Stability of error bounds for semi-infinite convex constraint systems

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Dedicated to Professor Hedy Attouch on his 60th birthday

Abstract

In this paper, we are concerned with the stability of the error bounds for semi-infinite convex constraint systems. Roughly speaking, the error bound of a system of inequalities is said to be stable if all its "small" perturbations admit a (local or global) error bound. We first establish subdifferential characterizations of the stability of error bounds for semi-infinite systems of convex inequalities. By applying these characterizations, we extend some results established by Azé & Corvellec [3] on the sensitivity analysis of Hoffman constants to semi-infinite linear constraint systems.

Mathematics Subject Classification: 49J52, 49J53, 90C30, 90C34

Key words: error bounds, Hoffman constants, subdifferential

1 Introduction

Our aim in this paper is to study the behavior of the error bounds under data perturbations. Error bounds which are considered here are for a system of semi-infinite constraints in $\mathbb{R}^n$, that is for the problem of finding $x \in \mathbb{R}^n$ satisfying:

$$f_t(x) \leq 0 \quad \text{for all} \quad t \in T, \tag{1}$$

where T is a compact, possibly infinite, Hausdorff space, $f_t : \mathbb{R}^n \to \mathbb{R}$, $t \in T$, are given convex functions such that $t \mapsto f_t(x)$ is continuous on T for each $x \in \mathbb{R}^n$. According to Rockafellar ([23], Thm. 7.10), in this case, $(t, x) \mapsto F(t, x) := f_t(x)$ is continuous on $T \times \mathbb{R}^n$, i.e., $F \in C(T \times \mathbb{R}^n, \mathbb{R})$, the set of continuous functions on $T \times \mathbb{R}^n$.

Set

$$f(x) := \max\{f_t(x) : \ t \in T\} \quad \text{and} \quad T_f(x) := \{t \in T : \ f_t(x) = f(x)\}.$$

We use the symbol $[f(x)]_+$ to denote $\max(f(x), 0)$. Let S_F denote the set of solutions to (1) and recall that the distance of an element x to S_F denoted by $d(x, S_F)$ is defined by $d(x, S_F) = \inf_{z \in S_F} \|x - z\|$ with the convention $d(x, S_F) = +\infty$ whenever S_F is empty.

We shall say that system (1) admits *an error bound* if there exists a real $c(F) > 0$ such that

$$d(x, S_F) \leq c(F)[f(x)]_+ \ \text{ for all } x \in \mathbb{R}^n. \tag{2}$$

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For $\bar{x} \in \text{Bdry } S_F$ (the topological boundary of S_F), we shall say that system (1) admits an error bound at $\bar{x}$, if there exist reals $c(F, \bar{x})$, $\varepsilon > 0$ such that

$$d(x, S_F) \leq c(F, \bar{x})[f(x)]_+ \text{ for all } x \in B(\bar{x}, \varepsilon), \tag{3}$$

where $B(\bar{x}, \varepsilon)$ denotes an open ball with center $\bar{x}$ and radius ε.

Since the pioneering work ([12]) by Hoffman on error bounds for systems of affine functions, error bounds have been intensively discussed and it is now well established that they have a large range of applications in different areas such as, for example, sensitivity analysis, convergence analysis of algorithms, and penalty functions methods in mathematical programming. For a detailed account the reader is referred to the works [3–6, 15, 16, 18–20, 24], and especially to the survey papers by Azé [2], Lewis & Pang [15], Pang [21], as well as the book by Auslender & Teboule [1] for the summary of the theory of error bounds and its various applications.

When dealing with the behavior of the set S_F when F is perturbed, a crucial key to this is the boundedness of the Hoffman constants $c(F)$ and $c(F, \bar{x})$ in relations (2) and (3). For systems of linear inequalities, this question has been considered by Luo & Tseng [17], Azé & Corvellec [3] (see also Zheng & Ng [25] for systems of linear inequalities in Banach spaces and by Deng [7] for systems of a finite number of convex inequalities).

In the present paper, we are concerned with the stability of error bounds for finite dimensional semi-infinite constraint systems with respect to perturbations of F. More precisely, we establish characterizations for the boundedness of Hoffman constants $c(F)$ under "small" perturbations of F. We use these characterizations to obtain new results on the sensitivity analysis of Hoffman constants for semi-infinite linear constraint systems. The infinite dimensional extensions will be considered in the forthcoming paper [14].

The paper is organized as follows. The characterizations for the stability of the local error bounds are presented in Section 2. In Section 3, we then derive the characterizations for the stability of the global error bounds. In the final section, we establish necessary and sufficient conditions for the local Lipschitz property of Hoffman constants for semi-infinite systems of linear inequalities.

2 Stability of local error bounds

In what follows, we will use the notation $\Gamma_0(\mathbb{R}^n)$ to denote the set of extended real-valued lower semicontinuous convex functions $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, which are supposed to be proper, that is such that $\text{Dom } f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$ is nonempty. Recall that the subdifferential of a convex function f at a point $x \in \text{Dom } f$ is defined by

$$\partial f(x) = \{x^* \in \mathbb{R}^n : \langle x^*, y - x\rangle \leq f(y) - f(x), \ \forall y \in \mathbb{R}^n\}.$$

For a given $f \in \Gamma_0(\mathbb{R}^n)$, we consider first the set of solutions of a single convex inequality:

$$S_f := \{x \in \mathbb{R}^n : f(x) \leq 0\}. \tag{4}$$

We will use notations $c(f)$ and $c(f, \bar{x})$ for its respectively global and local error bound (Hoffman) constants (see definitions (2) and (3)), while the *best bounds* (the exact lower bounds of all Hoffman constants) will be denoted $c_{\min}(f)$ and $c_{\min}(f, \bar{x})$ respectively. The latter coincides with $[\text{Er } f(\bar{x})]^{-1}$, where

$$\text{Er } f(\bar{x}) = \liminf_{\substack{x \to \bar{x} \\ f(x) > 0}} \frac{f(x)}{d(x, S_f)}$$

is the *error bound modulus* [9]) (also known as *conditioning rate* [22]) of f at $\bar{x}$.

The following characterizations of the global and local error bounds are well known (see, for instance, [3]). They are needed in the sequel.

Theorem 1 *Let $f \in \Gamma_0(\mathbb{R}^n)$. Then one has*

(i). S_f admits a global error bound if and only if

$$\tau(f) := \inf\{d(0, \partial f(x)) : \ x \in \mathbb{R}^n, \ f(x) > 0\} > 0.$$

Moreover, $c_{\min}(f) = [\tau(f)]^{-1}$.

(ii). S_f admits a local error bound at $\bar{x} \in \mathrm{Bdry}\ S_f$, if and only if

$$\tau(f, \bar{x}) := \liminf_{x \to \bar{x},\, f(x)>0} d(0, \partial f(x)) > 0.$$

Moreover $c_{\min}(f, \bar{x}) = [\tau(f, \bar{x})]^{-1}$.

(iii). (Relation between the global error bound and the local error bound) The following equality holds

$$c_{\min}(f) = \sup_{x \in \mathrm{Bdry}\ S_f} c_{\min}(f, x).$$

Constant $\tau(f, \bar{x})$ in part (ii) of the above theorem is also known as *limiting outer subdifferential slope* of f at $\bar{x}$ [9].

For a mapping $\varphi : X \to Y$ between two Banach spaces X, Y, denote by $\mathrm{Lip}(\varphi)$ its Lipschitz constant:

$$\mathrm{Lip}(\varphi) := \sup_{u,v \in X,\, u \neq v} \frac{\|\varphi(u) - \varphi(v)\|_Y}{\|u - v\|_X}.$$

the Lipschitz constant of φ near x is defined by

$$\mathrm{Lip}(\varphi, x) := \limsup_{u,v \to x,\, u \neq v} \frac{\|\varphi(u) - \varphi(v)\|_Y}{\|u - v\|_X}.$$

First we obtain the following characterization of the stability of local error bounds for system (4).

Theorem 2 *Let $f \in \Gamma_0(\mathbb{R}^n)$ and $\bar{x} \in \mathbb{R}^n$ such that $f(\bar{x}) = 0$. Then the following two statements are equivalent:*

(i). $0 \notin \mathrm{Bdry}\ \partial f(\bar{x})$;

(ii). There exist reals $c := c(f, \bar{x}) > 0$ and $\varepsilon > 0$ such that for all $g \in \Gamma_0(\mathbb{R}^n)$, satisfying $\bar{x} \in S(g)$ and

$$\limsup_{x \to \bar{x}} \frac{|(f(x) - g(x)) - (f(\bar{x}) - g(\bar{x}))|}{\|x - \bar{x}\|} \leq \varepsilon \tag{5}$$

one has $c_{\min}(g, \bar{x}) \leq c$.

