On bipartite graphs of diameter 3 and defect 2

Charles Delorme[1,*] Leif K. Jørgensen[2,†]

Mirka Miller[3,4, ‡] Guillermo Pineda-Villavicencio[5,6,§]

[1] *Lab. de Recherche en Informatique*

University de Paris-Sud

91405 Orsay, France

[2] *Department of Mathematics and Computer Science*

Aalborg University

F. Bajers Vej 7, DK-9220 Aalborg Ø, Denmark

[3] *School of Electrical Engineering and Computer Science*

The University of Newcastle, Australia

Newcastle, Australia

[4] *Department of Mathematics*

University of West Bohemia

Pilsen, Czech Republic

[5] *School of Information Technology and Mathematical Sciences*

University of Ballarat

Ballarat, Australia

[6] *Department of Computer Science*

University of Oriente

Santiago de Cuba, Cuba

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Abstract

We consider bipartite graphs of degree $\Delta \geq 2$, diameter $D = 3$, and defect 2 (having 2 vertices less than the bipartite Moore bound). Such graphs are called bipartite $(\Delta, 3, -2)$-graphs. We prove the uniqueness of the known bipartite $(3, 3, -2)$-graph and bipartite $(4, 3, -2)$-graph. We also prove several necessary conditions for the existence of bipartite $(\Delta, 3, -2)$-graphs. The most general of these conditions is that either Δ or $\Delta - 2$ must be a perfect square. Furthermore, in some cases for which the condition holds, in particular, when $\Delta = 6$ and $\Delta = 9$, we prove the non-existence of the corresponding bipartite $(\Delta, 3, -2)$-graphs, thus establishing that there are no bipartite $(\Delta, 3, -2)$-graphs, for $5 \leq \Delta \leq 10$.

Keywords: Degree/diameter problem for bipartite graphs, bipartite Moore bound, bipartite Moore graphs, defect.

1 Introduction

An upper bound on the maximum number of vertices that a bipartite graph of maximum degree Δ and diameter D can have is given by:

$$M^b_{\Delta,D} = 1+\Delta+\Delta(\Delta-1)+\cdots+\Delta(\Delta-1)^{D-2}+(\Delta-1)^{D-1} = 2\left[1+(\Delta-1)+\cdots+(\Delta-1)^{D-1}\right]$$

This expression is known as the *bipartite Moore bound* and the bipartite graphs attaining this bound are known as *bipartite Moore graphs*. Studies of bipartite Moore graphs can be found in [1, 2, 17]. See also the survey by Miller and Širáň [13].

We consider the following problem.

> *Degree/diameter problem for bipartite graphs*: Given natural numbers Δ and D, find the largest possible number $N^b_{\Delta,D}$ of vertices in a bipartite graph of maximum degree Δ and diameter D.

[*]cd@lri.fr
[†]leif@math.aau.dk
[‡]mirka.miller@newcastle.edu.au
[§]work@guillermo.com.au

A general upper bound on $N^b_{\Delta,D}$ is given by $M^b_{\Delta,D}$. At present, there are only a few exact values of $N^b_{\Delta,D}$ known. In particular, for $D=2$ and any $\Delta \geq 2$, and for $D=3,4$ and 6 whenever $\Delta-1$ is a prime power, we have $N^b_{\Delta,D}=M^b_{\Delta,D}$. Additionally, we know the value of $N^b_{3,5}=M^b_{3,5}-6$; it was found by Bond and Delorme [4], see also [10].

The vertex set of a graph Γ is denoted by $V(\Gamma)$, and its edge set by $E(\Gamma)$. We call a bipartite graph of maximum degree Δ, diameter D and order $M^b_{\Delta,D}-\delta$ (for $\delta>0$) a *bipartite* $(\Delta, D, -\delta)$-*graph*. The parameter δ is called the *defect*.

Let $x \in V(\Gamma)$ and $X \subseteq V(\Gamma)$. We say that X *dominates* x, if x either belongs to X or is adjacent to a vertex of X.

We first find conditions for δ under which a bipartite $(\Delta, D, -\delta)$-graph must be regular of degree Δ.

Let Γ be a bipartite graph of maximum degree $\Delta \geq 3$ and diameter $D \geq 3$, and suppose that Γ contains a vertex u of degree $\Delta-1$. We use the standard decomposition for a bipartite graph with respect to an edge uv [3]. For $0 \leq i \leq D-1$, we count the vertices at distance i from u and at distance $i+1$ from v, and the vertices at distance i from v and at distance $i+1$ from u. Then Γ has at most $M^b_{\Delta,D}-[1+(\Delta-1)+\ldots+(\Delta-1)^{D-2}]$ vertices. Consequently,

Proposition 1.1 *For* $\delta < \left[1+(\Delta-1)+(\Delta-1)^2+\ldots+(\Delta-1)^{D-2}\right]$, $\Delta \geq 3$ *and* $D \geq 3$, *a bipartite* $(\Delta, D, -\delta)$*-graph is regular.*

Moreover, for D odd, by using a more careful counting argument, we obtain Proposition 1.2.

