$. Recall that

$$(D_g): \min_{y \in B} f^g(y)$$

is the dual problem of (P) related to g. Define $L_1 : A \times B \to I\!R_{+\infty}$, as follows

$$L_1(x,y) = f(x) - g(x,y).$$

This function has some interesting properties:

Theorem 2.4.1

$$\sup_{y \in B} \inf_{x \in A} L_1(x,y) = \inf_{x \in A} \sup_{y \in B} L_1(x,y).$$

Proof: The inequality $\sup_{y \in B} \inf_{x \in A} L_1(x,y) \leq \inf_{x \in A} \sup_{y \in B} L_1(x,y)$ is always true. For the opposite, since g is non-negative we have:

$$f(x) \geq f(x) - g(x,y) = L_1(x,y), \ \forall (x,y) \in A \times B,$$

then

$$f(x) \geq \sup_{y \in B} L_1(x,y) \geq \inf_{z \in A} \sup_{y \in B} L_1(z,y), \ \forall x \in A.$$

It follows that

$$\inf_{x \in A} \sup_{y \in B} L_1(x,y) \leq \inf f.$$

But, since $g \in \mathcal{F}_f^{A,B}$, we have that

$$\inf f = -\inf_{y \in B} f^g(y) = -\left(\inf_{y \in B} \left\{ \sup_{x \in A} [g(x,y) - f(x)] \right\} \right)$$

$$\implies \inf f = \sup_{y\in B}\inf_{x\in A} L_1(x,y),$$

which means,

$$\inf_{x\in A}\sup_{y\in B} L_1(x,y) \le \sup_{y\in B}\inf_{x\in A} L_1(x,y).$$

Finally,

$$\sup_{y\in B}\inf_{x\in A} L_1(x,y) = \inf_{x\in A}\sup_{y\in B} L_1(x,y).\square$$

We are interested now in which properties are satisfied for every saddle-point of L_1. Remember that $(x_0, y_0) \in A\times B$ is a saddle point of L_1 if and only if

$$L_1(x_0,y) \le L_1(x_0,y_0) \le L_1(x,y_0),\ \forall (x,y)\in A\times B.$$

Proposition 2.4.2 *Let $f \in \mathcal{F}^A$, $g \in \mathcal{F}_f^{A,B}$ and L_1 be as before, if there exists $(x_0,y_0)\in A\times B$ saddle point of L_1, then:*

i) $x_0 \in dom(f)$.

ii) y_0 *is an optimal solution of* (D_g),

iii) $f^{gg}(x_0) = f(x_0)$.

Proof:

i) This is immediate thanks to the definition of saddle point. In fact, if $x_0 \notin dom(f)$ then $L_1(x_0,y_0) = +\infty$ and this will imply that

$$L_1(x_0,y_0) > L_1(x,y_0),\ \forall x\in dom(f)\subset A,$$

this last inequality is a contradiction to the definition of saddle point.

ii) From the previous theorem and the definition of saddle point, we have that

$$L_1(x_0,y_0) = \sup_{y\in B}\inf_{x\in A} L_1(x,y) = \inf_{x\in A}\sup_{y\in B} L(x,y).$$

But

$$\begin{aligned}\sup_{y\in B}\inf_{x\in A} L_1(x,y) &= \sup_{y\in B}\inf_{x\in A}[f(x)-g(x,y)]\\ &= \sup_{y\in B}\left[-\sup_{x\in A}[g(x,y)-f(x)]\right]\\ &= \sup_{y\in B}[-f^g(y)]\\ &= -\inf_{y\in B} f^g(y).\end{aligned}$$

Moreover

$$L_1(x_0, y_0) = \inf_{x \in A} L_1(x, y_0) = -f^g(y_0).$$

Altogether we have that

$$f^g(y_0) = \inf_{y \in B} f^g(y).$$

iii) $f^{gg}(x_0) = \sup_{y \in B}[g(x_0, y) - f^g(y)] = \sup_{y \in B}\left[g(x_0, y) - \sup_{z \in A}[g(z, y) - f(z)]\right]$. Which means,

$$f^{gg}(x_0) = \sup_{y \in B} \inf_{z \in A}[g(x_0, y) - g(z, y) + f(z)] = \sup_{y \in B} \inf_{z \in A}[g(x_0, y) + L_1(z, y)].$$

This implies

$$f^{gg}(x_0) \geq \inf_{z \in A}[g(x_0, y_0) + L_1(z, y_0)] = g(x_0, y_0) + \inf_{z \in A} L_1(z, y_0),$$

but since (x_0, y_0) is a saddle point of L_1, then $\inf_{z \in A} L_1(z, y_0) = L_1(x_0, y_0)$. With this, we have that

$$f^{gg}(x_0) \geq g(x_0, y_0) + L_1(x_0, y_0) = f(x_0),$$

which means $f^{gg}(x_0) \geq f(x_0)$. Since $f^{gg}(x) \leq f(x)$ is always true for every $x \in A$ (see Proposition 1.5.2 item 3.), the conclusion follows.

Proposition 2.4.3 *If x_0 is a solution of (P) and y_0 is a solution of (D_g), then (x_0, y_0) is a saddle point of L_1.*

Proof: Since x_0 and y_0 are solutions of (P) and (D_g) respectively, we have that $f(x_0) = -f^g(y_0)$ which implies that

$$0 \leq g(x_0, y_0) \leq f(x_0) + f^g(y_0) = 0$$

and thus $g(x_0, y_0) = 0$. Now take $y \in B$ (fixed arbitrarily)

$$L_1(x_0, y) = f(x_0) - g(x_0, y) \leq f(x_0) = f(x_0) - g(x_0, y_0) = L_1(x_0, y_0).$$

Since $y \in B$ was fixed arbitrarily, we can conclude that

$$L_1(x_0, y) \leq L_1(x_0, y_0), \ \forall y \in B. \tag{2.10}$$

On the other hand

$$\inf_{x \in A} L_1(x, y_0) = \inf_{x \in A}[f(x) - g(x, y_0)] = -f^g(y_0) = f(x_0) = L_1(x_0, y_0). \tag{2.11}$$

Thanks to (2.10) and (2.11) the conclusion follows.□

In Proposition 2.4.2 we would like to improve the fact that, in general, for every saddle point $(x_0, y_0) \in A \times B$ of L_1 we have that $f^{gg}(x_0) = f(x_0)$. For doing this, we impose an additional condition on g.

Proposition 2.4.4 *Let $g \in \mathcal{F}_f^{A,B}$ be such that $\inf_{y\in B} g(x,y) = 0$ for every $x \in A$. The following are equivalent:*

i) (x_0, y_0) is a saddle-point of L.

ii) x_0 is a solution of (P) and y_0 is a solution of (D_g).

Proof: The implication ii) ⇒ i) is true thanks to the previous Proposition.

Consider now (x_0, y_0) a saddle-point of L_1, then

$$L_1(x_0, y) \leq L_1(x_0, y_0),\ \forall y \in B,$$

which is equivalent to

$$f(x_0) - g(x_0, y) \leq f(x_0) - g(x_0, y_0),\ \forall y \in B.$$

In turn, this is equivalent to

$$g(x_0, y_0) \leq g(x_0, y),\ \forall y \in B.$$

The above expression yields

$$g(x_0, y_0) = \inf_{y\in B} g(x_0, y) = 0.$$

On the other hand

$$L_1(x_0, y_0) \leq L_1(x, y_0),\ \forall x \in A.$$

This implies that

$$f(x_0) \leq f(x) - g(x, y_0),\ \forall x \in A$$

(remember that $g(x_0, y_0) = 0$). Taking $\inf_{x\in A}$ we have

$$f(x_0) \leq -f^g(y_0).$$

And since $0 \leq g(x,y) \leq f(x) + f^g(y)$ is always true for every $(x,y) \in A \times B$ we have that $f(x_0) = -f^g(y_0)$. This implies that x_0 is a solution of (P) and y_0 is a solution of (D_g).

Remark: To prove that there exists a $g \in \mathcal{F}_f^{A,B}$ such that $\inf_{y\in B} g(x,y) = 0$ for every $x \in A$ just consider $g \equiv 0$ or $g(x,y) = \|y\|$.

2.5 Lagrange-type functions

In this section we consider the following:

$$(CP): \min_{x\in A} f(x),$$

where

$$A := \{x \in X : \ h_i(x) \leq 0, \ \forall i = 1, \ldots, m\},$$

$f \in \mathcal{F}^A, \ h_i : X \to I\!R,$ are arbitrary functions for every $i = 1, \ldots, m$ and $X = I\!R^n$.

Lagrangian-type functions are studied, for instance, in [36]. Remember that in Section 1.2 we have seen that these functions can be induced by a set of functions **K** which is defined as follows: a function $\chi : I\!R^m \times \Omega \to I\!R$ belongs to **K** if and only if

i) $\chi(\cdot, \omega)$ is l.s.c. for all $\omega \in \Omega$.

ii) $\sup_{\omega\in\Omega} \chi(v, \omega) = 0$ for all $v \in I\!R^m_-$.

The Lagrangian-type function is defined as follows: $L_\chi : I\!R^n \times \Omega \to I\!R$

$$L_\chi(x, \omega) := f(x) + \chi(h(x), \omega).$$

Now, in this Section, we introduce the set $\mathcal{H}^{C,B}$. The main difference with the set $\mathcal{F}_f^{A,B}$ is the fact that these G-coupling functions will depend directly to the constraints and not the objective function. With these coupling functions we will also generate Lagrange-type functions.

Motivation: Classical Lagrangian Duality

Let

$$(CP): \min_{x\in A} f(x)$$

be a typical minimization problem, where in this case,

$$A := \{x \in X : \ h_i(x) \leq 0, \ \forall i = 1, \ldots, m\},$$

$$f : A \to I\!R, \ h_i : X \to I\!R, \text{ are convex l.s.c functions with } i = 1, \ldots, m.$$

The classical Lagrangian is defined as follows:

$$L(x, y) := f(x) + \langle h(x), y\rangle.$$

Remember that (see [3] and [14]) the following is the well known dual problem:

$$(D_L): \min_{y\in I\!R^m_+} \sup_{x\in A}\{\langle y, -h(x)\rangle - f(x)\},$$

$h(x) = (h_1(x), \ldots, h_m(x))$. Moreover, x_0 is a solution of (CP) and y_0 is a solution of (D_L) if and only if (x_0, y_0) is a saddle point of the Lagrangian function L which means,

$$L(x_0, y) \leq L(x_0, y_0) \leq L(x, y_0),\ \forall x \in A,\ \forall y \in I\!R^m_+.$$

Define now $g : B \times B \to I\!R$, as follows:

$$g(z, y) := \langle z, y\rangle, \tag{2.12}$$

where $B = I\!R^m_+ \subset Y = I\!R^m$. We can see that $g \in \mathcal{F}^{B,B}$ satisfies the following property:

$$\inf_{y\in B} g(z, y) = g(z, 0) = 0,\ \text{for every } z \in B.$$

Let

$$g_h(x, y) = g(-h(x), y), \tag{2.13}$$

then $g_h \in \mathcal{F}^{A,B}$. Let us calculate $f^{g_h}(y)$:

$$f^{g_h}(y) = \sup_{x\in A}[g_h(x, y) - f(x)] = \sup_{x\in A}[\langle -h(x), y\rangle - f(x)],$$

which means

$$f^{g_h}(y) = \sup_{x\in A}\{\langle -h(x), y\rangle - f(x)\},\ \text{for every } y \in B = I\!R^m_+$$

and thus the classical Lagrangian duality is recovered. Furthermore, it is immediate to see that $f^{g_h}(0) = \sup_{x\in A}\{-f(x)\}$ which implies that $g_h \in \mathcal{F}^{A,B}_f$.

The set $\mathcal{H}^{C,B}$

We focus our attention on the general case, i.e. $f \in \mathcal{F}^A$ and $h_i : X \to I\!R$ $(i = 1, \ldots, m)$ are not necessarily convex functions.