Proof. For $(i) \Rightarrow (ii)$, suppose that $0 \notin \text{Bdry}\, \partial f(\bar{x})$. Consider first the case $0 \in \text{Int}\, \partial f(\bar{x})$. Then there exists $r > 0$ such that $rB^* \subseteq \partial f(\bar{x})$, and consequently

$$f(x) - f(\bar{x}) \geq r\|x - \bar{x}\| \text{ for all } x \in \mathbb{R}^n.$$

Take any $\varepsilon \in (0, r)$. For any $g \in \Gamma_0(\mathbb{R}^n)$ with $\bar{x} \in S(g)$ and satisfying relation (5) one has

$$\liminf_{x \to \bar{x}} \frac{g(x) - g(\bar{x})}{\|x - \bar{x}\|} \geq \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x})}{\|x - \bar{x}\|} - \varepsilon \geq r - \varepsilon.$$

Since g is convex it follows that

$$g(x) - g(\bar{x}) \geq (r - \varepsilon)\|x - \bar{x}\| \text{ for all } x \in \mathbb{R}^n. \tag{6}$$

Let $x \in \text{Dom}\, g \setminus S(g)$. Then the restriction of g to the segment $[\bar{x}, x]$ is continuous. Since $g(\bar{x}) \leq 0$, there exists $z := (1-t)\bar{x} + tx \in [\bar{x}, x]$ $(t \in [0,1])$ such that $g(z) = 0$. Therefore, by (6) and the convexity of g, one obtains

$$g(x) = g(x) - g(z) \geq (1-t)(g(x) - g(\bar{x})) \geq (r - \varepsilon)(1-t)\|x - \bar{x}\| = (r - \varepsilon)\|x - z\|,$$

and therefore, $c_{\min}(g, \bar{x}) \leq (r - \varepsilon)^{-1}$.

Suppose now that $0 \notin \partial f(\bar{x})$ and take any $\varepsilon \in (0, m(f))$, where $m(f) = d(0, \partial f(\bar{x}))$. Then for any $g \in \Gamma_0(\mathbb{R}^n)$ with $\bar{x} \in S(g)$ and satisfying relation (5), one has $m(g) > m(f) - \varepsilon$. On the other hand, from Theorem 1, $c_{\min}(g, \bar{x}) \leq [m(g)]^{-1}$. Hence $c_{\min}(g, \bar{x}) \leq (m(f) - \varepsilon)^{-1}$, which completes the proof of $(i) \Rightarrow (ii)$.

Let us prove $(ii) \Rightarrow (i)$. Assume to the contrary that $0 \in \text{Bdry}\, \partial f(\bar{x})$. This means that, firstly, $0 \in \partial f(\bar{x})$ and, secondly, for any $\varepsilon > 0$ there exists $u^*_\varepsilon \in \varepsilon B^* \setminus \partial f(\bar{x})$. The first condition implies that f attains its minimum at $\bar{x}$, while it follows from the second one that for any $\delta > 0$, we can find $x_\delta \in B(\bar{x}, \delta) \setminus \{\bar{x}\}$ such that

$$\langle u^*_\varepsilon, x_\delta - \bar{x} \rangle > f(x_\delta) - f(\bar{x}).$$

Hence

$$f(x_\delta) < f(\bar{x}) + \varepsilon\|x_\delta - \bar{x}\| \leq \inf_{x \in \mathbb{R}^n} f(x) + \varepsilon\|x_\delta - \bar{x}\|.$$

By virtue of the Ekeland variational principle [8], we can select $y_\delta \in \mathbb{R}^n$ satisfying $\|y_\delta - x_\delta\| \leq \|x_\delta - \bar{x}\|/2$ and $f(y_\delta) \leq f(x_\delta)$ such that the function

$$f(\cdot) + 2\varepsilon\| \cdot - y_\delta\|$$

attains a minimum at y_δ. Hence $y_\delta \neq \bar{x}$ and $0 \in \partial(f(\cdot) + 2\varepsilon\|\cdot - y_\delta\|)(y_\delta) = \partial f(y_\delta) + 2\varepsilon B^*$, that is, there exists $y^*_\delta \in \partial f(y_\delta)$ such that $\|y^*_\delta\| \leq 2\varepsilon$. Let us take a sequence of reals $(\delta_k)_{k \in \mathbb{N}}$ converging to 0 with $\delta_k > 0$. Without loss of generality, we can assume that the sequence $\{(y_{\delta_k} - \bar{x})/\|y_{\delta_k} - \bar{x}\|\}_{k \in \mathbb{N}}$ converges to some $z \in \mathbb{R}^n$ with $\|z\| = 1$. Let $z^* \in \mathbb{R}^n$ be such that $\|z^*\| = 1$ and $\langle z^*, z \rangle = 1$. For each $\varepsilon > 0$, let us consider a function $g_\varepsilon \in \Gamma_0(\mathbb{R}^n)$ defined by

$$g_\varepsilon(x) := f(x) - f(\bar{x}) + \varepsilon\langle z^*, x - \bar{x} \rangle, \; x \in \mathbb{R}^n.$$

Then, obviously, $g_\varepsilon(\bar{x}) = 0$, g satisfies (5), and $g_\varepsilon(y_{\delta_k}) > 0$ when k is sufficiently large. Since $y^*_{\delta_k} \in \partial f(y_{\delta_k})$ and $\|y^*_{\delta_k}\| \leq 2\varepsilon$, then $d(0, \partial g_\varepsilon(y_{\delta_k})) \leq 3\varepsilon$. Thanks to Theorem 1 (note that $(y_{\delta_k}) \to \bar{x}$ as $k \to \infty$), we obtain $c_{\min}(g_\varepsilon, \bar{x}) \geq \varepsilon^{-1}/3$, and as $\varepsilon > 0$ is arbitrary, the proof is completed. □

Remark 3 *Condition* (5) *in Theorem 2 means that* g *is an* ε-perturbation *[14] of* f *near* $\bar{x}$*. Analyzing the proof of Theorem 2 one can easily see that for characterizing the error bound property it is sufficient to require a weaker one-sided estimate:*

$$\limsup_{x\to\bar{x}} \frac{(f(x)-g(x))-(f(\bar{x})-g(\bar{x}))}{\|x-\bar{x}\|} \le \varepsilon.$$

We consider now semi-infinite convex constraint systems of the form (1) with the solution set

$$S_F := \{x \in \mathbb{R}^n : \;\; f_t(x) \le 0 \quad \text{for all} \quad t \in T\}, \tag{7}$$

where T is a compact, possibly infinite, Hausdorff space, $f_t : \mathbb{R}^n \to \mathbb{R}$, $t \in T$, are given convex functions such that $t \mapsto f_t(x)$ is continuous on T for each $x \in \mathbb{R}^n$ and $F \in C(T \times \mathbb{R}^n, \mathbb{R})$ is defined by $F(t,x) := f_t(x)$, $(t,x) \in T \times \mathbb{R}^n$.

As mentioned in the introduction, we set

$$f(x) := \max\{f_t(x) : \; t \in T\} \quad \text{and} \quad T_f(x) := \{t \in T : \; f_t(x) = f(x)\}.$$

Note that under the above assumption, the subdifferential of the function f at a point $x \in \mathbb{R}^n$ is given by (see, for instance, Ioffe & Tikhomirov [13], also in Hantoute & López [10] and Hantoute-López -Zălinescu [11])

$$\partial f(x) = \text{co}\left\{ \bigcup_{t\in T_f(x)} \partial f_t(x) \right\}, \tag{8}$$

where "co" stands for the convex hull of a set.

The following theorem gives a characterization of the stability of local error bounds for system (7).

Theorem 4 *Let* $\bar{x} \in \mathbb{R}^n$ *such that* $f(\bar{x}) = 0$*. The following two statements are equivalent:*

(i). $0 \notin \text{Bdry}\, \partial f(\bar{x})$;

(ii). There exist reals $c := c(F,\bar{x}) > 0$ *and* $\varepsilon > 0$ *such that if*

$$G \in C(T \times \mathbb{R}^n, \mathbb{R}); \quad g_t(x) := G(t,x); \quad g_t \text{ are convex}; \tag{9}$$

$$\bar{x} \in S_G; \tag{10}$$

$$\sup_{t\in T} |f_t(\bar{x}) - g_t(\bar{x})| < \varepsilon; \tag{11}$$

$$\sup_{t\in T, x\in\mathbb{R}^n} |(f_t(x)-g_t(x)) - (f_t(\bar{x})-g_t(\bar{x}))| \le \varepsilon\|x-\bar{x}\|; \tag{12}$$

$$g(x) := \max\{g_t(x) : \; t \in T\}; \quad T_g(x) := \{t \in T : \; g_t(x) = g(x)\}; \tag{13}$$

$$T_f(\bar{x}) \subseteq T_g(\bar{x}) \text{ whenever } 0 \in \text{Int}\, \partial f(\bar{x}), \tag{14}$$

then one has $c_{\min}(G,\bar{x}) \le c$.

Proof. *(i)* $\Rightarrow$ *(ii)*. If $g(\bar{x}) < 0$, then $c_{\min}(G,\bar{x}) = 0$ due to the continuity of G, and the conclusion holds true trivially. Therefore it suffices to consider the case $g(\bar{x}) = 0$. Suppose that $0 \notin \text{Bdry}\, \partial f(\bar{x})$. Consider first the case $0 \in \text{Int}\, \partial f(\bar{x})$. Then there exists $r > 0$ such that $rB^* \subseteq \partial f(\bar{x})$. Take any

$\varepsilon \in (0, r)$ and let G, g_t, and g satisfy (9)–(14). By relation (8), for each $u^* \in rB^* (\subseteq \partial f(\bar{x}))$, there exist elements $t_1, \ldots, t_k$ of $T_f(\bar{x})$; $u_i^* \in \partial f_{t_i}(\bar{x})$, and reals $\lambda_1, \ldots, \lambda_k$ such that

$$\lambda_i \geq 0 \ (i = 1, \cdots, k); \quad \sum_{i=1}^{k} \lambda_i = 1; \quad u^* = \sum_{i=1}^{k} \lambda_i u_i^*. \tag{15}$$

Hence, for any $x \in \mathbb{R}^n$,

$$\langle u^*, x \rangle = \sum_{i=1}^{k} \lambda_i \langle u_i^*, x \rangle \leq \sum_{i=1}^{k} \lambda_i f'_{t_i}(\bar{x}, x) \leq \sum_{i=1}^{k} \lambda_i g'_{t_i}(\bar{x}, x) + \varepsilon \leq g'(\bar{x}, x) + \varepsilon.$$

Consequently, $g(x) \geq (r - \varepsilon)\|x - \bar{x}\|$, for all $x \in \mathbb{R}^n$. This implies $c_{\min}(G, \bar{x}) \leq (r - \varepsilon)^{-1}$.

Suppose now that $0 \notin \partial f(\bar{x})$. Denote $m = m(f) := d(0, \partial f(\bar{x}))$. Then, for any $\eta \in (0, m/3)$, there exists $\delta > 0$ such that $d(0, \partial f(x)) > m - \eta$ for all $x \in B(\bar{x}, \delta)$. Take any $\varepsilon \in (0, \min\{\delta^2, \eta, \eta^2\})$ and let G, g_t, and g satisfy (9)–(14). For $u^* \in \partial g(\bar{x})$, by applying again relation (8) to the function g, we can find elements $t_1, \ldots, t_k$ of $T_g(\bar{x})$; $u_i^* \in \partial g_{t_i}(\bar{x})$, and reals $\lambda_1, \ldots, \lambda_k$ satisfying conditions (15). Therefore, for all $x \in \mathbb{R}^n$, one has

$$\begin{aligned} \langle u^*, x - \bar{x} \rangle = \textstyle\sum_{i=1}^{k} \lambda_i \langle u_i^*, x - \bar{x} \rangle \quad &\leq \textstyle\sum_{i=1}^{k} \lambda_i (g_{t_i}(x) - g_{t_i}(\bar{x})) \\ &\leq \textstyle\sum_{i=1}^{k} \lambda_i (f_{t_i}(x) - f_{t_i}(\bar{x})) + \varepsilon\|x - \bar{x}\| \\ &\leq f(x) - f(\bar{x}) + \varepsilon + \varepsilon\|x - \bar{x}\|. \end{aligned}$$

Note that for the last inequality, we use the fact that for any $t \in T_g(\bar{x})$, one has

$$f_t(\bar{x}) \geq g_t(\bar{x}) - \varepsilon = g(\bar{x}) - \varepsilon = f(\bar{x}) - \varepsilon.$$

By considering the function

$$\varphi(x) := f(x) - \langle u^*, x - \bar{x} \rangle + \varepsilon\|x - \bar{x}\|, \ \ x \in \mathbb{R}^n,$$

we have

$$\varphi(\bar{x}) \leq \inf_{x \in \mathbb{R}^n} \varphi(x) + \varepsilon.$$

By virtue of the Ekeland variational principle, we can select $z \in \mathbb{R}^n$ satisfying $\|z - \bar{x}\| \leq \varepsilon^{1/2}$ and

$$0 \in \partial(\varphi(\cdot) + \varepsilon^{1/2}\| \cdot - z\|)(z) \subseteq \partial f(z) - u^* + (\varepsilon^{1/2} + \varepsilon)B^*.$$

That is, $u^* \in \partial f(z) + (\varepsilon^{1/2} + \varepsilon)B^*$. Moreover, $z \in B(\bar{x}, \delta)$, and by the definition of ε, one obtains $\|u^*\| > m - 3\eta$. Hence $d(0, \partial g(\bar{x})) \geq m - 3\eta$, and by Theorem 1, we derive the desired conclusion $c_{\min}(G, \bar{x}) < (m - 3\eta)^{-1}$.

For $(ii) \Rightarrow (i)$, assume to the contrary that $0 \in \text{Bdry } f(\bar{x})$. Observe from the proof of Theorem 2 that, for each $\varepsilon > 0$, one can find an element $z^* \in \mathbb{R}^n$ with $\|z^*\| = 1$ and construct a function (note that $f(\bar{x}) = 0$)

$$g_\varepsilon(x) := f(x) + \varepsilon\langle z^*, x - \bar{x} \rangle, \ x \in \mathbb{R}^n$$

satisfying $g_\varepsilon(\bar{x}) = 0$ and $c_{\min}(g_\varepsilon, \bar{x}) \geq \varepsilon^{-1}/3$. For $t \in T$, we define the function $g_t : \mathbb{R}^n \to \mathbb{R}$ by

$$g_t(x) := f_t(x) + \varepsilon\langle z^*, x - \bar{x} \rangle, \ \ x \in \mathbb{R}^n.$$

Then

$$g_\varepsilon(x) = \max\{g_t(x) : \ t \in T\}; \quad T_g(\bar{x}) = T_f(\bar{x}); \quad g_t(\bar{x}) = f_t(\bar{x}) \text{ for all } \ t \in T;$$

and

$$\sup_{x \in \mathbb{R}^n} |f_t(x) - g_t(x)| = \varepsilon \text{ (hence relation (12) is verified).}$$

Moreover, $c_{\min}(G, \bar{x}) = c_{\min}(g_\varepsilon, \bar{x}) \geq \varepsilon^{-1}/3$. The proof is completed. □

Remark 5 *In the proof of (ii) ⇒ (i), a stronger assertion has been established: If* $0 \in \text{Bdry } f(\bar{x})$*, then for any* $\varepsilon > 0$*, there exists* $a_\varepsilon \in \mathbb{R}^n$ *with* $\|a_\varepsilon\| \leq \varepsilon$ *such that if*

$$G_\varepsilon(t, x) := F(t, x) + \langle a_\varepsilon, x - \bar{x} \rangle, \quad (t, x) \in T \times \mathbb{R}^n,$$

then $c_{\min}(G_\varepsilon, \bar{x}) \geq \varepsilon^{-1}$.

Remark 6 *It is important to note that the condition* $T_f(\bar{x}) \subseteq T_g(\bar{x})$ *in the case* $0 \in \text{Int}\, \partial f(\bar{x})$ *is crucial. To see this, let us consider the following example. Let* $\mathbb{R}^2$ *be endowed with any norm satisfying* $\|(0, x)\| = |x|$ *and let* $f_i : \mathbb{R}^2 \to \mathbb{R}$ *be defined by* $f_i(x_1, x_2) := |x_i|$, $i = 1, 2$*;* $F := (f_1, f_2)$*;* $f := \max\{f_1, f_2\}$*. Then*

$$S_F = \{x \in \mathbb{R}^2 : \ f_i(x) \leq 0, \ i = 1, 2\} = \{(0, 0)\},$$

and $\partial f((0, 0)) = B^*_{\mathbb{R}^2}$*. For each* $\varepsilon > 0$*, we define the functions* $g_{i,\varepsilon}$ *(*$i = 1, 2$*) by*

$$g_{1,\varepsilon}(x_1, x_2) := |x_1| + \varepsilon |x_2|; \quad g_{2,\varepsilon}(x_1, x_2) := |x_2| - \varepsilon;$$

$G_\varepsilon := (g_{1,\varepsilon}, g_{2,\varepsilon})$*;* $g_\varepsilon := \max\{g_{1,\varepsilon}, g_{2,\varepsilon}\}$*. Obviously,*

$$S(G_\varepsilon) = \{(0, 0)\} \text{ and } \max\{Lip(f_1 - g_{1,\varepsilon}), Lip(f_2 - g_{2,\varepsilon})\} \leq \varepsilon.$$

For any positive $\delta < \varepsilon^{-1}$ *set* $z_\delta = (0, \delta) \in \mathbb{R}^2$*. Then* $d(z_\delta, S(G_\varepsilon)) = \delta$*;* $g_\varepsilon(z_\delta) = \varepsilon\delta$*. Hence,* $c_{\min}(G_\varepsilon, (0, 0)) \geq \varepsilon^{-1}$.

3 Stability of global error bounds

In this section, we deal with the stability of Hoffman global error bounds for semi-infinite convex constraint systems. First, we establish a characterization for the global stability for the case of a single inequality (4):

$$S_f := \{x \in \mathbb{R}^n : \ f(x) \leq 0\}. \tag{4'}$$

Theorem 7 *Let* $f \in \Gamma_0(\mathbb{R}^n)$*,* $\emptyset \neq S_f \subseteq \text{Int}(\text{Dom } f)$*. Then the following two statements are equivalent:*

(i). There exists $\tau > 0$ *such that*

$$\inf\{d(0, \text{Bdry}\,(\partial f(x))) : \ x \in \mathbb{R}^n, \ f(x) = 0\} > \tau, \tag{16}$$

and the following asymptotic qualification condition is satisfied:

($\mathcal{AQC}$) *For any sequences* $(x_k)_{k\in\mathbb{N}} \subseteq S_f$, $(x_k^*)_{k\in\mathbb{N}} \subseteq \mathbb{R}^n$ *satisfying*

$$\lim_{k\to\infty} \|x_k\| = \infty, \quad \lim_{k\to\infty} f(x_k)/\|x_k\| = 0, \quad x_k^* \in \partial f(x_k) \tag{17}$$

one has $\liminf_{k\to\infty} \|x_k^*\| > \tau$;

(ii). For any $\bar{x} \in \mathbb{R}^n$ *there exist reals* $c := c(f) > 0$ *and* $\varepsilon > 0$ *such that for all* $g \in \Gamma_0(\mathbb{R}^n)$ *satisfying*

$$S(g) \neq \emptyset \text{ if } 0 \in \mathrm{Int}(\partial f(z)) \text{ for some } z \in S_f;$$
$$|f(\bar{x}) - g(\bar{x})| < \varepsilon; \quad Lip(f-g) < \varepsilon$$

one has $c_{\min}(g) \leq c$.