We have seen, in equation (2.12), that there exist $q \in I\!N$ and $g \in \mathcal{F}^{C,B}$, with $C \subset I\!R^m$ and $B \subset I\!R^q$ which satisfies:

$$(A) \quad \inf_{y\in B} g(z, y) = 0,\ \text{for every } z \in C.$$

Furthermore the function g_h defined in (2.13) satisfies:

$$(B) \quad g_h(x, y) := g(-h(x), y),\ x \in A,\ y \in B \text{ belongs to } \mathcal{F}^{A,B}_f.$$

In fact we considered the case $n = m$ and $C = B$. From (A), g_h satisfies a similar property

$$\inf_{y \in B} g_2(x, y) = 0, \text{ for every } x \in A.$$

Let it be $\mathcal{H}^{C,B} \subset \mathcal{F}^{C,B}$ the set of every $g \in \mathcal{F}^{C,B}$ which satisfies (A) and its associated function g_h satisfies (B). We say that $\mathcal{H}^{C,B}$ are the G-coupling functions related to the constraints.

Given $g \in \mathcal{H}^{C,B}$ we define

$$(D_{g_h}) : \min_{y \in B} f^{g_h}(y),$$

with

$$f^{g_h}(y) = \sup_{x \in A}[g_h(x, y) - f(x)] = \sup_{x \in A}[g(-h(x), y) - f(x)].$$

Thanks to (B), this problem is well defined (there is no duality gap).

Theorem 2.5.1 *Let $g \in \mathcal{H}^{C,B}$. Then (x_0, y_0) is a saddle point of*

$$L_2(x, y) := f(x) - g(-h(x), y), \; x \in A, \; y \in B$$

if and only if x_0 is a solution of (CP) and y_0 is a solution of (D_{g_h}).

Proof: Suppose that (x_0, y_0) is a saddle point of L_2. Thanks to Proposition 2.4.2, we have that $x_0 \in dom(f) \subset A$ and y_0 is a solution of (D_{g_h}). We only need to prove now that x_0 is a solution of (CP), for doing this it is enough to see that since (x_0, y_0) is a saddle point of L_2, this implies:

$$L_2(x_0, y) \leq L_2(x_0, y_0), \; \forall y \in B$$

which is equivalent to

$$f(x_0) - g(-h(x_0), y) \leq f(x_0) - g(-h(x_0), y_0), \; \forall y \in B.$$

Then

$$g(-h(x_0), y_0) \leq g(-h(x_0), y), \; \forall y \in B.$$

And thus,

$$g(-h(x_0), y_0) = \inf_{y \in B} g(-h(x_0), y).$$

Thanks to (A), $\inf_{y \in B} g(-h(x_0), y) = 0$. Using (B), we can say that

$$\inf f = L_2(x_0, y_0)$$

(check the proof of Proposition 2.4.2 item ii) and therefore

$$\inf f = L_2(x_0, y_0) = f(x_0) - g(-h(x_0), y_0) = f(x_0).$$

Altogether x_0 is a solution of (CP).

On the other hand, let x_0 be a solution of (CP) and y_0 a solution of (D_{g_h}), we have then:

$$f(x_0) + f^{g_h}(y_0) = 0,$$

but

$$0 \leq g(-h(x_0), y_0) \leq f(x_0) + f^{g_h}(y_0) = 0,$$

then $g(-h(x_0), y_0) = 0$. Finally

$$L_2(x_0, y) = f(x_0) - g(-h(x_0), y) \leq f(x_0) = f(x_0) - g(-h(x_0), y_0) = L_2(x_0, y_0) =$$

$$-f^{g_h}(y_0) = \inf_{x \in A} L_2(x, y_0) \leq L_2(x, y_0).$$

This means

$$L_2(x_0, y) \leq L_2(x_0, y_0) \leq L_2(x, y_0), \ \forall x \in A, \ \forall y \in B. \square$$

We can re-formulate this result such that we can put in evidence *Kuhn-Tucker* conditions:

Lemma 2.5.2 *Take $g \in \mathcal{H}^{C,B}$ as before. Then (x_0, y_0) is a saddle point of*

$$L_2(x, y) := f(x) - g(-h(x), y), \ x \in A, \ y \in B$$

if and only if the following conditions are satisfied:

i) $\inf[f(x) - g(-h(x), y_0) : \ x \in A] < +\infty$.

ii) $g(-h(x_0), y_0) = 0$.

iii) $x_0 \in \arg\min[f(x) - g(-h(x), y_0) : \ x \in A]$.

Remark: The dual feasibility is implied by conditions ii) and iii) together, since thanks to them $-f^g(y_0) = f(x_0) \in I\!R$.

It is easy to check that every $g \in \mathcal{F}_h^{C,\Omega}$ which satisfies that $g(\cdot, \omega)$ is u.s.c. for all $\omega \in \Omega$, can give us a function in $\mathbf{K}$ defined by

$$\chi(v, \omega) := -g(-v, \omega).$$

Recall the following sets:

$$\mathcal{L}_\chi^+(\omega) := \{(u, v) \in I\!R^{1+m} : u + \chi(v, \omega) \geq 0\},$$

$$\mathcal{T}_\eta(A) := \{(f(x) - \eta, h(x)) : x \in A\}$$

and

$$\mathcal{H}^- := \{(u, v) \in I\!R^{1+m} : u < 0,\ v \leq 0\}.$$

We are considering $\mathcal{T}_\eta(A)$ instead of $\mathcal{T}_\eta$, because we are interested only in the set A.

The following result establishes necessary and sufficient conditions for saddle points in the context of G-coupling functions (recall Proposition 1.2.4).

Proposition 2.5.3 *Let χ be defined in terms of a function $\psi \in \mathcal{F}_h^{C,\Omega}$ (i.e. $\chi(v, \omega) := -\psi(-v, \omega)$). Then $(x_*, \omega_*) \in A \times \Omega$ is a saddle point of L_χ if and only if*

$$\mathcal{T}_\eta(A) \subset \mathcal{L}_\chi^+(\omega_*),$$

where $\eta = f(x_*)$ and

$$L_\chi(x, \omega) := f(x) + \chi(h(x), \omega),\ (x, \omega) \in A \times \Omega.$$

Proof:

$\Leftarrow$ Suppose that there exists $(x_*, \omega_*) \in A \times \Omega$ such that $\mathcal{T}_\eta(A) \subset \mathcal{L}_\chi^+(\omega_*)$, with $\eta = f(x_*)$. Then

$$f(x) - \eta + \chi(h(x), \omega_*) \geq 0, \text{ for all } x \in A. \tag{2.14}$$

Using $x = x_*$ in this inequality we have that $\chi(h(x_*), \omega_*) \geq 0$. Since $\chi(v, \omega) = -\psi(-v, \omega)$ and ψ is a G-coupling function, then $\chi(h(x), \omega) \leq 0$ for every $(x, \omega) \in A \times \Omega$, thus $\chi(h(x_*), \omega_*) = 0$. With this, (2.14) can be written as follows

$$L_2(x, \omega_*) = f(x) + \chi(h(x), \omega_*) \geq \eta = f(x_*) = L_2(x_*, \omega_*),\ \forall x \in A. \tag{2.15}$$

On the other hand, $\chi(h(x_*), \omega_*) = 0$ also implies that

$$f(x_*) + \chi(h(x_*), \omega) \leq f(x_*) = f(x_*) + \chi(h(x_*), \omega_*),\ \forall \omega \in \Omega$$

which is equivalent to

$$L_2(x_*,\omega) \le L_2(x_*,\omega_*),\ \forall \omega \in \Omega. \tag{2.16}$$

Thanks to (2.15) and (2.16) the statement follows.

$\Rightarrow$ Suppose that $(x_*,\omega_*) \in I\!R^n \times \Omega$ is a saddle point of L_χ. From Lemma 2.5.2 we have that $x_* \in A$, $\chi(h(x_*),\omega_*) = 0$ and

$$x_* \in \arg\min\{f(x) + \chi(h(x),\omega_*) :\ x \in A\}.$$

Therefore $f(x_*) + \chi(h(x_*),\omega_*) = f(x_*) \le f(x) + \chi(h(x),\omega_*)$ for every $x \in A$. Making $\eta = f(x_*) \in I\!R$ we have that

$$f(x) - \eta + \chi(h(x),\omega_*) \ge 0 \text{ for all } x \in A$$

which is equivalent to say that $\mathcal{T}_\eta(A) \subset \mathcal{L}^+_\chi(\omega_*)$.□

Example: Let $p \in I\!N$ and define $\psi \in \mathcal{F}^{C,\Omega}_h$ as follows: $\Omega := (I\!R^m_+)^{1+p}$ $(C = I\!R^m_+)$ and

$$\psi(z,y) := \min(\langle z, y_0\rangle, \dots, \langle z, y_p\rangle),\ (z,y) \in C \times \Omega. \tag{2.17}$$

Then L_2 is given by:

$$L_2(x,y) := f(x) - \psi(-h(x),y) = f(x) + \max(\langle h(x), y_0\rangle, \dots, \langle h(x), y_p\rangle),$$

for every $(x,y) \in A \times \Omega$.

This example defines a G-coupling function which allows us to recover the Lagrangian function defined in (1.8). Propositions 1.2.5, 1.2.6 and Theorem 1.2.7 remain valid under this setting.

In the following Theorem, given a (CP), we will fix a function $g \in \mathcal{H}^{C,B}$ and we will point out what kind of behavior there is around the functions f and f^{g_h} when we approach the function f by a family of functions which will be represented by a bi-function F.

Theorem 2.5.4 *Consider (CP). Let $q \in I\!N$, $C \times B \subset I\!R^m \times I\!R^q$ and $g \in \mathcal{H}^{C,B}$. Take a function $F : I\!R^n \times I\!R^p \to I\!R_{+\infty}$ which satisfies:*

- *There exists $z_0 \in I\!R^p$ such that for every $\varepsilon > 0$ we can find $\delta > 0$ satisfying $|F(x,z) - f(x)| < \varepsilon$ for all $x \in A$ and $z \in B(z_0, \delta)$.*

Then the following are true:

i) $\lim_{z \to z_0} \left(\inf_{x \in A} F(x,z) \right) = \inf_{x \in A} f(x)$.

ii) $\lim_{z \to z_0} \left(\inf_{y \in B} F^{g_h}(y,z) \right) = \inf_{y \in B} f^{g_h}(y)$.

Where

$$F^{g_h}(y,z) := \sup_{x \in A} [g(-h(x), y) - F(x,z)].$$

Proof:

i) Take $\varepsilon > 0$ then there exists $\delta > 0$ such that

$$f(x) - \varepsilon < F(x,z) < f(x) + \varepsilon, \ \forall x \in A, \ z \in B(z_0, \delta). \tag{2.18}$$

The first inequality of (2.18) implies

$$\inf_{y \in A} f(y) - \varepsilon < F(x,z), \ \forall x \in A, \ z \in B(z_0, \delta),$$

hence

$$\inf_{y \in A} f(y) - \varepsilon \leq \inf_{x \in A} F(x,z), \ \forall z \in B(z_0, \delta).$$

Taking $\liminf_{z \to z_0}$

$$\inf_{y \in A} f(y) - \varepsilon \leq \liminf_{z \to z_0} \left(\inf_{x \in A} F(x,z) \right). \tag{2.19}$$

From the second inequality of (2.18) we have

$$\inf_{y \in A} F(y,z) < f(x) + \varepsilon, \ \forall x \in A, \ z \in B(z_0, \delta),$$

thus

$$\inf_{y \in A} F(y,z) \leq \inf_{x \in A} f(x) + \varepsilon, \ \forall z \in B(z_0, \delta).$$

Taking $\limsup_{z \to z_0}$

$$\limsup_{z \to z_0} \left(\inf_{y \in A} F(y,z) \right) \leq \inf_{x \in A} f(x) + \varepsilon. \tag{2.20}$$

Since $\varepsilon > 0$ was chosen arbitrarily making $\varepsilon \to 0$ in (2.19) and (2.20) we have

$$\inf_{x \in A} f(x) \leq \liminf_{z \to z_0} \left(\inf_{x \in A} F(x,z) \right) \leq \limsup_{z \to z_0} \left(\inf_{y \in A} F(y,z) \right) \leq \inf_{x \in A} f(x).$$