Proof. *(i)* ⇒ *(ii)*. First, if $0 \in \mathrm{Int}\,\partial f(z)$ for some $z \in S_f$ then $S_f = \{z\}$ and the conclusion follows as in the proof of Theorem 2. Let us consider now the case $0 \notin \partial f(x)$ for all $x \in \mathrm{Bdry}\ S_f$. Let the statement (i) be fulfilled. We first prove the following claim.

Claim. For any $\bar{x} \in \mathbb{R}^n$ *there exists* $\varepsilon > 0$ *such that*

$$\inf\{\|x^*\| : \ x^* \in \partial f(x),\ x \in \mathbb{R}^n \text{ with } f(x) \geq -\varepsilon\|x - \bar{x}\| - \varepsilon\} \geq \tau. \tag{18}$$

Indeed, suppose by contradiction that for some $\bar{x} \in \mathbb{R}^n$ relation (18) does not hold. Then, there exist a sequence of reals $(\varepsilon_k) \downarrow 0_+$; sequences $(x_k)_{k\in\mathbb{N}}$, $(x_k^*)_{k\in\mathbb{N}}$ of points in $\mathbb{R}^n$ such that $(\forall k)\ f(x_k) \geq -\varepsilon_k\|x_k - \bar{x}\| - \varepsilon_k$; $x_k^* \in \partial f(x_k)$; and $\|x_k^*\| < \tau$.

For any $x \in \mathbb{R}^n$ with $f(x) > 0$, and any $x^* \in \partial f(x)$, we can select $z \in \mathrm{Bdry}\ S_f$ such that $\|x - z\| = d(x, S_f)$. Then by the convexity of f, $f(z) = 0$, and by (16), $\tau(f, z) \geq d(0, \partial f(z)) > \tau$. In virtue of Theorem 1 (ii), there exists $0 < \delta < \|x - z\|$ such that

$$\tau d(y, S_f) \leq [f(y)]_+ \text{ for all } y \in B(z, \delta).$$

By taking $r = \delta\|x - z\|^{-1}/2 \in (0, 1)$; $y := z + r(x - z) \in B(z, \delta) \cap [z, x]$, one obtains $f(y) > 0$, $\|y - z\| = d(y, S_f)$, and

$$\tau r\|x - z\| = \tau d(y, S_f) \leq f(y) \leq rf(x) + (1 - r)f(z) = rf(x) \leq r\langle x^*, x - z\rangle.$$

Consequently, $\|x^*\| \geq \tau$.

Hence, $f(x_k) \leq 0$ when k is sufficiently large. Without loss of generality, assume that $f(x_k) \leq 0$ for all indexes k. If $(x_k)_{k\in\mathbb{N}}$ is bounded, by relabeling if necessary, we can assume that $(x_k)_{k\in\mathbb{N}}$, $(x_k^*)_{k\in\mathbb{N}}$ converge to some points x_0, $x_0^* \in \mathbb{R}^n$, respectively. Then, $f(x_0) \leq 0$; $\|x_0^*\|_* \leq \tau$ as well as $x_0^* \in \partial f(x_0)$. Moreover, since $S_f \subseteq \mathrm{Int}(\mathrm{Dom}\ f)$, then $f(x_0) = \lim_{k\to\infty} f(x_k) = 0$. This contradicts condition (16). If $(x_k)_{k\in\mathbb{N}}$ is unbounded we have a contradiction with ($\mathcal{AQC}$) since (after relabeling) $\lim_{k\to\infty} \|x_k\| = +\infty$ and $\lim_{k\to\infty} f(x_k)/\|x_k\| = \lim_{k\to\infty} f(x_k)/\|x_k - \bar{x}\| = 0$. The claim is proved.

Let $\bar{x} \in \mathbb{R}^n$ and let $\varepsilon \in (0, \tau)$ be as in the claim. Suppose $g \in \Gamma_0(\mathbb{R}^n)$ satisfies $|f(\bar{x}) - g(\bar{x})| < \varepsilon$ and $\mathrm{Lip}(f - g) < \varepsilon$. For any $x \in \mathbb{R}^n$ with $g(x) > 0$, one has $\partial g(x) \subseteq \partial f(x) + \varepsilon B^*$. Hence,

$$d(0, \partial g(x)) \geq d(0, \partial f(x)) - \varepsilon.$$

On the other hand, by

$$f(x) \geq g(x) + (f(\bar{x}) - g(\bar{x})) - \varepsilon\|x - \bar{x}\| \geq -\varepsilon\|x - \bar{x}\| - \varepsilon,$$

taking into account the claim one obtains $d(0, \partial f(x)) \geq \tau$, and consequently

$$d(0, \partial g(x)) \geq \tau - \varepsilon.$$

In virtue of Theorem 1 (*ii*), we derive the desired inequality $c_{\min}(g) \leq (\tau - \varepsilon)^{-1}$.

(*ii*) $\Rightarrow$ (*i*). Assume to the contrary that (*i*) does not hold. Then, one of the following two cases can occur:

Case 1. There exist sequences $(x_k)_{k\in\mathbb{N}}$, $(x_k^*)_{k\in\mathbb{N}}$ such that $f(x_k) = 0$; $x_k^* \in \mathrm{Bdry}\,(\partial f(x_k))$ $(\forall k)$ and $\lim_{k\to\infty} \|x_k^*\| = 0$.

Let $\varepsilon > 0$ be given arbitrarily. Pick a sequence of reals $(\delta_k) \downarrow 0$. Since $x_k^* \in \mathrm{Bdry}\,(\partial f(x_k))$, we have, firstly, $x_k^* \in \partial f(x_k)$ and, secondly, there exists $u_k^* \in \varepsilon B^*$ such that $x_k^* + u_k^* \notin \partial f(x_k)$. The first condition implies that

$$f(x) - \langle x_k^*, x - x_k\rangle \geq 0 \quad \text{for all} \;\; x \in \mathbb{R}^n,$$

while it follows from the second one that we can find $y_k \in B(x_k, \delta_k) \setminus \{x_k\}$ satisfying

$$\langle x_k^* + u_k^*, y_k - x_k\rangle > f(y_k) - f(x_k) = f(y_k).$$

Hence

$$f(y_k) - \langle x_k^*, y_k - x_k\rangle < \varepsilon\|y_k - x_k\|.$$

By virtue of the Ekeland variational principle [8], we can select $z_k \in \mathbb{R}^n$ satisfying $\|y_k - z_k\| \leq \|y_k - x_k\|/2$ and $f(z_k) \leq f(y_k) + \langle x_k^*, z_k - y_k\rangle$ such that the function

$$x \longmapsto f(x) - \langle x_k^*, x - x_k\rangle + 2\varepsilon\|x - z_k\|$$

attains its minimum at z_k. Hence $z_k \neq x_k$ and $0 \in \partial f(z_k) - x_k^* + 2\varepsilon B^*$. That is, there exists $z_k^* \in \partial f(z_k)$ such that $\|z_k^*\| \leq \|x_k^*\| + 2\varepsilon$. We distinguish the following two subcases:

Subcase 1.1. The sequence $(x_k)_{k\in\mathbb{N}}$ is bounded. Take any $\bar{x} \in \mathbb{R}^n$ and chose $M > \max\{\sup_k \|x_k - \bar{x}\|, 1\}$. Without loss of generality, we can assume that the sequence $\{(z_k - x_k)/\|z_k - x_k\|\}_{k\in\mathbb{N}}$ converges to some $u \in \mathbb{R}^n$ with $\|u\| = 1$. Let $u^* \in \mathbb{R}^n$ be such that $\|u^*\| = 1$ and $\langle u^*, u\rangle = 1$. Let us consider a function $g_\varepsilon \in \Gamma_0(\mathbb{R}^n)$ defined by

$$g_\varepsilon(x) := f(x) + \frac{\varepsilon}{M}\langle u^*, x - x_k\rangle, \; x \in \mathbb{R}^n.$$

Then, obviously, $\mathrm{Lip}(f - g_\varepsilon) = \varepsilon/M < \varepsilon$, $|f(\bar{x}) - g_\varepsilon(\bar{x})| < \varepsilon$, and $g_\varepsilon(z_k) > 0$ when k is sufficiently large. Since $z_k^* \in \partial f(z_k)$ and $\|z_k^*\| \leq \|x_k^*\| + 2\varepsilon$, then

$$d(0, \partial g_\varepsilon(z_k)) \leq \|x_k^*\| + 3\varepsilon \leq 4\varepsilon$$

when k is sufficiently large, and consequently $c_{\min}(g_\varepsilon) \geq (4\varepsilon)^{-1}$.

Subcase 1.2. $\lim_{k\to\infty} \|x_k\| = \infty$. Pick $x_0 \in S_f$. We can assume that the sequence $\{(x_k - x_0)/\|x_k - x_0\|\}_{k\in\mathbb{N}}$ converges to some $u \in \mathbb{R}^n$ with $\|u\| = 1$. Let us pick $u^* \in \mathbb{R}^n$ such that $\|u^*\| = 1$ and $\langle u^*, u\rangle = 1$ and consider the function $g_\varepsilon \in \Gamma_0(\mathbb{R}^n)$ defined by

$$g_\varepsilon(x) := f(x) + \varepsilon\langle u^*, x - x_0\rangle,\ x \in \mathbb{R}^n.$$

One has $x_0 \in S_{g_\varepsilon}$; $|f(\bar{x}) - g_\varepsilon(\bar{x})| \le \varepsilon\|\bar{x} - x_0\|$; $\mathrm{Lip}(f - g_\varepsilon) = \varepsilon$. Moreover, $g_\varepsilon(x_k) > 0$ when k is sufficiently large; $x_k^* + \varepsilon u^* \in \partial g_\varepsilon(x_k)$. Hence $c_{\min}(g_\varepsilon) \ge \varepsilon^{-1}$.

Case 2. There exist sequences $(x_k)_{k\in\mathbb{N}} \subseteq S_f$, $(x_k^*)_{k\in\mathbb{N}} \subseteq \mathbb{R}^n$ satisfying (17), and $\lim_{k\to\infty} \|x_k^*\| = 0$. In this case, for each $\varepsilon > 0$, we consider the function g_ε defined as in Subcase 1.2. Then, $g(x_k) > 0$ when k is sufficiently large. Moreover, $d(0, \partial g_\varepsilon(x_k)) \le \|x_k^*\| + \varepsilon$, which completes the proof. □

We turn our attention now to semi-infinite convex constraint systems of the form

$$S_F := \{x \in \mathbb{R}^n :\ \ f_t(x) \le 0 \quad \text{for all} \quad t \in T\}, \tag{7'}$$

where T is a compact, possibly infinite, Hausdorff space, $f_t : \mathbb{R}^n \to \mathbb{R}$, $t \in T$, are given convex functions such that $t \mapsto f_t(x)$ is continuous on T for each $x \in \mathbb{R}^n$, and $F \in C(T \times \mathbb{R}^n, \mathbb{R})$ is defined by $F(t,x) := f_t(x)$, $(t,x) \in T \times \mathbb{R}^n$. As in Section 2, we set

$$f(x) := \max\{f_t(x) :\ t \in T\} \quad \text{and} \quad T_f(x) := \{t \in T :\ f_t(x) = f(x)\}.$$

A characterization of the stability of global error bounds for the semi-infinite constraint system (7) is given in the following theorem.

Theorem 8 *The following two statements are equivalent:*

(i). There exists $\tau > 0$ such that

$$\inf\{d(0, \mathrm{Bdry}\,(\partial f(x))) :\ \ x \in \mathbb{R}^n,\ f(x) = 0\} > \tau, \tag{16'}$$

and asymptotic qualification condition $(\mathcal{AQC})$is satisfied.

(ii). For any $\bar{x} \in \mathbb{R}^n$ there exist reals $c := c(F, \bar{x}) > 0$ and $\varepsilon > 0$ such that if

$$G \in C(T \times \mathbb{R}^n, \mathbb{R});\quad g_t(x) := G(t,x);\quad g_t \textit{ are convex}; \tag{9'}$$

$$S_G \ne \emptyset; \tag{19}$$

$$\sup_{t\in T} |f_t(\bar{x}) - g_t(\bar{x})| < \varepsilon; \tag{20}$$

$$\sup_{t\in T} Lip(f_t - g_t) < \varepsilon; \tag{21}$$

$$g(x) := \max\{g_t(x) :\ t \in T\};\quad T_g(x) := \{t \in T :\ g_t(x) = g(x)\}; \tag{13'}$$

$$T_f(x) \subseteq T_g(x) \textit{ whenever } 0 \in \mathrm{Int}(\partial f(x)) \textit{ for some } x \in S_F, \tag{22}$$

then one has $c_{\min}(G, \bar{x}) \le c$.

Proof. (*i*) $\Rightarrow$ (*ii*). When $x \in \operatorname{Int}(\partial f(x))$ for some $x \in S_F$, then obviously $S_F = \{x\}$ and the proof follows as in Theorem 4. Suppose now that $0 \notin \partial f(x)$ for all $x \in S_F$ with $f(x) = 0$. Thanks to the claim in the proof of Theorem 7, for any $\bar{x} \in \mathbb{R}^n$ there exists $\eta > 0$ such that

$$\inf\{\|x^*\| : \ \ x^* \in \partial f(x), \ x \in \mathbb{R}^n \text{ with } f(x) \geq -\eta\|x - \bar{x}\| - \eta\} \geq \tau. \tag{23}$$

Let $\varepsilon > 0$ be given (it will be made precise later) and let G, g_t, and g satisfy (9), (13), (19)–(22). Let $x \in \mathbb{R}^n$ with $g(x) > 0$ and let $x^* \in \partial g(x)$. It follows from (20) and (21) that

$$|f_t(x) - g_t(x)| \leq |f_t(\bar{x}) - g_t(\bar{x})| + \varepsilon\|x - \bar{x}\| < \varepsilon(\|x - \bar{x}\| + 1) \tag{24}$$

for some $\bar{x} \in \mathbb{R}^n$ and all $t \in T$, and consequently

$$|f(x) - g(x)| \leq \varepsilon(\|x - \bar{x}\| + 1). \tag{25}$$

When $t \in T_g(x)$ it also follows from (24) that

$$f_t(x) > g(x) - \varepsilon(\|x - \bar{x}\| + 1). \tag{26}$$

Combining (25) and (26) we obtain for $t \in T_g(x)$:

$$f_t(x) > f(x) - 2\varepsilon(\|x - \bar{x}\| + 1).$$

By relation (8), there exist elements $t_1, \ldots, t_k$ of $T_g(x)$; $x_i^* \in \partial g_{t_i}(x)$, and reals $\lambda_1, \ldots, \lambda_k$ such that

$$\lambda_i \geq 0 \ (i = 1, \cdots, k); \quad \sum_{i=1}^{k} \lambda_i = 1; \quad x^* = \sum_{i=1}^{k} \lambda_i x_i^*.$$

For all $y \in \mathbb{R}^n$, one has

$$\begin{aligned} \langle x^*, y - x\rangle &= \textstyle\sum_{i=1}^k \lambda_i \langle x_i^*, y - x\rangle \leq \sum_{i=1}^k \lambda_i (g_{t_i}(y) - g_{t_i}(x)) \\ &\leq \textstyle\sum_{i=1}^k \lambda_i (f_{t_i}(y) - f_{t_i}(x)) + \varepsilon\|y - x\| \\ &< f(y) - f(x) + 2\varepsilon(\|x - \bar{x}\| + 1) + \varepsilon\|y - x\|. \end{aligned} \tag{27}$$

Let us consider the function $\varphi : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\varphi(y) := f(y) - \langle x^*, y - x\rangle + \varepsilon\|y - x\|, \ \ y \in \mathbb{R}^n.$$

Then,

$$\varphi(x) \leq \inf_{y \in \mathbb{R}^n} \varphi(y) + 2\varepsilon(\|x - \bar{x}\| + 1).$$

Let us apply again the Ekeland variational principle to find $z \in \mathbb{R}^n$ such that $\|z - x\| \leq \varepsilon^{1/2}(\|x - \bar{x}\| + 1)$ and

$$0 \in \partial(\varphi(\cdot) + 2\varepsilon^{1/2}\| \cdot - z\|)(z) \subseteq \partial f(z) - x^* + (2\varepsilon^{1/2} + \varepsilon)B^*.$$

That is,

$$x^* \in \partial f(z) + (2\varepsilon^{1/2} + \varepsilon)B^*. \tag{28}$$

On the other hand, since $\|z - x\| \leq \varepsilon^{1/2}(\|x - \bar{x}\| + 1)$, then

$$\|z - \bar{x}\| \geq \|x - \bar{x}\| - \|z - x\| \geq (1 - \varepsilon^{1/2})\|x - \bar{x}\| - \varepsilon^{1/2}.$$

Hence, when $\|x^*\| < \tau$, from relations (25) and (27), one has

$$\begin{aligned} f(z) &\geq f(x) - 2\varepsilon(\|x-\bar{x}\|+1) - (\tau+\varepsilon)\|z-x\| \\ &\geq g(x) - 3\varepsilon(\|x-\bar{x}\|+1) - (\tau+\varepsilon)\|z-x\| \\ &\geq -\big(3\varepsilon + (\tau+\varepsilon)\varepsilon^{1/2}\big)(\|x-\bar{x}\|+1) \\ &\geq -\big(3\varepsilon + (\tau+\varepsilon)\varepsilon^{1/2}\big)\big((1-\varepsilon^{1/2})^{-1}(\|z-\bar{x}\|+\varepsilon^{1/2})+1\big) \\ &= -\big(3\varepsilon + (\tau+\varepsilon)\varepsilon^{1/2}\big)(1-\varepsilon^{1/2})^{-1}(\|z-\bar{x}\|+1). \end{aligned}$$

Consequently, by taking $\varepsilon > 0$ sufficiently small such that

$$\big(3\varepsilon + (\tau+\varepsilon)\varepsilon^{1/2}\big)(1-\varepsilon^{1/2})^{-1} < \eta, \tag{29}$$

one can ensure $f(z) \geq -\eta\|z-\bar{x}\| - \eta$, and therefore by relations (23) and (28), one derives $\|x^*\| \geq \tau - (2\varepsilon^{1/2} + \varepsilon)$. Thus, when $\varepsilon > 0$ is sufficiently small such that, in addition to (29), $2\varepsilon^{1/2} + \varepsilon < \tau$, then

$$d(0, \partial g(x)) \geq \tau - 2\varepsilon^{1/2} - \varepsilon \;\text{ for all } x \in \mathbb{R}^n \;\text{ with }\; g(x) > 0.$$

Thanks to Theorem 1, we derive the desired conclusion $c_{\min}(G) < (\tau - 2\varepsilon^{1/2} - \varepsilon)^{-1}$.