Hence

$$\lim_{z \to z_0} \left(\inf_{x \in A} F(x,z) \right) = \inf_{x \in A} f(x).$$

ii) Take $\varepsilon > 0$. From (2.18) we have for every $(x, y) \in A \times B$ and $z \in B(z_0, \delta)$:

$$\begin{aligned}(g(-h(x), y) - f(x)) - \varepsilon &< (g(-h(x), y) - F(x, z)) \\ &< (g(-h(x), y) - f(x)) + \varepsilon. \end{aligned} \tag{2.21}$$

The first inequality of (2.21) gives us

$$(g(-h(x), y) - f(x)) - \varepsilon < F^{g_h}(y, z), \ \forall (x, y) \in A \times B, \ z \in B(z_0, \delta),$$

taking $\sup\limits_{x \in A}$

$$f^{g_h}(y) - \varepsilon < F^{g_h}(y, z), \ \forall y \in B, \ z \in B(z_0, \delta). \tag{2.22}$$

In a similar way we can prove the following

$$F^{g_h}(y, z) < f^{g_h}(y) + \varepsilon, \ \forall y \in B, \ z \in B(z_0, \delta). \tag{2.23}$$

(2.22) and (2.23) together gives us

$$f^{g_h}(y) - \varepsilon < F^{g_h}(y, z) < f^{g_h}(y) + \varepsilon, \ \forall y \in B, \ z \in B(z_0, \delta).$$

This last chain of inequalities is similar to (2.18), therefore, repeating the steps in item i) we will have

$$\lim_{z \to z_0} \left(\inf_{y \in B} F^{g_h}(y, z) \right) = \inf_{y \in B} f^{g_h}(y). \square$$

Corollary 2.5.1 *Consider (CP) and take $g \in \mathcal{H}^{C,B}$. Take a sequence of functions $\{f_k\}_{k \in \mathbb{N}}$ such that*

- *For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_k(x) - f(x)| < \varepsilon$ for all $x \in A$ and $k \geq N$.*

Then the following are true:

i) $\lim\limits_{k \to +\infty} \left(\inf\limits_{x \in A} f_k(x) \right) = \inf\limits_{x} f(x).$

ii) $\lim\limits_{k \to +\infty} \left(\inf\limits_{y \in B} f_k^{g_h}(y) \right) = \inf\limits_{y \in B} f^{g_h}(y).$

Where

$$f_k^{g_h}(y) := \sup_{x \in A} [g(-h(x), y) - f_k(x)].$$

Proof: Define $J := \{z \in [0,1] : z = 1/k, \text{ for some } k \in \mathbb{N}\}$. Use Theorem 2.5.4 with $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}_{+\infty}$ defined by

$$F(x,z) = f_{\frac{1}{z}}(x), \ \forall x \in \mathbb{R}^n, \ z \in J,$$

and $z_0 = 0$.

Example: The following sequence of functions satisfies conditions i) and ii) of Corollary 2.5.1:

$$f_k(x) := f(x) + k \max\{0, h_1(x), \ldots, h_m(x)\}.$$

2.6 The Equilibrium Problem

Let X be a Banach space. Take $f : X \times X \to \overline{\mathbb{R}}$ amd $K \subset X$ a non-empty closed convex set. Assume that

i) $f(x,\cdot) : K \to \overline{\mathbb{R}}$ is a convex l.s.c. function for every $x \in K$.

ii) $f(x,x) = 0, \ \forall x \in K$.

The Equilibrium Problem is defined as follows:

$$(EP): \ \text{Find } x \in K \text{ such that } f(x,z) \geq 0, \forall z \in K.$$

Let us define the sets F and $\widetilde{F}$:

$$F := \left\{ x \in K : \ \inf_{z \in K} f(x,z) \neq -\infty \right\}, \tag{2.24}$$

$$\widetilde{F} := \left\{ x \in K : \ \inf_{z \in K} f(x,z) = 0 \right\}. \tag{2.25}$$

Notice that if F is empty, then the (EP) will have no solutions, therefore, F represents the set of feasible points of problem (EP). It is immediate to see that $\widetilde{F} \subset F$ and $\widetilde{F}$ is the set of solutions of (EP).

Take now $B \subset Y$ with Y an arbitrary Banach space and $g \in \mathcal{F}^{K,B}$. Consider now for every $x \in K$:

$$(P^x): \ \inf_{z \in K} f(x,z) \qquad\qquad (D_g^x): \ \inf_{y \in B} f_x^g(y)$$

where $f_x^g(y) = \sup_{z \in K}\{g(z,y) - f(x,z)\}$ for every $y \in B$.

Where it is needed, we will use the following notation for the function $f(x, \cdot)$ for a given x:

$$f_x(y) := f(x,y), \ \forall y.$$

Lemma 2.6.1 *Consider (EP) and $g \in \mathcal{F}^{K,B}$ as above. If there exists $\overline{x} \in K$ such that $\inf_y f_{\overline{x}}^g(y) = 0$ then $\overline{x} \in \widetilde{F}$.*

Proof: Let us assume that $\overline{x} \in K$ but $\overline{x} \notin \widetilde{F}$. Since $f(x,x) = 0$ for all $x \in K$, then $\inf_{z \in K} f(\overline{x}, z)$ is always non-positive. Hence, there must exist $\overline{z} \in K$ such that $f(\overline{x}, \overline{z}) < -\varepsilon$ for some $\varepsilon > 0$. On the other hand,

$$f_{\overline{x}}^g(y) = \sup_{z \in K}[g(z,y) - f(\overline{x}, z)] \geq g(\overline{z}, y) - f(\overline{x}, \overline{z}) > g(\overline{z}, y) + \varepsilon.$$

This implies that

$$f_{\overline{x}}^g(y) > \varepsilon, \ \forall y \in B$$

and therefore $\inf f_{\overline{x}}^g \neq 0$, a contradiction. We must have that $\overline{x} \in \widetilde{F}$.

Example: Let us put in evidence that the converse of Lemma 2.6.1 is not true. Take $K = [0, +\infty)$, $f(x,z) = z^2 - x^2$ and $g \in \mathcal{F}^{K,K}$ equals to $g(z,y) = \frac{1}{zy+1}$. It is clear that $\overline{x} = 0$ is a solution of (EP). On the other hand,

$$f_{\overline{x}}^g(y) = \sup_{z \in K}[g(z,y) - f(\overline{x}, z)] = \sup_{z \geq 0}\left[\frac{1}{zy+1} - z^2\right], \ \forall y \geq 0.$$

Thus,

$$f_{\overline{x}}^g(y) = 1, \ \forall y \geq 0.$$

Therefore $\inf f_{\overline{x}}^g > 0$.

Lemma 2.6.1 motivates the following Definition.

Definition 2.6.1 *Consider (EP) as above and let $\overline{x} \in K$. Given Y an arbitrary Banach space, $B \subset Y$ and a $g \in \mathcal{F}^{K,B}$, we will say that g satisfies the* Zero Duality Gap Property *(ZDGP) if the following are equivalent:*

i) $\overline{x} \in \widetilde{F}$.

ii) $\inf_{y \in B} f_{\overline{x}}^g(y) = 0$.

The next Lemma tells us that there exist G-coupling functions which satisfy the ZDGP.

Lemma 2.6.2 *Let $B \subset Y$, $0 \in B$ and Y be an arbitrary Banach space. Take $g \in \mathcal{F}^{K,B}$ such that $g(z,0) = 0$, for every $z \in K$. Then g satisfies the ZDGP.*

Proof: Let us calculate $f_x^g(y)$, with $x \in F$, $y \in B$:

$$f_x^g(y) = \sup_{z \in K}[g(z,y) - f(x,z)].$$

It is clear that

$$\inf_{y \in B} f_x^g(y) \leq f_x^g(0) = \sup_{z \in K}[g(z,0) - f(x,z)] = -\inf_{z \in K} f(x,z). \tag{2.26}$$

On the other hand, for every $x \in F$, $y \in B$ and $z \in K$ one has that $f_x^g(y) + f(x,z) \geq 0$, which implies that we always have

$$\inf_{y \in B} f_x^g(y) \geq -\inf_{z \in K} f(x,z). \tag{2.27}$$

Therefore, if $x \in \widetilde{F}$

$$\inf_{y \in B} f_x^g(y) = -\inf_{z \in K} f(x,z) = 0 \Longrightarrow \inf_{y \in B} f_x^g(y) = 0.$$

Hence, there exists $g \in \mathcal{F}^{K,B}$ which satisfies the ZDGP.

Example: Thanks to Lemma 2.6.2, if $Y = X'$, $B = K^+$ and $g(z,y) = y(z)$ then g satisfies the ZDGP.

Let us give now a function g that will generate a duality scheme which has been already studied in [25]. It is important to mention that the authors of [25] work with a function f which satisfies that $f(x,\cdot) : X \to \overline{I\!R}$ is convex and l.s.c. for every $x \in K$. Since our function f satisfies a weaker assumption we need to show that despite this, their results remain valid in this setting. For sake of completeness we will include their proofs when necessary.

Consider X and arbitrary reflexive Banach space and $Y = X'$. Let $i_K : X' \to \overline{I\!R}$ be defined by $i_K(y) := \inf_{x \in K} \langle y, x \rangle$, where $\langle y, x \rangle := y(x)$ and $K^* := \{y \in I\!R^n : \ i_K(y) > -\infty\}$ is the effective domain of i_K. (Since i_K is a concave u.s.c function, then K^* is a closed convex set.) Take $g \in \mathcal{F}^{K,K^*}$ defined by:

$$g(z,y) = \langle y, z \rangle - i_K(y). \tag{2.28}$$

Clearly, $g(z,0) = 0$ for every $z \in K$. Calculate now $f_x^g(y)$ for $x \in F$:

$$\begin{aligned} f_x^g(y) &= \sup_{z\in K}[g(z,y) - f(x,z)] \\ &= \sup_{z\in K}[\langle y,z\rangle - i_K(y) - f(x,z)] \\ &= \sup_{z\in K}[\langle y,z\rangle - f(x,z)] - i_K(y). \end{aligned}$$

Proposition 2.6.3 *The function g defined in* (2.28) *satisfies the ZDGP.*

Proof: This function g satisfies the conditions of Lemma 2.6.2, therefore the statement follows.

Theorem 2.6.4 *[25] Assume that $dom(f(\overline{x},\cdot)) \cap intK \neq \emptyset$ for every $\overline{x} \in \widetilde{F}$. Then if $\overline{x} \in \widetilde{F}$ then there exists $y^* \in K^*$ such that $\sup_{z\in K}[\langle y^*,z\rangle - f(\overline{x},z)] - i_K(y^*) = 0$. Conversely, if there exist $\overline{x} \in F$ and $y^* \in K^*$ such that $\sup_{z\in K}[\langle y^*,z\rangle - f(\overline{x},z)] - i_K(y^*) = 0$, then $\overline{x}$ is a solution of (EP).*

This result says not only that $\inf_{y\in K^*} f_{\overline{x}}^g(y) = 0$, but also that the dual problem $(D_g^{\overline{x}})$ has a solution. In order to recover this Theorem in our context of G-coupling functions, we need the following lemma.

Lemma 2.6.5 *For every $y \in K^*$ and $x \in F$, one has:*

$$\sup_{z\in K}[\langle y,z\rangle - f(x,z)] - i_K(y) \geq 0.$$

Proof: We have for every $x \in F$ and $(z,y) \in K \times K^*$ fixed

$$i_K(y) - \sup_{a\in K}[\langle y,a\rangle - f(x,a)] \leq \langle y,z\rangle - \sup_{a\in K}[\langle y,a\rangle - f(x,a)] \leq f(x,z).$$

Then,

$$i_K(y) - \sup_{a\in K}[\langle y,a\rangle - f(x,a)] \leq f(x,z),\ \forall (z,y) \in K \times K^*.$$

Taking $\inf_{z\in K}$:

$$i_K(y) - \sup_{a\in K}[\langle y,a\rangle - f(x,a)] \leq \inf_{z\in K} f(x,z) \leq 0,$$

where the last inequality occurs, since $\inf_{z\in K} f(x,z) \leq f(x,x) = 0$. This means, $\sup_{z\in K}[\langle y,z\rangle - f(x,z)] - i_K(y) \geq 0$.

Before presenting the proof of Theorem 2.6.4, we need the notion of normal cone.