$(ii) \Rightarrow (i)$. Assume that (i) does not hold. Take any $\bar{x} \in \mathbb{R}^n$. Observe from the proof of Theorem 7 that for each $\varepsilon > 0$, we can find $a_\varepsilon \in \mathbb{R}^n$, $b_\varepsilon \in \mathbb{R}$ such that the function

$$g_\varepsilon(x) = f(x) + \langle a_\varepsilon, x\rangle + b_\varepsilon, \; x \in \mathbb{R}^n$$

verifies the following conditions

$$\|a_\varepsilon\| < \varepsilon; \; S_{g_\varepsilon} \neq \emptyset; \; |\langle a_\varepsilon, \bar{x}\rangle + b_\varepsilon| < \varepsilon; \text{ and } c_{\min}(g_\varepsilon) \geq \varepsilon^{-1}.$$

For $t \in T$, we define the function $g_t : \mathbb{R}^n \to \mathbb{R}$ by

$$g_t(x) := f_t(x) + \langle a_\varepsilon, x\rangle + b_\varepsilon, \;\; x \in \mathbb{R}^n;$$

Then $g_\varepsilon(x) = \max\{g_t(x) : \; t \in T\}$, $T_g(x) = T_f(x)$ for all $x \in \mathbb{R}^n$; $|g_t(\bar{x}) - f_t(\bar{x})| < \varepsilon$ for all $t \in T$; and $\sup_{t\in T} \operatorname{Lip}(f_t - g_t) < \varepsilon$ as well as $c_{\min}(G_\varepsilon) = c_{\min}(g_\varepsilon) \geq \varepsilon^{-1}$. The proof is completed. □

From this proof of $(ii) \Rightarrow (i)$, observe that if the condition (i) of the theorem is not satisfied, we can find a sequence of affine perturbations $(g_t^k)_{k\in\mathbb{N}}$ of (f_t) such that

$$\lim_{k\to\infty} \sup_{t\in T} |g_t^k(\bar{x}) - f_t(\bar{x})| = 0; \quad \lim_{k\to\infty} \operatorname{Lip}(g_t^k - f_t) = 0; \quad \text{and} \quad \lim_{k\to\infty} c_{\min}(G^k) = 0.$$

4 Application to the sensitivity analysis of Hoffman constants for semi-infinite linear constraint systems

In this section, by using the results established in the preceding section, we generalize the results on the sensitivity analysis of Hoffman constants established by Azé & Corvellec in [3] for systems of finitely many linear inequalities to semi-infinite linear systems.

We consider now semi-infinite linear systems in $\mathbb{R}^n$ defined by

$$\langle a(t), x\rangle \leq b(t) \quad \text{for all} \quad t \in T, \tag{30}$$

where T is a compact, possibly infinite, metric space and the functions $a : T \to \mathbb{R}^n$ and $b : T \to \mathbb{R}$ are continuous on T.

Consider spaces $C(T, \mathbb{R}^n)$ and $C(T, \mathbb{R})$ of continuous functions $a : T \to \mathbb{R}^n$ and $b : T \to \mathbb{R}$ respectively, endowed with the norms

$$\|a\| := \max_{t\in T} \|a(t)\| \text{ and } \|b\| := \max_{t\in T} |b(t)|.$$

Denote by $S_{a,b}$ the set of solutions to system (30). We will also use the following notations:

$$f_{a,b}(x) := \max_{t\in T} \big(\langle a(t), x\rangle - b(t)\big); \quad J_{a,b}(x) := \{t \in T : \langle a(t), x\rangle - b(t) = f_{a,b}(x)\} \quad \text{for each } x \in \mathbb{R}^n.$$

Obviously, $J_{a,b}(x)$ is a compact subset of T for each $x \in \mathbb{R}^n$ and we have

$$\partial f_{a,b}(x) = \text{co}(a_{J_{a,b}(x)}),$$

where we use the notation $a_J := \{a(t) : \ t \in J\}$.

According to Theorem 1, $S_{a,b}$ admits a *global error bound* if and only if

$$\tau(a, b) := \inf\{d(0, \partial f_{a,b}(x)) : x \in \mathbb{R}^n, f_{a,b}(x) > 0\} = \inf_{x\notin S_{a,b}} d(0, \text{co}(a_{J_{a,b}(x)})) > 0. \tag{31}$$

Moreover, the *best bound* is given by $c_{\min}(a, b) = [\tau(a, b)]^{-1}$.

Let us first consider the Hoffman constant $c_1(a) = [\sigma_1(a)]^{-1}$, where

$$\sigma_1(a) := \inf\{d(0, \text{co}(a_J)) : J \subseteq T, J \text{ is compact}, 0 \notin \text{co}(a_J)\}, \tag{32}$$

which is an extension of the one in [3]. It is obvious that $\sigma_1(a) \le \tau(a, b)$. That is,

$$d(x, S_{a,b}) \le c_1(a)[f_{a,b}(x)]_+ \quad \text{for all } x \in \mathbb{R}^n.$$

Theorem 9 *Suppose that $a \in C(T, \mathbb{R}^n)$ satisfies*

$$0 \notin \text{Bdry}\,(co(a_J)) \quad \textit{for all compact subsets } J \subseteq T. \tag{33}$$

Then function σ_1 defined by (32) is positive and Lipschitz near a.

Conversely, if $0 \in \text{Bdry}\,(co(a_J))$ for some compact subset $J \subset T$, then for any $x \in \mathbb{R}^n$ and $\varepsilon > 0$ there exist $a_\varepsilon \in C(T, \mathbb{R}^n)$; $b_\varepsilon \in C(T, \mathbb{R})$ such that

$$x \in S_{a_\varepsilon,b_\varepsilon}, \quad \|a_\varepsilon - a\| \le \varepsilon, \quad \|b_\varepsilon - b\| \le \varepsilon, \quad \textit{and} \quad \tau(a_\varepsilon, b_\varepsilon) < \varepsilon,$$

where $b(t) = \langle a(t), x\rangle$ for all $t \in T$.

Proof. We first prove that the infimum in the definition of $\sigma_1(a)$ is actually the minimum, that is,

$$\sigma_1(a) := \min\{d(0, \text{co}(a_J)) : J \subseteq T, J \text{ is compact}, 0 \notin \text{co}(a_J)\}, \tag{34}$$

which implies immediately $\sigma_1(a') > 0$ for all a' near a.

Indeed, by the definition of $\sigma_1(a)$ and according to Carathéodory's theorem, there exist sequences $(t_i^k) \subseteq T$, and $(\lambda_i^k) \subseteq \mathbb{R}_+$ $(i = 1, \ldots, n+1)$ such that

$$\sum_{i=1}^{n+1} \lambda_i^k = 1;\ 0 \notin \operatorname{co}\{a(t_i^k) : i = 1, \ldots, n+1\} \text{ and } \lim_{k\to\infty} \left\| \sum_{i=1}^{n+1} \lambda_i^k a(t_i^k) \right\| = \sigma_1(a).$$

By the compactness of T, without loss of generality, we can assume that $(t_i^k) \to t_i$; $(\lambda_i^k) \to \lambda_i$ $(i = 1, \ldots, n+1)$. Therefore, by the continuity of a, one obtains

$$\left\| \sum_{i=1}^{n+1} \lambda_i a(t_i) \right\| = \sigma_1(a).$$

Moreover, since $0 \notin \operatorname{co}\{a(t_i^k) : i = 1, \ldots, n+1\}$, then $0 \notin \operatorname{Int}(\operatorname{co}\{a(t_i) : i = 1, \ldots, n+1\})$. This together with the assumption (33) yields $0 \notin \operatorname{co}\{a(t_i) : i = 1, \ldots, n+1\}$, and the relation (34) is shown.