Definition 2.6.2 *Given a convex set $C \subset X$ and $x \in C$, we say that $x^* \in X'$ is normal to C at x if and only if*

$$0 \geq \langle x^*, z - x \rangle, \ \forall z \in C.$$

The set of all vectors which are normal to C at x will be denoted by $N_C(x)$ and it will be called the normal cone of C at x. If $x \notin C$ we say that $N_C(x) = \emptyset$.

Proof of Theorem 2.6.4:

- If $\overline{x} \in \widetilde{F}$, then
$$f_{\overline{x}}(\overline{x}) = 0 = \min_{z \in K} f(\overline{x}, z).$$
Then, using Theorem 3.5.7 of [9], there exists $y \in (-N_K(\overline{x}))$ such that $f_{\overline{x}}(\overline{x}) + f^*_{\overline{x}}(y) = \langle y, \overline{x} \rangle$. And thus
$$\sup_{z \in K}[\langle y, z \rangle - f(\overline{x}, z)] \leq f^*_{\overline{x}}(y) = f_{\overline{x}}(\overline{x}) + f^*_{\overline{x}}(y) = \langle y, \overline{x} \rangle = i_K(y),$$
which means $\sup_{z \in K}[\langle y, z \rangle - f(\overline{x}, z)] - i_K(y) \leq 0$ (here $f^*_{\overline{x}}$ stands for the Fenchel conjugate of the function $f^*_{\overline{x}}$, see Definition 1.1.8). But thanks to the previous lemma, this implies that $\sup_{z \in K}[\langle y, z \rangle - f(\overline{x}, z)] - i_K(y) = 0$.

- Take $z_0 \in K$ (arbitrarily fixed) and suppose that there exist $\overline{x} \in F$ and $y^* \in K^*$ such that $\sup_{z \in K}[\langle y^*, z \rangle - f(\overline{x}, z)] = i_K(y^*)$, then
$$f(\overline{x}, z_0) \geq \langle y^*, z_0 \rangle - \sup_{z \in K}[\langle y^*, z \rangle - f(x, z)] = \langle y^*, z_0 \rangle - i_K(y^*) \geq 0.$$
And thus, since $z_0 \in K$ was fixed arbitrarily, $f(\overline{x}, z_0) \geq 0$ for every $z \in K$. Thus, $\overline{x}$ belongs to $\widetilde{F}$.□

Remark: In Theorem 3.5.7 of [9] the concept of *subdifferential* is used. For simplicity and readability we have not included this definition.

Definition 2.6.3 *[25] For $\varepsilon > 0$, we say that $x_\varepsilon \in K$ is an ε-solution to (EP) if it satisfies:*

$$f(x_\varepsilon, z) \geq -\varepsilon, \quad \textit{for all } z \in K.$$

The following Theorem presents a relationship between ε-solutions to (EP) and the optimal value of (D^x_g).

Theorem 2.6.6 *Consider (EP). Let $\varepsilon > 0$ and $g \in \mathcal{F}^{K,B}$ be a function which satisfies the ZDGP. If there exists $x \in F$ such that $\inf_{y \in B} f_x^g(y) \leq \varepsilon$ then $x \in K$ is an ε-solution to (EP). If in addition, $g(z,0) = 0$ for every $z \in K$, then the converse is also true.*

Proof: Suppose that for a given $x \in F$, $\inf_{y \in B} f_x^g(y) \leq \varepsilon$. We have that

$$g(z,y) - f(x,z) \geq -f(x,z), \ \forall y \in B, z \in K.$$

Then

$$\sup_{z' \in K} [g(z',y) - f(x,z')] \geq -f(x,z),$$

hence

$$\sup_{z' \in K} [g(z',y) - f(x,z')] \geq \sup_{z \in K} [-f(x,z)].$$

Taking $\inf_{y \in B}$:

$$\inf_{y \in B} f_x^g(y) \geq \sup_{z \in K} [-f(x,z)].$$

Therefore

$$\varepsilon \geq \inf_{y \in B} f_x^g(y) \geq \sup_{z \in K} [-f(x,z)],$$

which implies

$$\inf_{z \in K} f(x,z) \geq -\varepsilon$$

and x is an ε-solution to (EP).

For the converse, let x_ε be an ε-solution to (EP). Since $g(z,0) = 0$ for every $z \in K$, Equation (2.26) is valid and since Equation (2.27) is always true, we have that

$$\inf_{y \in B} f_{x_\varepsilon}^g(y) = -\inf_{z \in K} f(x_\varepsilon, z). \tag{2.29}$$

On the other hand, since x_ε is an ε-solution to (EP), we have that $\inf_{z \in K} f(x_\varepsilon, z) \geq -\varepsilon$ and therefore $-\inf_{z \in K} f(x_\varepsilon, z) \leq \varepsilon$. This together with Equation (2.29) imply that

$$\inf_{y \in B} f_{x_\varepsilon}^g(y) \leq \varepsilon. \square$$

2.6.1 The Complementarity Problem

We define the Complementarity Problem as follows:

$$(CP): \text{ Find } x \in K \text{ such that } T(x) \in K^+ \text{ and } \langle T(x), x \rangle = 0,$$

where $K \subset \mathbb{R}^n$ is a closed convex cone and $T : K \to \mathbb{R}^n$ is a continuous function. The following lemma tells us that this problem is a particular case of the (EP) (see [24] and references therein).

Lemma 2.6.7 *Consider (CP) as above. Define $f(x,z) = \langle T(x), z-x\rangle$ with x and z in K. The solution set of the (CP) is equal to the solution set of (EP) related to f.*

Proof: Take x' a solution of (CP), then $x' \in K$, $Tx' \in K^+$ and $\langle Tx', x'\rangle = 0$. On the other hand, for any $z \in K$ we have that

$$f(x', z) = \langle Tx', z - x'\rangle = \langle Tx', z\rangle \geq 0.$$

Whence, x' is a solution to (EP).
Consider now x' as a solution to (EP) for f. Then $f(x', z) \geq 0$, for every $z \in K$. This implies that

$$\langle Tx', z\rangle \geq \langle Tx', x'\rangle, \ \forall z \in K. \tag{2.30}$$

By making $z = 0$, we have that $\langle Tx', x'\rangle \leq 0$. By making now $z = 2x'$ in (2.30), we have

$$\langle Tx', 2x' - x'\rangle \geq 0,$$

which implies

$$\langle Tx', x'\rangle \geq 0.$$

This together with (2.30) we have that $\langle Tx', x'\rangle = 0$. But this, in (2.30), implies that $\langle Tx', z\rangle \geq 0$ for every $z \in K$. Therefore, if x' is a solution to (EP) then $\langle Tx', x'\rangle = 0$ and $Tx' \in K^+$ which is equivalent to say that x' is a solution to (CP).$\square$

Let us take f as in the previous lemma and $g \in \mathcal{F}^{K,K^+}$ defined as:

$$g(z, y) = \langle y, z\rangle.$$

Calculate $f_x^g(y)$ $(x \in K, \ y \in K^+)$:

$$f_x^g(y) = \sup_{z\in K}[g(z,y) - f(x,z)]$$

$$\Rightarrow f_x^g(y) = \sup_{z\in K}[\langle y - T(x), z\rangle] + \langle T(x), x\rangle$$

$$\Rightarrow f_x^g(y) = \begin{cases} \langle T(x), x\rangle & y - T(x) \in K^- \\ +\infty & \text{otherwise,} \end{cases}$$

where $K^- := \{w \in I\!\!R^n : \langle w, x\rangle \leq 0, \ \forall x \in K\}$. But $y - T(x) \in K^-$ is equivalent to the statement that

$$x \in T^{-1}(y + K^+) \subset T^{-1}(K^+),$$

and this inclusion is true since K^+ is a closed convex cone and $y \in K^+$. Then

$$f^g_x(y) = \begin{cases} \langle T(x), x\rangle & x \in T^{-1}(y + K^+) \\ +\infty & \text{otherwise.} \end{cases}$$

Therefore,

$$f(x, z) + f^g_x(y) = \begin{cases} \langle T(x), z\rangle & x \in T^{-1}(y + K^+),\ z \in K \\ +\infty & \text{otherwise.} \end{cases}$$

Calculate now the set F given by equation (2.24):

- If $x \in K$ is such that $T(x) \in K^+$ then

  $$\inf_{z\in K} f(x, z) = \inf_{z\in K}\langle T(x), z - x\rangle = -\langle T(x), x\rangle \neq -\infty,$$

  which means $x \in F$.

- If $x \in K$ is such that $T(x) \notin K^+$ then there exists $z_0 \in K$ satisfying $\langle T(x), z_0\rangle < 0$. Thus

  $$\lim_{n\to+\infty} f(x, nz_0) = -\infty = \inf_{z\in K} f(x, z),$$

  which means $x \notin F$.

All these imply that $F = T^{-1}(K^+)$.

Since g satisfies ZDGP (g satisfies the conditions of Lemma 2.6.2), $\overline{x} \in K$ is a solution of (EP) (related to f) if and only if (Definition 2.6.1)

$$\inf_{y\in K^+} f^g_{\overline{x}}(y) = \inf_{y\in K^+}[\langle T(\overline{x}), \overline{x}\rangle : \overline{x} \in T^{-1}(y + K^+)] = 0$$

which in turn is equivalent to

$$T(\overline{x}) \in K^+ \text{ and } \langle T(\overline{x}), \overline{x}\rangle = 0. \tag{2.31}$$

If we look closely to the function $f^g_x : K^+ \to I\!R_{+\infty}$, we will see that this function is non-negative for every $x \in F$. Therefore, finding a solution of (EP) is equivalent to minimize $\langle T(x), x\rangle$ subject to $x \in F$ ($T(x) \in K^+$ and $x \in K$).

In [13] the (CP) is considered, when $K = K^+ = I\!R^n_+$ and T is an affine operator, in other words, the case of the Linear Complementarity Problem (LCP). It is studied in the following way.

$\overline{x}$ is a solution of (LCP), if and only if $\overline{x}$ solves:

$$\text{minimize } \langle T(x), x\rangle, \text{ subject to } T(x) \in I\!R^n_+, \ x \in I\!R^n_+, \text{ and } \langle T(x), x\rangle = 0.$$

It is immediate to see that this equivalent problem is identical to (2.31), therefore by using this $g \in \mathcal{F}^{K,K^+}$ (the one used at the beginning of this section) we can generate a dual problem of (LCP) which has been treated in [13].

2.7 Fenchel's Duality and abstract convexity

In [18] (see Theorem 1.1.13), a special duality was discovered. This duality relates the minimization of the convex function $f - h$ (f is a convex function and h is a concave function) and the maximization of the concave function $h_* - f^*$ (see Definitions 1.1.8 and 1.1.10), where the functions f and h are defined in a finite dimensional vector space.

Fenchel's duality is central to the study of constrained optimization. The key condition in Fenchel's duality is the following

$$ri(dom(f)) \cap ri(dom(h)) \neq \emptyset. \tag{2.32}$$

In some works ([10], [11] and [26]), instead of working with the function $f - h$, they make the change of $-h = \widetilde{h}$ and they work with the function $f + \widetilde{h}$ where f and $\widetilde{h}$ are proper l.s.c. convex functions. Under this setting, we have:

$$(P) : \min_{x \in X}[f(x) + \widetilde{h}(x)]$$

and

$$(D) : \max_{x^* \in X'}[-f^*(x^*) - \widetilde{h}^*(-x^*)].$$

Fenchel's condition, (2.32), remains valid in the following sense:

If (2.32) *is satisfied with* $h = -\widetilde{h}$ *then*

$$\inf_{x \in X}[f(x) + \widetilde{h}(x)] = \max_{x^* \in X'}[-f^*(x^*) - \widetilde{h}^*(-x^*)].$$

In [10] and [11] a new condition which is not an interior point condition is found.

Theorem 2.7.1 *([11], Theorem 3.2) (Stable Fenchel-Rockafellar Duality) Let* $f, \widetilde{h} : X \to I\!R_{+\infty}$ *be proper and l.s.c. convex functions such that* $dom(f) \cap dom(\widetilde{h}) \neq \emptyset$*. Then the following statements are equivalent:*

i) $\inf_{x\in X}[f(x) + \widetilde{h}(x) + x^*(x)] = \max_{v\in X'}[-f^*(v - x^*) - \widetilde{h}^*(-v)], \ \forall x^* \in X'.$

ii) $epi(f^*) + epi(\widetilde{h}^*)$ *is weak* closed.*

Corollary 2.7.1 *([11], Corollary 3.2) (Generalized Fenchel-Rockafellar Duality) Let* $f, \widetilde{h} : X \to I\!R_{+\infty}$ *be proper and l.s.c. convex functions such that* $dom(f) \cap dom(\widetilde{h}) \neq \emptyset$*. If the set* $epi(f^*) + epi(\widetilde{h}^*)$ *is weak* closed, then*

$$\inf_{x\in X}[f(x) + \widetilde{h}(x)] = \max_{v\in X'}[-f^*(v) - \widetilde{h}^*(-v)].$$

Remark: Corollary 2.7.1 can be found as well in [10], Corollary 3.

The authors of [26] generalize the results above. The latter work considers a conjugation which uses a different coupling function, namely the one induced by an additive set of functions. The inner product in finite dimensional spaces and the evaluation function for Banach spaces ($\varphi(x, x^*) = x^*(x)$, for every $x \in X$ and $x^* \in X'$) are particular cases of additive set of functions. Corollary 5.2 of [26] extends Corollary 2.7.1 to this more general setting.

Let us mention now our generalized version of Fenchel's Duality.

Take $f : X \to I\!R_{+\infty}$, $h : X \to I\!R_{-\infty}$ two arbitrary functions and $g : X \times Y \to I\!R$ a coupling function (X and Y are two arbitrary Banach spaces).

Recall an important definition given in Section 1.5:

Abstract convex and concave functions: We say that f is abstract-convex if and only if $f \equiv f^{gg}$. We say that h is abstract-concave if and only if $h \equiv h_{gg}$.

Consider

$$(P) : \min_{x\in X}[f(x) - h(x)]$$

and

$$(D) : \max_{y\in Y}[h_g(y) - f^g(y)].$$

The following is always true:

$$\alpha = \inf_{x \in X}(f(x) - h(x)) \geq \sup_{y \in Y}(h_g(y) - f^g(y)) = \beta. \tag{2.33}$$

The following theorem tells us where we should look for a condition such that $\alpha = \beta$.

Theorem 2.7.2 *Let f be an abstract-convex function and h an abstract-concave function (with respect to g).*

i) If $\alpha \in I\!R$ and $supp_l(f, H) \cap supp^u(h + \alpha, H) \neq \emptyset$ then $\alpha = \beta$ and (D) has a solution.

ii) If $\beta \in I\!R$ and $supp_l(f^g, H) \cap supp^u(h_g - \beta, H) \neq \emptyset$ then $\alpha = \beta$ and (P) has a solution.

Proof:

i) $supp_l(f, H) \cap supp^u(h + \alpha, H) \neq \emptyset$ if and only if there exists $(y_0, \mu_0) \in Y \times I\!R$ such that

$$f(x) \geq g(x, y_0) - \mu_0 \geq h(x) + \alpha,\ \forall x \in X.$$

The first inequality implies $\mu_0 \geq f^g(y_0)$ and the second one implies $h_g(y_0) \geq \mu_0 + \alpha$. From this

$$h_g(y_0) \geq f^g(y_0) + \alpha,$$

which implies, together with (2.33),

$$\beta \geq h_g(y_0) - f^g(y_0) \geq \alpha \geq \beta.$$

Therefore $\alpha = \beta$ and (D) has a solution.

ii) It follows by duality, since $f \equiv f^{gg}$ and $h \equiv h_{gg}$. In fact, consider for now

$$(P) : \min_{y \in Y}[f^g(y) - h_g(y)].$$

It is clear that the optimal value is $\alpha' = -\beta$ and that $supp^u(h_g - \beta, H) = supp^u(h_g + \alpha', H)$. Now, since $\alpha' \in I\!R$ and $supp_l(f^g, H) \cap supp^u(h_g + \alpha', H) \neq \emptyset$ we are in the same conditions of item i) and we can conclude that $\alpha' = \beta'$ and the dual problem has a solution, which in this case is as follows

$$(D) : \max_{x \in X}[h_{gg}(x) - f^{gg}(x)]$$

and $\beta' = \sup_{x\in X}[h_{gg}(x) - f^{gg}(x)]$. But f is an abstract-convex function and h is an abstract-concave function, therefore

$$(D) : \max_{x\in X}[h(x) - f(x)] \equiv \min_{x\in X}[f(x) - h(x)]$$

and $-\beta' = \alpha = \inf_{x\in X}[f(x) - h(x)]$.□

The condition $supp_l(f, H) \cap supp^u(h + \alpha, H) \neq \emptyset$ geometrically means that the epigraph of f and the hypograph of $h+\alpha$ (see Proposition 1.5.3) can be *separated* by an *abstract-affine* function. This means: there exists $(y, \mu) \in Y \times I\!R$ such that

$$f(x) \geq g(x, y) - \mu \geq h(x) + \alpha, \ \forall x \in X.$$

If we take $X = Y = I\!R^n$ and $g(x, y) = \langle x, y\rangle$, we would be in the case of the classical Fenchel's Duality theorem. As we mentioned at the beginning of this Section, in Theorem 1.1.13, we found that instead of the condition $supp_l(f, H)\cap supp^u(h+\alpha, H) \neq \emptyset$ there is the following one: $ri(dom(f)) \cap ri(dom(h)) \neq \emptyset$. These conditions are not equivalent. In fact, Rockafellar's condition implies ours: since $ri(epi(f))\cap ri(hyp(h+\alpha)) = \emptyset$, they can be separated *properly* by a hyperplane (Theorem 11.3, in [31]). If $ri(dom(f)) \cap ri(dom(h)) \neq \emptyset$ then this hyperplane can not be a vertical hyperplane and therefore is the graph of an affine function. This affine function belongs to $supp_l(f_H) \cap supp^u(h + \alpha, H)$.

In the following example, we will have that α is finite, $supp_l(f, H)\cap supp^u(h+\alpha, H) \neq \emptyset$ and $ri(dom(f)) \cap ri(dom(h)) = \emptyset$.

Example: Consider $g : I\!R^2 \to I\!R$, $f, h : I\!R \to \overline{I\!R}$ as follows:

$$g(x, y) := xy,$$

$$f(x) := \begin{cases} x^2, & x \leq 0 \\ +\infty, & x > 0. \end{cases}$$

and

$$h(x) := \begin{cases} -\infty, & x < 0 \\ -x^2, & x \geq 0. \end{cases}$$

We are in the classical convex duality. It is immediate to see that f is closed convex and proper, h is closed concave and proper, even more, $ri(dom(f))\cap ri(dom(h)) = \emptyset$. Since $dom(f) \cap dom(h) = \{0\}$ we can consider

$$(P) : \min_{x\in I\!R}[f(x) - h(x)].$$

It is clear that

$$f(x) - h(x) := \begin{cases} 0, & x = 0 \\ +\infty, & x \neq 0 \end{cases}$$

therefore, $\alpha = 0$ (even more, it is attained at $x = 0$). If we want to calculate $supp_l(f, H)$ and $supp^u(h, H)$, thanks to Proposition 1.5.3 we have that

$$supp_l(f, H) = epi(f^*) \text{ and } supp^u(h, H) = hyp(h_*).$$

It is not difficult to prove that f^* and h_* are given by:

$$f^*(x^*) := \begin{cases} \dfrac{x^{*2}}{4}, & x^* \leq 0 \\ \\ 0, & x^* > 0 \end{cases}$$

and

$$h_*(x^*) := \begin{cases} -\dfrac{x^{*2}}{4}, & x^* \leq 0 \\ \\ 0, & x^* > 0. \end{cases}$$

With this

$$epi(f^*) \cap hyp(h_*) = \{(x^*, \mu) \in I\!R^2 : x^* \geq 0, \mu = 0\}.$$

Furthermore, any point (x_0^*, μ_0) in $epi(f^*) \cap hyp(h_*) \equiv supp_l(f, H) \cap supp^u(h, H)$ provides us with a solution of

$$(D) : \max_{x^* \in I\!R} [h_*(x^*) - f^*(x^*)],$$

namely x_0^*.

From Theorem 2.7.1 and Corollary 2.7.1 it is clear that the condition $epi(f^*) + epi(\widetilde{h}^*)$ being weak* closed implies that $\alpha = \beta$ and (D) has a solution. From this, it is not difficult to show that $supp_l(f, H) \cap supp^u(h + \alpha, H) \neq \emptyset$ where $h = -\widetilde{h}$.

Chapter 3

Strongly Star-Shaped cones

As was mentioned in Section 1.7, a sub-family of the strongly star-shaped cones is exploited in [35]. By using this sub-family, the authors of [35] extend the results in [28].

In this Chapter we will show that the sub-family considered in [35] is, in fact, formed by every proper strongly star-shaped cone in the space X. We will prove it using the notion of recession cone, see Equation (1.9).

We will give as well a characterisation for every strongly star-shaped cone with non-empty interior of its kern_*. For doing this, we will need to establish first many results from [34] in arbitrary Banach spaces.

3.1 The set $\mathcal{K}(X)$

The set $\mathcal{K}(X)$ is defined in terms of the set $U(K)$ which is a subset of $\text{kern}_* K$ (see Section 1.7, Definitions 1.7.2 and 1.7.3). This implies that every $K \in \mathcal{K}(X)$ is a strongly star-shaped cone. We will show now that in fact, if $K \subset X$ is a strongly star-shaped cone such that $\text{cl}K \neq X$ then $K \in \mathcal{K}(X)$. For doing this we need the following:

Theorem 3.1.1 *Let $K \subsetneq X$ be a cone. Then*

$$U(K) = kern_* K.$$

Proof: By definition, one has $U(K) \subset \text{kern}_*(K)$ which is equivalent to $(\text{kern}_*(K))^c \subset (U(K))^c$. The proof of the opposite inclusion is divided in two parts:

1) $U(K)^c \subset (\text{kern}_* K)^c \cup (\text{kern}_* K \cap (\text{cl}K \cap (-\text{cl}K)))$: in fact, take $u \in U(K)^c$, such that $u \in \text{kern}_* K$. Since $u \notin U(K)$ there exists $x_u \in X$ such that $x_u + \{tu : t \in I\!R\} \subset K$. It is immediate to see that this last inclusion implies that u and $-u$ belong to K^∞ and, thanks to Lemma 1.4.2 item v), it can be seen that u and $-u$ belong to cl$K$.

2) $\text{kern}_* K \cap (\text{cl}K \cap (-\text{cl}K)) = \emptyset$: suppose that there exists $u \in \text{kern}_* K \cap (\text{cl}K \cap (-\text{cl}K))$. Then, $-u \in \text{cl}K$. If $-u \in \text{bd}(K)$, then $\{0, -u\} \subset \text{bd}(K) \cap (u + R_{-u})$, and since $u + R_{-u}$ intersects bd(K) more than once, $u \notin \text{kern}_* K$. This contradiction proves that $-u \in \text{int}K$. Consider $\varepsilon > 0$ such that $B(u, \varepsilon)$, $B(-u, \varepsilon) \subset K$ ($u \in \text{int}K$ thanks to the definition of kern$_*$). Since $0 \notin \text{int}K$, there exists $y \in B\left(0, \frac{\varepsilon}{4}\right)$ such that $y \notin \text{cl}K$. Define $d = y - u$, with this $u + d \notin \text{cl}K$. There exists $t_1 \in (0, 1)$ such that $u + t_1 d \in \text{bd}K$. Now $u + 2d = u + 2(y - u) = -u + 2y$ and $-u + 2y \in B(-u, \varepsilon) \subset K$ imply that $u + 2d \in \text{int}K$. Therefore there exists $t_2 \in (1, 2)$ such that $u + t_2 d \in \text{bd}K$. We have found that $u + R_d$ intersects the bd(K) more than once, hence $u \notin \text{kern}_* K$, which is a contradiction.