Let us now prove that σ_1 is Lipschitz near a. For each $a' \in C(T, \mathbb{R}^n)$, set

$$\mathcal{T}(a') = \{J \subseteq T : \quad J \text{ is compact},\ 0 \notin \operatorname{co}(a'_J)\}.$$

Then, by (34), we can find a neighborhood $\mathcal{U}$ of a such that

$$\sigma_1(a') > \sigma_1(a)/2 \quad \text{for all } a' \in \mathcal{U};$$

$$\operatorname{co}(a^1_J) \subseteq \operatorname{co}(a^2_J) + (\sigma_1(a)/4)B_{\mathbb{R}^n} \quad \text{for all compact } J \subseteq T \text{ and all } a^1,\ a^2 \in \mathcal{U},$$

where $B_{\mathbb{R}^n}$ stands for the unit ball in $\mathbb{R}^n$. These relations imply immediately that $\mathcal{T}(a^1) = \mathcal{T}(a^2) = \mathcal{T}(a)$ for all a^1, $a^2 \in \mathcal{U}$. Therefore, we have by a simple computation

$$d(0, \operatorname{co}(a^1_J)) \leq d(0, \operatorname{co}(a^2_J)) + \|a^1 - a^2\| \quad \text{for all } a^1,\ a^2 \in \mathcal{U};\ \text{all } J \in \mathcal{T}(a),$$

consequently, $\sigma_1(a^1) \leq \sigma_1(a^2) + \|a^1 - a^2\|$. Thus, σ_1 is Lipschitz (of rank 1) near a.

Conversely, assume now that $0 \in \operatorname{Bdry}(\operatorname{co}(a_J))$ for some compact subset $J \subseteq T$. Let $x \in \mathbb{R}^n$ and $\varepsilon > 0$. Define $b'_\varepsilon \in C(T, \mathbb{R})$ by $b'_\varepsilon(t) := \langle a(t), x\rangle + \varepsilon d(t, J)/(2\max_{t' \in T} d(t', J))$, where $d(t, J)$ stands for the distance from t to J with respect to the metric on T, and

$$f_{a,b'_\varepsilon}(z) := \max_{t\in T} \big(\langle a(t), z\rangle - b'_\varepsilon(t)\big).$$

Then, we obviously have $\|b'_\varepsilon - b\| \leq \varepsilon/2$, $f_{a,b'_\varepsilon}(x) = 0$, and $\partial f_{a,b'_\varepsilon}(x) = \operatorname{co}(a_J)$. Thus, $0 \in \operatorname{Bdry} \partial f_\varepsilon(x)$. Thanks to Theorem 4 and by observing from its proof, there exist $a_\varepsilon \in C(T, \mathbb{R}^n)$; $b_\varepsilon \in C(T, \mathbb{R})$ such that

$$x \in S_{a_\varepsilon, b_\varepsilon}, \quad \|a_\varepsilon - a\| \leq \varepsilon, \quad \|b_\varepsilon - b'_\varepsilon\| \leq \varepsilon/2, \quad \text{and} \quad \tau(a_\varepsilon, b_\varepsilon) < \varepsilon.$$

To complete the proof it is sufficient to notice that $\|b_\varepsilon - b\| \leq \varepsilon$. □

Let $a \in C(T, \mathbb{R}^n)$ and $b \in C(T, \mathbb{R})$ be such that $S_{a,b} \neq \emptyset$. For $J \subseteq T$ denote $a_J^{-1}(b_J) := \{x \in \mathbb{R}^n : \langle a(t), x\rangle = b(t) \quad \text{for all} \quad t \in J\}$. Set

$$\mathcal{T}_{a,b} := \{J \subseteq T : \ J \text{ is compact},\ \ S_{a,b} \cap a_J^{-1}(b_J) \neq \emptyset \ \text{ or } \ S_{a,0} \cap a_J^{-1}(0) \neq \{0\}\}. \tag{35}$$

Fix a pair $\bar{a} \in C(T, \mathbb{R}^n)$ and $\bar{b} \in C(T, \mathbb{R})$ such that $S_{\bar{a},\bar{b}} \neq \emptyset$. For (a,b) in a neighborhood $\mathcal{U}$ of $(\bar{a}, \bar{b})$ define

$$c_2(a) = [\sigma_2(a)]^{-1}, \ \sigma_2(a) := \inf\{d(0, \mathrm{co}(a_J)) : \ \ J \in \mathcal{T}_{\bar{a},\bar{b}}, \ \ 0 \notin \mathrm{co}(a_J)\}. \tag{36}$$

The number $\sigma_2(a)$ is an extension of a constant denoted by $\tau(a)$ in [3]. The following lemma shows that $c_2(a)$ is also a Hoffman constant for $S_{a,b}$.

Lemma 10 *There exists a neighborhood $\mathcal{U}$ of $(\bar{a}, \bar{b})$ such that $\sigma_2(a) \leq \tau(a,b)$, for all $(a,b) \in \mathcal{U}$, where $\tau(a,b)$ is defined by (31), that is, $c_2(a)$ is a Hoffman constant for $S_{a,b}$.*

Proof. $J_{a,b}(x) \in \mathcal{T}_{\bar{a},\bar{b}}$ for all $(a,b) \in \mathcal{U}$ and all $x \in \mathbb{R}^n$ with $|f_{\bar{a},\bar{b}}(x)| < \delta(\|x\| + 1)$. Indeed, if this does not hold, we can find sequences $(a_k, b_k) \subseteq C(T, \mathbb{R}^n) \times C(T, \mathbb{R})$; $(x_k) \subseteq \mathbb{R}^n$ such that $(a_k, b_k) \to (\bar{a}, \bar{b})$ and $f_{\bar{a},\bar{b}}(x_k)/(\|x_k\| + 1) \to 0$ and for all indexes k, $J_{a_k,b_k}(x_k) \notin \mathcal{T}_{\bar{a},\bar{b}}$. If (x_k) is bounded, then, by relabeling if necessary, we can assume that (x_k) converges to $x_0 \in \mathbb{R}^n$ with $f_{\bar{a},\bar{b}}(x_0) = 0$. We obtain thus when k is sufficiently large:

$$J_{a_k,b_k}(x_k) \subseteq J_{\bar{a},\bar{b}}(x_0) \in \{J \subseteq T : \ \ J \text{ is compact}, \ \ S_{\bar{a},\bar{b}} \cap a_J^{-1}(b_J) \neq \emptyset\} \subseteq \mathcal{T}_{\bar{a},\bar{b}}.$$

Otherwise, we can assume that $\|x_k\| \to \infty$ and $x_k/\|x_k\| \to u$ ($\|u\| = 1$); then, when k is sufficiently large,

$$J_{a_k,b_k}(x_k) \subseteq J_{\bar{a},0}(u) \in \{J \subseteq T : \ \ J \text{ is compact}, \ \ S_{\bar{a},0} \cap a_J^{-1}(0) \neq \{0\}\} \subseteq \mathcal{T}_{\bar{a},\bar{b}},$$

a contradiction.

Let $\mathcal{V} \subseteq \mathcal{U}$ be a neighborhood of $(\bar{a}, \bar{b})$ such that

$$|f_{a,b}(x) - f_{\bar{a},\bar{b}}(x)| < \delta(\|x\| + 1) \quad \text{for all } x \in \mathbb{R}^n, \ \ (a,b) \in \mathcal{V}.$$

Observe from Theorem 1 (ii) and (iii) that

$$\tau(a,b) = \inf_{x \in \mathbb{R}^n, \, f_{a,b}(x)=0} \ \sup_{\varepsilon > 0} \inf\{d(0, \mathrm{co}(a_{J_{a,b}(z)})) : \ z \in B(x, \varepsilon) \setminus S_{a,b}\}.$$

Let $(a,b) \in \mathcal{V}$ be given. Then, for any $x \in \mathbb{R}^n$ with $f_{a,b}(x) = 0$, we have $|f_{\bar{a},\bar{b}}(x)| < \delta(\|x\| + 1)$. Therefore, there exists $\varepsilon > 0$ such that

$$|f_{\bar{a},\bar{b}}(z)| < \delta(\|z\| + 1) \quad \text{for all } z \in B(x, \varepsilon).$$

Hence

$$J_{a,b}(z) \in \mathcal{T}_{\bar{a},\bar{b}} \quad \text{for all } \ z \in B(x, \varepsilon).$$

Obviously, $0 \notin \mathrm{co}(a_{J_{a,b}(z)})$ for any $z \notin S_{a,b}$. Thus

$$\sigma_2(a) \leq d(0, \mathrm{co}(a_{J_{a,b}(z)})) \quad \text{for all } z \in B(x, \varepsilon) \setminus S_{a,b},$$

which implies clearly that $\sigma_2(a) \leq \tau(a,b)$. $\square$

The following theorem is an extension to semi-infinite linear constraint systems of Theorem 4.2 in Azé & Corvellec [3].

Theorem 11 *Suppose that $a \in C(T, \mathbb{R}^n)$ and $b \in C(T, \mathbb{R})$ are such that*

$$0 \notin \operatorname{Bdry}(co(a_J)) \quad \textit{for all } J \in \mathcal{T}_{a,b}. \tag{37}$$

Then function σ_2 defined by (36) is positive and Lipschitz near a.