Finally, $U(K)^c \subset (\text{kern}_* K)^c$ and $U(K) = \text{kern}_* K$.□

This theorem implies that

$$\mathcal{K}(X) = \{K \subsetneq X : K \text{ is a closed conic set, with nonempty kern}_* K\}.$$

However K is a closed conic set with nonempty kern$_*$ if and only if K is a strongly star-shaped cone by definition. Henceforth, we can re-define $\mathcal{K}(X)$ as

$$\mathcal{K}(X) = \{K \subsetneq X : K \text{ is a closed conic set, which is a strongly star-shaped cone}\}.$$

Remark: The only strongly star-shaped conic set not in $\mathcal{K}(X)$ is the whole space X. This is in fact good. Remember that the cones in $\mathcal{K}(X)$ are used (in [35]) for generating pre-order relations (see Definition 1.7.1), therefore if $K = X$ the order relation $\succeq_K$ has no practical use, since in this case one would have that $x \succeq_K y$ for every $x, y \in X$.

We have mentioned in Section 1.7 that in [35] the set $U(K)$ for a $K \in \mathcal{K}(X)$ is very important for finding a characterization of minimal points and two different ways of scalarization for some Vector Optimization Problems (see Sections 5-8 of the referred paper). Thanks to Theorem 3.1.1, it is clear now that the strongly star-shaped property of K is actually the most important one.

Now, it would be interesting to see under which additional conditions Proposition 1.7.5 will give us that $\text{kern}_* K = \bigcap_{i=1}^{m} \text{kern}_* K_i$. For example in $\mathbb{R}^2$ consider $K_1 = \{x : 2x_1 \geq x_2, x_1 \geq 0, x_2 \geq 0\}$ and $K_2 = \{x : \frac{1}{2}x_1 \leq x_2, x_1 \geq 0, x_2 \geq 0\}$. It is immediate to see that $K = K_1 \cup K_2 = \mathbb{R}^2_+$ and $\text{kern}_* K \neq \text{kern}_* K_1 \cap \text{kern}_* K_2$ (see Figure 3.1).

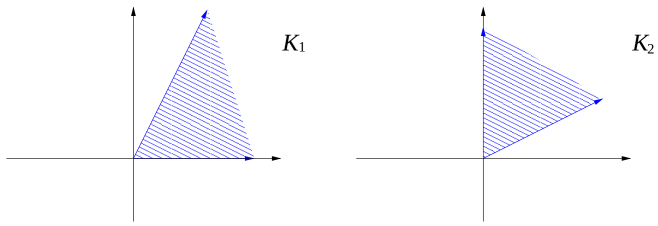

Figure 3.1: K_1 and K_2

The following result gives a sufficient condition that one may impose to the sets K_i.

Proposition 3.1.2 *Let $K_i \in \mathcal{K}(X)$, $i = 1, \ldots, m$ and $\bigcap_{i=1}^{m}$ kern$_* K_i \neq \emptyset$. Let $K = \bigcup_{i=1}^{m} K_i$. If for every $u \notin$ kern$_* K_i$ there exists $x_{i,u} \in X$ such that $p_{u,K_i}(x_{i,u}) = -\infty$ for each $i \in \{1, \ldots, m\}$ then $K \in \mathcal{K}(X)$ and*

$$\bigcap_{i=1}^{m} \textit{kern}_* K_i = \textit{kern}_* K.$$

Proof: Thanks to Proposition 1.7.5 we already have that $\bigcap_{i=1}^{m} \text{kern}_* K_i \subset \text{kern}_* K$. Take now $u \in \text{kern}_* K$ and suppose that $u \notin \bigcap_{i=1}^{m} \text{kern}_* K_i$. There exists $i_0 \in \{1, \ldots, m\}$ such that $u \notin \text{kern}_* K_{i_0}$, even more, there exists $x_{i_0,u} \in X$ such that $p_{u,K_{i_0}}(x_{i_0,u}) = -\infty$. However, from Proposition 1.7.2 item ii), $-\infty < p_{u,K}(x_{i_0,u})$ but

$$-\infty < p_{u,K}(x_{i_0,u}) = \min_i p_{u,K_i}(x_{i_0,u}) \leq p_{u,K_{i_0}}(x_{i_0,u}) = -\infty$$

and thus a contradiction occurs. Therefore $\bigcap_{i=1}^{m} \text{kern}_* K_i = \text{kern}_* K$.□

Remark: If we consider an arbitrary family of strongly star-shaped cones instead of a finite one, we may not have that $\bigcap_{i\in I} \text{kern}_* K_i \subset \text{kern}_* K$. The reason for this is that for the proof of Proposition 1.7.5, in [35], it is important that the function $p_{u,K}$ is continuous. Since this function is the *minimum* of a finite number of continuous functions, namely $p_{u,K} = \min_i p_{u,K_i}$, then the affirmation follows. By using an arbitrary family of cones, this minimum becomes an *infimum* of functions and we will not necessarily have that the function $p_{u,K}$ is continuous.

Let us put in evidence that there exists a strongly star-shaped cone, K, in $\mathcal{K}(X)$ such that for every $u \notin \text{kern}_* K$ there exists $x_u \in X$ such that $p_{u,K}(x_u) = -\infty$.

Consider for example, $H \subset X$ a half space of X, which means that there exists a continuous linear functional a^* such that

$$H := \{x \in X : a^*(x) \geq 0\}.$$

It is immediate to see that every half space $H \subset X$ is a convex closed cone and $\text{kern}_* H = \text{int} H = \{x \in X : a^*(x) > 0\}$. For $u \in \text{int} H$ calculate $p_{u,H}(x)$, for $x \in X$ fixed:

$$p_{u,H}(x) = \inf[\lambda \in I\!R : \lambda u - x \in H].$$

Nevertheless, $\lambda u - x \in H$ if and only if

$$a^*(\lambda u - x) \geq 0 \Longrightarrow \lambda a^*(u) \geq a^*(x),$$

since $u \in \text{int} H$, then $a^*(u) > 0$ and

$$\lambda \geq \frac{a^*(x)}{a^*(u)}.$$

Therefore $p_{u,H}(x) = \dfrac{a^*(x)}{a^*(u)}$. Since $x \in X$ was fixed arbitrarily,

$$p_{u,H}(x) = \frac{a^*(x)}{a^*(u)}, \ \forall x \in X.$$

Take now $u \notin \text{int} H$ and calculate $p_{u,H}(u)$:

$$p_{u,H}(u) = \inf\{\lambda \in I\!R : (\lambda - 1)u \in H\}.$$

However, $a^*(u) \leq 0$, thus $\lambda_1 a^*(u) = a^*(\lambda_1 u) \geq 0$, for every $\lambda_1 < 0$. Therefore $p_{u,H}(u) = -\infty$.

We can summarize this as follows:

Corollary 3.1.1 *Let $H_i \subset X$ be half spaces for every $i = 1, \ldots, m$ with $\bigcap_{i=1}^{m} intH_i \neq \emptyset$. Take $K = \bigcup_{i=1}^{m} H_i$. Then $K \in \mathcal{K}(X)$ and*

$$\bigcap_{i=1}^{m} intH_i = kern_* K.$$

It is easy to express the additional assumption imposed on each K_i in another way.

Proposition 3.1.3 *Consider $K \in \mathcal{K}(X)$. The following are true:*

1) Take $u \in kern_* K$, then $K = \{-x : p_{u,K}(x) \leq 0\}$.

2) For each $u \notin kern_* K$ there exists x_u such that $p_{u,K}(x_u) = -\infty$ if and only if $\{x : p_{u,K}(x) > 0\} \subset kern_* K$.

Proof:

1) Thanks to Proposition 1.7.1 item 3

$$-x \in K \Longleftrightarrow \lambda u - x \in K,\ \forall \lambda \geq 0 \Longleftrightarrow p_{u,K}(x) \leq 0.$$

Thus $K = \{-x : p_{u,K}(x) \leq 0\}$.

2) Assume that for each $u \notin \text{kern}_* K$ there exists x_u such that $p_{u,K}(x_u) = -\infty$. The latter equality implies that for every $\lambda > 0$, $-x_u + \lambda(-u) \in K$. Then $-u \in K^\infty = K$ (see Equation (1.9)). This occurs for every $u \notin \text{kern}_* K$, therefore

$$-((\text{kern}_* K)^c) \subset K$$

which is equivalent to

$$\{x : p_{u,K}(x) > 0\} = (-K)^c \subset \text{kern}_* K.$$

For the converse

$$\{x : p_{u,K}(x) > 0\} = (-K)^c \subset \text{kern}_* K \Longrightarrow (\text{kern}_* K)^c \subset -K,$$

it is immediate then that $p_{u,K}(u) = -\infty$ for every $u \notin \text{kern}_* K$.

3.2 Some technical results

We would like to give a characterization for a sub-family of $\mathcal{K}(X)$. In order to do it, we need to point out first that Propositions 1.7.6 - 1.7.10 are still valid in Banach spaces. Remark 5.8 of [34] gives a definition of *min-sublinear* function valid in Banach spaces which is important for the next results:

Definition 3.2.1 *A function $f : X \to I\!R$ is called min-sublinear if for each $x \in X$ there exists a continuous sublinear function p_x (i.e. $\sup\{|p_x(z)| : \|z\| = 1\} < +\infty$) such that $p_x(y) \geq f(y)$ for all $y \in X$ and $p_x(x) = f(x)$.*

The following propositions extend Propositions 1.7.6-1.7.8 to Banach spaces. Because the same proofs as those given in [34] can be used for the more general setting of Banach spaces, we omit them here. We are taking into consideration Definition 3.2.1.

Proposition 3.2.1 *A positively homogeneous function $f : X \to I\!R$ is min-sublinear if and only if for any $z \in X$ there exists a number $k_z > 0$ such that*

$$f(x) - f(z) \leq k_z\|x - z\| \text{ for all } x \in X.$$

Proposition 3.2.2 *Let f be a positively homogeneous Lipschitz function defined on X with a Lipschitz constant L. Let $S = \{x : \|x\| = 1\}$ be the unit sphere. Then there exists a family $(p_z)_{z\in S}$ of sublinear functions such that:*

1) $f(x) = \min_{z\in S} p_z(x)$ for all $x \in X$.

2) $p_z(x) \leq f(z) + L\|x - z\|$ for all $x \in X$.

Proposition 3.2.3 *Let U be a radiant subset of X and $0 \in int\ kernU$. Then there exist $\varepsilon > 0$ and a family of convex closed sets $(K_u)_{u\in U}$ such that:*

1) $u \in K_u$ and $B(0, \varepsilon) \subset K_u$ for all $u \in U$.

2) $U = \bigcup_{u\in U} K_u$.

Now, we present the extension of Proposition 1.7.9. We will follow the proof of Theorem 5.2 of [34] considering Definition 3.2.1. The main difference of this proof with the one in [34] is the fact that a sublinear function defined in the unit Sphere, of an arbitrary Banach space, could not attain its supremum.

Proposition 3.2.4 *Let U be a closed radiant subset of X. The Minkowski gauge of the set U, $\mu_U(x) = \inf\{\lambda > 0 : x \in \lambda U\}$, is Lipschitz if and only if there exists $\varepsilon > 0$, a set of indices T and a family $(U_t)_{t\in T}$ of convex sets containing the ball $B(0, \varepsilon)$ such that*

$$U = cl \bigcup_{t\in T} U_t.$$

Proof: Assume that there exist $\varepsilon > 0$, a set of indices T and a family of convex sets U_t such that

$$U = cl \bigcup_{t\in T} U_t$$

and $B(0, \varepsilon) \subset U_t$ for all $t \in T$. Let μ_t be the Minkowski gauge of the set U_t. The convexity of U_t implies the sublinearity of μ_t for every t. Since $B(0, \varepsilon) \subset U_t$, from Definition 1.7.6 it follows that that

$$\mu_t(x) \le \mu_{B(0,\varepsilon)}(x) = \frac{1}{\varepsilon}\|x\|$$

for every $t \in T$ and therefore every μ_t, $t \in T$, is a Lipschitz function with Lipschitz constants uniformly bounded by $1/\varepsilon$. Thanks to Proposition 5.4 of [34] the function μ_U coincides with the closure (Definition 1.1.6) of $\inf_{t\in T} \mu_t$. Since the Lipschitz constants of the functions μ_t are uniformly bounded, it follows that the function $\inf_{t\in T} \mu_t$ is Lipschitz. Thus μ_U coincides with this function and therefore μ_U is Lipschitz.