Conversely, if $0 \in \operatorname{Bdry}(co(a_J))$ for some $J \in \mathcal{T}_{a,b}$, then there exist sequences $(a^k) \subseteq C(T, \mathbb{R}^n)$; $(b^k) \subseteq C(T, \mathbb{R})$ such that for all $k \in \mathbb{N}$,

$$S_{a^k, b^k} \neq \emptyset, \quad \lim_{k \to \infty} (a^k, b^k) = (a, b), \quad \textit{and} \quad \lim_{k \to \infty} \tau(a^k, b^k) = 0.$$

Proof. The proof of the first part is similar to that of Theorem 9. We prove the converse part. Assume that $0 \in \operatorname{Bdry}(\operatorname{co}(a_J))$ for some $J \in \mathcal{T}_{a,b}$. According to the definition of $\mathcal{T}_{a,b}$, we consider the following two cases.

Case 1. $S_{a,b} \cap a_J^{-1}(b_J) \neq \emptyset$. Let $\bar{x} \in S_{a,b} \cap a_J^{-1}(b_J)$. For $\varepsilon > 0$, let $b'_\varepsilon \in C(T, \mathbb{R})$ be defined by $b'_\varepsilon(t) := b(t) + \varepsilon d(t, J)/(2 \max_{t' \in T} d(t', J))$,

$$f_{a, b'_\varepsilon}(x) := \max_{t \in T} \big(\langle a(t), x \rangle - b'_\varepsilon(t) \big).$$

Then, obviously $\|b'_\varepsilon - b\| \leq \varepsilon/2$, $f_{a,b'_\varepsilon}(\bar{x}) = 0$, and $\partial f_{a,b'_\varepsilon}(\bar{x}) = \operatorname{co}(a_J)$. Thus, $0 \in \operatorname{Bdry} \partial f_{a,b'}(\bar{x})$. By observing from the proof of Theorem 4, there exist $a_\varepsilon \in C(T, \mathbb{R}^n)$; $b_\varepsilon \in C(T, \mathbb{R})$ such that

$$\bar{x} \in S_{a_\varepsilon, b_\varepsilon}, \quad \|a_\varepsilon - a\| \leq \varepsilon, \quad \|b_\varepsilon - b'_\varepsilon\| \leq \varepsilon/2, \quad \text{and} \quad \tau(a_\varepsilon, b_\varepsilon) < \varepsilon.$$

To complete the proof it is sufficient to notice that $\|b_\varepsilon - b\| \leq \varepsilon$.

Case 2. $S_{a,0} \cap a_J^{-1}(0) \neq \{0\}$. Pick some $\bar{x} \in S_{a,b}$ and $z \in S_{a,0} \cap a_J^{-1}(0)$ with $\|z\|_2 := \langle z, z \rangle^{1/2} = 1$. For each $k \in \mathbb{N}^*$ and each $t \in J$, set

$$r_k(t) := \frac{b(t) + k^{-1} - \langle a(t), \bar{x} \rangle}{k}.$$

Then,

$$\langle a(t) + r_k(t) z, \bar{x} + kz \rangle = b(t) + k^{-1} + r_k(t) \langle z, \bar{x} \rangle \quad \text{for all } t \in J. \tag{38}$$

Since r_k is a continuous function on the compact subset $J \subseteq T$, by the Tietze-Uryson theorem, there exists a continuous function $\varphi_k \in C(T, \mathbb{R})$ such that

$$\varphi_k(t) = r_k(t) \ \ \forall\, t \in J \quad \text{and} \quad \sup_{t \in T} |\varphi_k(t)| = \sup_{t \in J} |r_k(t)|.$$

For every $k \in \mathbb{N}^*$, let us define $(a^k, b^k) \in C(T, \mathbb{R}^n) \times C(T, \mathbb{R})$ by

$$a^k(t) := a + \varphi_k(t) z; \quad b^k(t) := b(t) + \varphi_k(t) \langle z, \bar{x} \rangle, \ \ t \in T.$$

Then, $\lim_{k \to \infty} (a^k, b^k) = (a, b)$ and for all $k \in \mathbb{N}^*$, $\bar{x} \in S_{a^k, b^k}$. Moreover, by relation (38), when k is sufficiently large, one has

$$\bar{x} + kz \notin S_{a^k, b^k} \quad \text{and} \quad \operatorname{co}(a_J^k) \subseteq \partial f_{a^k, b^k}(\bar{x} + kz),$$

where

$$f_{a^k, b^k}(x) := \max_{t \in T} (\langle a^k(t), x \rangle - b^k(t)).$$

Since $0 \in \operatorname{co}(a_J)$, then (when k is sufficiently large) thanks again to Theorem 1, one has

$$\tau(a^k, b^k) \leq d(0, \operatorname{co}(a_J^k)) \leq \sup_{t \in T} |\varphi_k(t)| \cdot \|z\|.$$

Consequently, $\lim_{k \to \infty} \tau(a^k, b^k) = 0$, which completes the proof. □

Acknowledgement

The authors wish to thank the anonymous referees for careful reading of the paper and valuable comments and suggestions, which helped us improve the presentation.

The research of Huynh Van Ngai was supported by XLIM (Departement of Mathematics and Informatics), UMR 6172, University of Limoges and has been partially supported by NAFOSTED.

References

[1] Auslender, A., and Teboulle, M. *Asymptotic Cones and Functions in Optimization and Variational Inequalities*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2003.

[2] Azé, D. A survey on error bounds for lower semicontinuous functions. In *Proceedings of 2003 MODE-SMAI Conference* (2003), vol. 13 of *ESAIM Proc.*, EDP Sci., Les Ulis, pp. 1–17.

[3] Azé, D., and Corvellec, J.-N. On the sensitivity analysis of Hoffman constants for systems of linear inequalities. *SIAM J. Optim. 12*, 4 (2002), 913–927.

[4] Azé, D., and Corvellec, J.-N. Characterizations of error bounds for lower semicontinuous functions on metric spaces. *ESAIM Control Optim. Calc. Var. 10*, 3 (2004), 409–425.

[5] Burke, J. V., and Deng, S. Weak sharp minima revisited. I. Basic theory. *Control Cybernet. 31*, 3 (2002), 439–469. Well-Posedness in Optimization and Related Topics (Warsaw, 2001).

[6] Burke, J. V., and Deng, S. Weak sharp minima revisited. II. Application to linear regularity and error bounds. *Math. Program., Ser. B 104*, 2-3 (2005), 235–261.

[7] Deng, S. Perturbation analysis of a condition number for convex inequality systems and global error bounds for analytic systems. *Math. Programming, Ser. A 83*, 2 (1998), 263–276.

[8] Ekeland, I. On the variational principle. *J. Math. Anal. Appl. 47* (1974), 324–353.

[9] Fabian, M., Henrion, R., Kruger, A. Y., and Outrata, J. V. Error bounds: necessary and sufficient conditions. To be published.

[10] Hantoute, A., and López, M. A. A complete characterization of the subdifferential set of the supremum of an arbitrary family of convex functions. *J. Convex Anal. 15*, 4 (2008), 831–858.

[11] Hantoute, A., López, M. A., and Zălinescu, C. Subdifferential calculus rules in convex analysis: a unifying approach via pointwise supremum functions. *SIAM J. Optim. 19*, 2 (2008), 863–882.

[12] Hoffman, A. J. On approximate solutions of systems of linear inequalities. *J. Research Nat. Bur. Standards 49* (1952), 263–265.

[13] Ioffe, A. D., and Tikhomirov, V. M. *Theory of Extremal Problems*, vol. 6 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1979.

[14] Kruger, A. Y., Ngai, H. V., and Théra, M. Stability of error bounds for convex constraint systems in Banach spaces. To be published.

[15] Lewis, A. S., and Pang, J.-S. Error bounds for convex inequality systems. In *Generalized Convexity, Generalized Monotonicity: Recent Results (Luminy, 1996)*, vol. 27 of *Nonconvex Optim. Appl.* Kluwer Acad. Publ., Dordrecht, 1998, pp. 75–110.

[16] Luo, X.-D., and Luo, Z.-Q. Extension of Hoffman's error bound to polynomial systems. *SIAM J. Optim. 4*, 2 (1994), 383–392.

[17] Luo, Z.-Q., and Tseng, P. Perturbation analysis of a condition number for linear systems. *SIAM J. Matrix Anal. Appl. 15*, 2 (1994), 636–660.

[18] Ng, K. F., and Zheng, X. Y. Error bounds for lower semicontinuous functions in normed spaces. *SIAM J. Optim. 12*, 1 (2001), 1–17.

[19] Ngai, H. V., and Théra, M. Error bounds for convex differentiable inequality systems in Banach spaces. *Math. Program., Ser. B 104*, 2-3 (2005), 465–482.

[20] Ngai, H. V., and Théra, M. Error bounds for systems of lower semicontinuous functions in Asplund spaces. *Math. Program., Ser. B 116*, 1-2 (2009), 397–427.

[21] Pang, J.-S. Error bounds in mathematical programming. *Math. Programming, Ser. B 79*, 1-3 (1997), 299–332. Lectures on Mathematical Programming (ISMP97) (Lausanne, 1997).

[22] Penot, J.-P. Error bounds, calmness and their applications in nonsmooth analysis. To be published.

[23] Rockafellar, R. T. *Convex Analysis.* Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

[24] Wu, Z., and Ye, J. J. On error bounds for lower semicontinuous functions. *Math. Program., Ser. A 92*, 2 (2002), 301–314.

[25] Zheng, X. Y., and Ng, K. F. Perturbation analysis of error bounds for systems of conic linear inequalities in Banach spaces. *SIAM J. Optim. 15*, 4 (2005), 1026–1041.