On the other hand, let the Minkowski gauge μ_U of the set U be Lipschitz with a Lipschitz constant L. Applying Proposition 3.2.2, we can find a finite sublinear function p_z for each $z \in S$ (the unit sphere), such that

$$\mu_U(x) = \min_{z\in S} p_z(x), \text{ for all } x \in X \tag{3.1}$$

and

$$\mu_U(z) + L\|x - z\| \ge p_z(x), \text{ for all } x \in X. \tag{3.2}$$

Consider the convex set $U_z = \{y : p_z(y) \le 1\}$. It follows from Proposition 5.2 of [34] that p_z is the Minkowski gauge of U_z. Combining (3.1) and Proposition 5.4 of [34] we conclude that

$$U = \mathrm{cl} \bigcup_{z\in S} U_z.$$

Applying (3.2) we deduce that for $z \in S$

$$\|p_z\| = \sup_{y\in S} p_z(y) \le \sup_{y\in S}(\mu_U(z) + L\|y - z\|) \le M,$$

where $M = \|\mu_U\| + 2L$ with $\|\mu_U\| = \sup_{y \in S} \mu_U(y)$ (which is not $+\infty$ since $0 \leq \mu_U(x) \leq L\|x\|$ because μ_U is Lipschitz). Thus the Lipschitz constants $\|p_z\|$ of the functions p_z are uniformly bounded for all $z \in S$. Since $p_z(y) \leq M\|y\|$ for all $y \in X$ and the function $y \mapsto M\|y\|$ is the Minkowski gauge of the ball $B(0, 1/M)$ it follows that

$$B(0, 1/M) \subset U_z, \text{ for all } z \in S. \square$$

Proposition 1.7.10 corresponds to Theorem 5.3 of [34]. In their proof it is needed Theorem 5.2 which we have just extended to Banach spaces (Proposition 3.2.4). Before we give Proposition 3.2.6 which is the extension of Proposition 1.7.10, let us quote Proposition 5.16 of [34] which is still valid in Banach spaces.

Proposition 3.2.5 *Let A a star-shaped set. Then cl$A$ is star-shaped as well and cl(kern A) $\subset$ kern(cl$A$).*

Proposition 3.2.6 *Let $U \subset X$ be a closed radiant set. Then the Minkowski gauge μ_U of the set U is Lipschitz if and only if $0 \in$ int kern$U$.*

Proof: Let U be a closed radiant subset of X and there exists $\varepsilon > 0$ such that $B(0, \varepsilon) \subset \text{kern}U$. It follows from Proposition 3.2.3 that

$$U = \bigcup_{u \in U} K_u, \tag{3.3}$$

where K_u is convex and contains the ball $B(0, \varepsilon)$. It follows from Proposition 3.2.4 and Equation (3.3) that μ_U is Lipschitz.

Assume now that μ_U is Lipschitz. Applying Proposition 3.2.4 we can find $\varepsilon > 0$ and a family $(U_t)_{t \in T}$ of convex sets such that $U = \text{cl}U'$, where $U' = \bigcup_{t \in T} U_t$ and $B(0, \varepsilon) \subset U_t$. Let $u \in U'$. Then there exists $t \in T$ such that $u \in U_t$. Since U_t is convex, $\lambda u + (1 - \lambda)y \in U_t \subset U$ for all $y \in B(0, \varepsilon)$ and $\lambda \in (0, 1)$. Thus $B(0, \varepsilon) \subset \text{kern}U' \subset \text{cl}(\text{kern}U')$. It follows from Proposition 3.2.5 that $\text{cl}(\text{kern}U') \subset \text{kern}(\text{cl}U') = \text{kern}U$. Hence $B(0, \varepsilon) \subset \text{kern}U$. $\square$

Let us show now some more properties which are going to be useful for the main result of next Section, Theorem 3.3.1. Recall the set $CK := (\text{int}K)^c$ (from Proposition 1.7.4).

Proposition 3.2.7 *Let K, K_1, K_2 be conic sets of X such that their closures belong to $\mathcal{K}(X)$ (i.e. cl$K \in \mathcal{K}(X)$).*

1) *Take* $u \in X$. *If* $K_1 \subset K_2$ *then* $p_{u,K_2} \leq p_{u,K_1}$. *Furthermore, if* $u \in kern_* K_1 \cap kern_* K_2$ *then the converse is true* ($p_{u,K_2} \leq p_{u,K_1}$ *implies* $K_1 \subset K_2$).

2) $kern_* K = kern_*(clK)$.

3) *If* $u \in kern_* K$ *then* $p_{u,K} \equiv p_{u,clK}$.

Proof:

1) Given $x \in X$ and any $\lambda > p_{u,K_1}(x)$, we have thanks to Proposition 1.7.1 item 3 and Equation (1.12) that $\lambda u - x \in K_1 \subset K_2$. Then $p_{u,K_2}(x) \leq p_{u,K_1}(x)$. Since $x \in X$ was chosen arbitrarily this inequality is valid for every $x \in X$.

   Now consider $u \in kern_* K_1 \cap kern_* K_2$ and $p_{u,K_2} \leq p_{u,K_1}$. Thanks to Proposition 3.1.3 item 1
$$K_i = \{-x : p_{u,K_i}(x) \leq 0\}, \text{ for } i = 1, 2.$$
   Thus, take $-x \in K_1$ then
$$0 \geq p_{u,K_1}(x) \geq p_{u,K_2}(x)$$
   which means that $-x \in K_2$ and we have $K_1 \subset K_2$.

2) This is immediate from the definition of kern_* (Definition 1.7.2 item ii) and from the following: $\text{int}K = \text{int cl}K$ and $\text{bd}K = \text{bd cl}K$.

3) Since $K \subset \text{cl}K$, then (item 1) $p_{u,\text{cl}K} \leq p_{u,K}$. Let us suppose that there exists $x_0 \in X$ such that $p_{u,\text{cl}K}(x_0) < p_{u,K}(x_0)$. Then
$$-p_{u,K}(x_0) < -p_{u,\text{cl}K}(x_0) \tag{3.4}$$
   from Proposition 1.7.4
$$-p_{u,\text{cl}K}(x_0) = p_{-u,CK}(x_0). \tag{3.5}$$
   Thanks to Proposition 1.7.1 item 3 we have the following:
$$p_{u,K}(x_0) = \sup[\lambda : \lambda u - x_0 \in K^c] = \sup[-\lambda : -\lambda u - x_0 \in K^c]$$
$$= -\inf[\lambda : \lambda(-u) - x_0 \in K^c] = -p_{-u,K^c}(x_0). \tag{3.6}$$
   Using (3.5) and (3.6) in (3.4), we have that $p_{-u,K^c}(x_0) < p_{-u,CK}(x_0)$. However, since $\text{int}K \subset K$ then $K^c \subset CK$ and thanks to item 1, $p_{-u,CK}(x) \leq p_{-u,K^c}(x)$ for every $x \in X$, in particular for x_0, a contradiction. From this, $p_{u,K} \equiv p_{u,\text{cl}K}$.□

Remark: Since $U(K) = \text{kern}_*K$ (Theorem 3.1.1), Proposition 1.7.1 remains valid even if $u \in \text{kern}_*K$ and K is a conic set not necessarily closed such that $\text{cl}K \subsetneq X$. The only difference we need to point out appears in item 3: the set $\Lambda_x = \{\lambda \in I\!R : \lambda u - x \in K$ is a segment of the form $(\lambda_x, +\infty)$ with $\lambda_x > -\infty$.

The following example shows that the converse of the first statement in Proposition 3.2.7(1) is not true in general.

Example: Consider $u = (1,0) \in I\!R^2$ and $K_1, K_2 \in \mathcal{K}(I\!R^2)$ given by Figure 3.2. In this case, we have that $p_{u,K_2}(x) \leq p_{u,K_1}(x)$ and $K_1 \cap K_2 = \{0\}$.

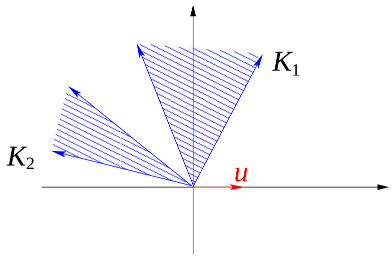

Figure 3.2: Example were $u \notin \text{kern}_*K_1 \cap \text{kern}_*K_2$

3.3 Characterization of a sub-family of $\mathcal{K}(X)$

Thanks to all the results given in the previous Section, we can finally show the following:

Theorem 3.3.1 *Let $K \in \mathcal{K}(X)$. The interior of kern_*K is non-empty, if and only if there exist $u \in X$, $\varepsilon > 0$ and $(K_\lambda)_{\lambda \in I}$ non-empty closed convex cones with*

$$B(u,\varepsilon) \subset K_\lambda, \text{ for all } \lambda \in I,$$

such that

$$K = cl \bigcup_\lambda K_\lambda.$$

Furthermore, $u \in int(kern_*K)$ and

$$p_{u,K} \equiv \inf_\lambda p_{u,K_\lambda}.$$

Proof: Let $u \in \text{int}(\text{kern}_*K)$. There exists $\varepsilon > 0$ such that $B(u,\varepsilon) \subset \text{kern}_*K \subset \text{kern}K$. It follows from the definition of $\text{kern}K$ (Definition 1.7.2 item i) that

$$\lambda a + (1-\lambda)x \in K$$

for all $a \in \text{kern}K$, $x \in K$ and $\lambda \in (0,1)$. Since $B(u,\varepsilon) \subset \text{kern}K$ and the latter set is convex (see [34] and references therein), then for every $x \in K$ the closed convex hull of the set $\{x\} \cup B(u,\varepsilon)$ is contained in $\text{kern}K$. Thus

$$\overline{co}\,(\{x\} \cup B(u,\varepsilon)) \subset \text{kern}K \subset K.$$

Hence, consider $K_x \subset K$ the cone generated by this closed convex hull. Then $x \in K_x$, for all $x \in K$ and therefore

$$K \subset \bigcup_{x\in K} K_x \subset K,$$

$$K = cl\ K \subset cl \bigcup_{x\in K} K_x \subset cl\ K = K.$$

This means $K = cl \bigcup_x K_x$ and $B(u,\varepsilon) \subset (K_x)$, for all $x \in K$.

For the converse, K is a closed cone, it remains to prove that $\text{int}(\text{kern}_* K) \neq \emptyset$. Propositions 3.2.4 and 3.2.6 imply that the Minkowski gauge of $K - u$ is a Lipschitz function and $0 \in \text{int}(\text{kern}(K-u))$, respectively. But $\text{int}(\text{kern}(K-u)) = \text{int}(\text{kern}_*(K-u))$ (Proposition 5.18 and Remark 5.9 in [34]), therefore $u \in \text{int}(\text{kern}_* K)$ and $K \in \mathcal{K}(X)$. Finally, since $K = cl \bigcup_\lambda K_\lambda$ and $u \in \text{kern}_* K = \text{kern}_* \bigcup_\lambda K_\lambda$, thanks to Proposition 3.2.7 item 3 we have the following:

$$p_{u,K} \equiv p_{u,\bigcup_\lambda K_\lambda} \equiv \inf_\lambda p_{u,K_\lambda}.\square$$

The next result gives a relationship between $\text{kern}_*(\text{cl}(K^c))$ and $\text{kern}_* K$ when $K \subset X$ is a closed convex cone with nonempty interior.

Lemma 3.3.2 *Let $K \subsetneq X$ be a closed convex cone with nonempty interior and consider $CK = (intK)^c$. Then*

$$kern_* CK = int(-K).$$

Proof: Since $K \neq X$, $-K = \{x \in X : -x \in K\}$ is a closed convex cone ($\text{int}(-K) = -\text{int}K$) which is contained entirely in CK. In fact, if $-K \not\subset CK$ then there exists $y \in K$ such that $-y \in -K \cap \text{int}K$ and $0 \in \text{int}K$ (see Theorem Theorem 1.1.2) and therefore $K = X$ which is a contradiction. We need to show that $\text{int}(-K) \subset \text{kern}_* CK$. We need that $u + R_x$ does not intersect $\text{bd}CK$ more than once for every $u \in \text{int}(-K)$ and $x \in X$. Fix $u \in \text{int}(-K)$ and consider the following cases:

a) if $x \in -K$: since $-K$ is a convex cone it is easy to prove that $u + R_x \subset \text{int}(-K) \subset \text{int}CK$.

b) if $x \in CK \setminus (-K)$: suppose that $u + R_x$ intersects bdCK. This implies that there exists $t > 0$ and $y \in \text{bd}CK = \text{bd}K \subset K$ such that $u+tx = y$, which means that $x = \dfrac{y-u}{t}$. However since $u \in \text{int}(-K)$, then $-u \in \text{int}K$ and therefore $y - u \in \text{int}K$ (there exists $\varepsilon > 0$ such that $B(-u, \varepsilon) \subset K$ then $B(y - u, \varepsilon) = y + B(-u, \varepsilon) \subset y + K \subset K$). This leads to $x \in \text{int}K$, a contradiction. Thus, $u + R_x \subset \text{int}CK$.

c) if $x \in CK^c = \text{int}K$: $u + R_x$ intersects bdCK, if not $x \in (CK)^\infty = CK$, see Equation (1.9). Now suppose that there exist $y_1, y_2 \in u + R_x \cap$ bdCK, $y_1 \neq y_2$. This implies that there exist $0 < t_1 < t_2$ such that $y_1 = u + t_1 x$, $y_2 = u + t_2 x = (u + t_1 x) + (t_2 - t_1)x$ the latter implies $y_2 = y_1 + (t_2 - t_1)x$. Since $x \in \text{int}K$ and $y_1 \in \text{bd}CK = \text{bd}K$ then $y_2 \in \text{int}K$, a contradiction.

These three items imply that $\text{int}(-K) \subset \text{kern}_* CK$. Now take $u \in \text{kern}_* CK \setminus \text{int}(-K)$. Then $u \notin \text{bd}(-K)$, otherwise $-u \in \text{bd}K = \text{bd}CK$ and $u + R_{-u}$ will intersect bdCK more than once, namely in 0 and $-u$. Nevertheless, $\text{kern}_* CK$ is convex (see [37]) therefore take $u_1 \in \text{int}(-K)$ and $[u_1, u]$ the segment which goes from u_1 to u, we have that $[u_1, u] \in \text{kern}_* CK$ but since $u_1 \in \text{int}(-K)$ and $u \notin \text{int}(-K)$ then$[u_1, u] \cap \text{bd}(-K) \neq \emptyset$, a contradiction. Finally, $\text{kern}_* CK = \text{int}(-K)$.□

The next Proposition answers, under certain assumptions, an interesting question: how many cones have the same kern_*?

Proposition 3.3.3 *Let $K \subsetneq X$ be a closed convex cone with non empty interior and L a closed cone, such that $-K \subset L \subset (intK)^c$ and $kern_* L = int(-K)$. If L is convex then $L = -K$ and if L^c is convex then $L = (intK)^c$.*

Proof: Suppose first that L is convex, then $\text{kern}_* L = \text{int}L = \text{int}(-K)$, which implies $L = -K$. On the other hand, if L^c is a convex cone, since L is closed then L^c is an open convex cone. Consider $Q = \text{cl } L^c$, then $L = (\text{int } Q)^c$. Hence, (thanks to the previous Lemma) $\text{kern}_* L = \text{int}(-Q) = \text{int}(-K)$, which implies that $K = Q$ and thus $L = (\text{int}K)^c$.□

Chapter 4

Conclusions and Further Research

At the beginning of our work we were looking for a duality scheme for non-convex optimization problems which resembles to the scheme for convex problems introduced by Fenchel and Rockafellar. By using the generalized conjugation theory and our G-coupling functions, we have accomplished this goal.

Strong points about this scheme are the fact that it is stable and it naturally induces a generalized Lagrangian (Theorems 2.3.6 and 2.4.2). While this scheme does not generate the one by Fenchel and Rockafellar, it is important to notice that if the problem is convex, the best dual approach or Lagrangian approach is the one generated by the Fenchel conjugate.

Another good characteristic of our duality scheme is that it can be used for the Equilibrium problem. By doing this, we can now study different optimization problems (like the Variational Inequality Problem and the Vector Optimization Problem) by means of a Dual problem and a generalized Lagrangian. In this work we put in evidence the first steps in this direction.

We wanted to find an equivalent result of Fenchel's Duality by means of our G-coupling functions. Even though we did not get this result, we presented in this work our geometrical interpretation of what this duality theorem should satisfy in a more general setting.

We were interested as well in the strongly star shaped cones. A sub-family of these cones have been used in [35] for solving vector optimization problems. We proved in this work that this sub-family in fact represents the whole family of proper closed strongly star-shaped cones (Theorem 3.1.1). We have even showed a characterization for a sub-family of these cones.

In the future, we would like to exploit further the duality scheme for the Equilibrium problem to study the Variational Inequality problem. We would like as well to find the connection between our scheme for the minimization problem and a generalized version of Fenchel's duality theorem. In the study of the strongly star-shaped cones, we intend to complete the characterization of them and to refine the techniques used by [35] for the vector optimization problem.

Bibliography

[1] Aubin, J. P. *Optima and Equilibria. An Introduction to Nonlinear Analysis.* Springer-Verlag Berlin Heidelberg (1993).

[2] Auslender, A.; Teboulle, M. *Asymptotic Cones and Functions in Optimization and Variational Inequalities.* Springer-Verlag New York, INC. (2003).

[3] Avriel, M. *Nonlinear Programming.* Prentice-Hall, INC., Englewood Cliffs, New Jersey (1976).

[4] Berge, C. *Topological Spaces.* Dover Publications, INC., Mineola, New York (1997).

[5] Bertsekas, D. P. *Constrained Optimization and Lagrange Multiplier Methods* Academic Press, New York (1982).

[6] Brézis, H. *Analyse Fonctionnelle: Théorie et applications.* Masson, Paris (1983).

[7] Borwein, J. M.; Lewis, A. S. *Convex analysis and nonlinear optimization.* Springer-Verlag New York Berlin Heidelberg (2000).

[8] Burachik, R. S. *Generalized Proximal Point Methods for the Variational Inequality Problem.* Tese de Doutorado, IMPA (1995).

[9] Burachik, R. S.; Iusem, A. N. *Set-valued mappings and enlargements of monotone operators.* Springer Optimization and Its Applications, Vol. 8 (2008).

[10] Burachik, R. S.; Jeyakumar, V. *A new geometric condition for Fenchel duality in infinite dimensions.* Mathematical Programming, Vol 104, N2-N3, pp. 229-233 (2005).

[11] Burachik, R. S.; Jeyakumar, V.; Wu, Z. Y. *Necessary and sufficient conditions for stable conjugate duality.* Journal of Nonlinear Analysis, 64, pp. 1998-2006 (2006).

[12] Burachik, R. S.; Rubinov, A. M. *Abstract Convexity and Augmented Lagrangians.* SIAM Journal on Optimization, Vol. 18, pp. 413-436 (2007).

[13] Cottle, R. W.; Pang, J. S.; Venkateswaran, V. *Sufficient matrices and the Linear Complementarity Problem.* Linear Algebra and its Applications 114/115, pp. 231-249 (1989).

[14] Crouzeix, J. P.; Ocaña, E.; Sosa, W. *Análisis Convexo.* Monogafía del IMCA (2003).

[15] Evtushcenko, Y. G.; Rubinov A. M.; Zhadan, V. G. *General Lagrange-type functions in constrained Global Optimization. Part 1: Auxiliary functions and optimality conditions.* Optimization Methods and Software, Vol. 16, pp. 193-230 (2001).

[16] Facchinei, F.; Pang, J. S. *Finite-dimensional variational inequalities and complementarity problems.* Springer-Verlag New York Berlin Heidelberg (2003).

[17] Fenchel, W. *On conjugate convex functions.* Canadian journal of mathematics, Vol. 1, pp. 73-77 (1949).

[18] Fenchel, W. *Convex cones, sets and functions.* Lectures Notes, Princeton University, Princeton (1953).

[19] Giannessi, F. *Theorems of the alternative and optimality conditions.* Journal of Optimization Theory and Applications, Vol. 42, pp. 331-365 (1984).

[20] Giannessi, F.; Mastroeni, M. *On the theory of vector optimization and variational inequalities. Image space analysis and separation.* Vector Variational Inequalities and Vector Equilibria. Mathematical Theories. Giannessi, F.; Kluwer Academic Publishers, Dordrecht, pp. 153-215 (1999).

[21] Giles, J. R. *Convex analysis with application in the differentiation of convex functions.* Pitman Advanced Publishing Program, Boston, London, Melbourne (1982).

[22] Harker, P. T.; Pang, J. S. *Finite-Dimensional Variational Inequality and Nonlinear Complementarity Problems: A survey of theory, Algorithms and Applications.* Mathematical Programming 48: 161-220 (1989).

[23] Hiriart-Urruty, J.-B.; Lemaréchal, C. *Fundamentals of Convex Analysis.* Springer-Verlag Berlin Heidelberg (2001).

[24] Iusem, A. N.; Sosa, W. *New existence results for Equilibrium Problems.* Nonlinear Analysis 52, pp. 621-635 (2003).

[25] Martínez-Legaz, J. E.; Sosa, W. *Duality for Equilibrium Problems.* Journal of Global Optimization, Vol. 35, N2, pp. 311-319 (2006).

[26] Jeyakumar V.; Rubinov A. M.; Wu Z. Y. *Generalized Fenchel's Conjugation Formulas and Duality for Abstract Convex Functions.* Journal of Optimization Theory and Applications, Vol. 132, N3, pp. 441-458 (2007).

[27] Kreyszig, E. *Introductory functional analysis with applications.* John Wiley & Sons. Inc. New York (1978).

[28] Luc, D. T. *Theory of vector optimization.* Lecture Notes Econ. and Math. Systems 319, Springer-Verlag, Berlin, New York (1989).

[29] Moreau, J. J. *Proximité et Dualité dans un espace Hilbertien.* Bulletin de la Societé Mathematique de France, Vol. 93, pp 273-299 (1965).

[30] Phelps, R. R. *Convex Functions, Monotone Operators and Differentiability* Springer-Verlag Berlin Heidelberg (1993, 2nd Edition).

[31] Rockafellar, R. T. *Convex Analysis.* Princeton University Press, Princeton, New Jersey (1970).

[32] Rockafellar, R. T. *Conjugate Duality and Optimization.* SIAM (1987).

[33] Rockafellar, R. T., Wets, R. J-B. *Variational Analysis* Springer-Verlag Berlin Heidelberg (1998).

[34] Rubinov, A. M. *Abstract Convexity and Global Optimization.* Kluwer Academic Publishers, Dordrecht/Boston/London (2000).

[35] Rubinov, A. M.; Gasimov, R. N. *Scalarization and Nonlinear Scalar Duality for Vector Optimization with Preferences that are not necessarily a Pre-order Relation.* Journal of Global Optimization, 29: 455-477 (2004).

[36] Rubinov, A. M.; Yang, X. Q. *Lagrange-type Functions in Constrained non-convex Optimization.* Kluwer Academic Publishers, Dordrecht/Boston/London (2002).

[37] Shveidel, A. *Separability of star-shaped sets and its application to an optimization problem* Optimization, 40: 207-227 (1997).

[38] Singer, I. *Abstract convex analysis.* John Wiley & Sons, Inc., New York, (1997).

[39] Sosa, W. *Iterative Algorithms for the abstract Equilibrium Problem.* Tese de Doutorado, IMPA. (1999).

[40] Sosa, W. *Introducción a la Optimización, Programación Lineal.* XVIII Coloquio de la Sociedad Matemática Peruana (2000).

[41] Taylor, A. E.; Lay, D. C. *Introduction to Functional Analysis. Second Edition.* Krieger Publishing Company, Malabar, Florida (1980).

[42] van Tiel, J. *Convex Analysis. An introductory text.* John Wiley & Sons, Inc., New York, (1984).

[43] Wright, S. J. *Primal-Dual Interior Point Methods.* Society for Industrial and Applied Mathematics (S.I.A.M.), Philadelphia, (1997).

[44] Yang, X. Q. *On the GAP functions of Prevariational Inequalties.* Journal of Optimization Theory and Applications Vol. 116, No 2, pp. 437-452 (2003).

[45] Zangwill, W. *Nonlinear Programming. A unified Approach* Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1969).

Index