Proposition 1.2 *For* $\delta < 2\left[(\Delta-1)+(\Delta-1)^3+\ldots+(\Delta-1)^{D-2}\right]$, $\Delta \geq 3$ *and odd* $D \geq 3$, *a bipartite* $(\Delta, D, -\delta)$*-graph is regular.*

Proof. Suppose that Γ contains a vertex u of degree $\Delta-1$. Then we may assume that the larger partite set of Γ contains u. Therefore, the number of vertices in the larger partite set is at most $1+(\Delta-1)^2+(\Delta-1)^4+\ldots+(\Delta-1)^{D-1}$. Thus, Γ would have at most $2[1+(\Delta-1)^2+(\Delta-1)^4+\ldots+(\Delta-1)^{D-1}]$ vertices, that is, $M^b_{\Delta,D}-2\left[(\Delta-1)+(\Delta-1)^3+\ldots+(\Delta-1)^{D-2}\right]$ vertices. $\square$

In this paper, we consider bipartite graphs of degree $\Delta \geq 2$ and diameter 3, having 2 vertices less than the bipartite Moore bound, that is, bipartite $(\Delta, 3, -2)$-graphs.

When $\Delta = 2$, bipartite $(2, 3, -2)$-graphs need not be regular; the unique bipartite $(2, 3, -2)$-graph is the path of length 3.

From now on, we assume $\Delta \geq 3$. By Proposition 1.1, a bipartite $(\Delta, 3, -2)$-graph is regular.

For $\Delta \geq 3$ and $D \geq 3$, there are only two known examples of bipartite $(\Delta, D, -2)$-graphs, namely, a bipartite $(3, 3, -2)$-graph and a bipartite $(4, 3, -2)$-graph, both shown in Fig. 1. In this paper we prove the uniqueness of these two graphs.

Moreover, we derive necessary conditions for the existence of bipartite $(\Delta, 3, -2)$-graphs. The most general of these conditions is that either Δ or $\Delta - 2$ must be a perfect square. However, these conditions are not sufficient. This is evidenced for example by the non-existence (proved in this paper) of bipartite $(6, 3, -2)$-graphs and bipartite $(9, 3, -2)$-graphs.

Our results are obtained using three different methods. We establish two interesting one-to-one correspondences involving bipartite $(\Delta, 3, -2)$-graphs; the first with normally regular digraphs (NRD), introduced by Jørgensen in [11]; and the second with symmetric group divisible designs [5]. Additionally, we make use of the one-to-one correspondence between symmetric matrices and quadratic forms.

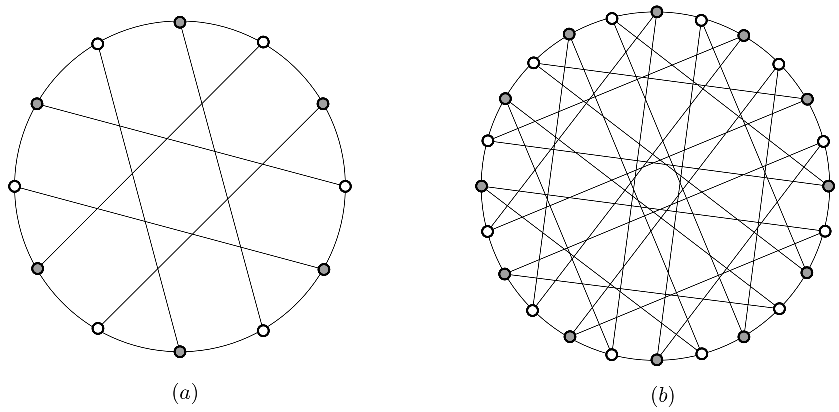


Figure 1: Two known bipartite $(\Delta, D, -2)$-graphs, for $\Delta \geq 3$ and $D \geq 3$, the unique bipartite $(3, 3, -2)$-graph (a) and the unique bipartite $(4, 3, -2)$-graph (b).

2 Preliminaries

Let Γ be a bipartite $(\Delta, D, -2)$-graph of order n, for $\Delta \geq 3$ and $D \geq 3$. Then the girth of Γ is $2r = 2(D-1) \geq 4$, and every vertex v of Γ is contained in exactly one cycle of length $2D-2$, denoted by C_{2D-2}. We call the vertex at distance r from v and contained in the C_{2r} the *repeat* of v and we denote it by $rep(v)$. Let then B be the so-called *defect matrix* of Γ, a permutation matrix satisfying $B^2 = I_n$ and defined by

$$(B)_{\alpha,\beta} = \begin{cases} 1 & \text{if } \beta = rep(\alpha) \\ 0 & \text{otherwise} \end{cases}$$

where I_n is the identity matrix of order n.

In Proposition 2.1, we present an interesting property of the function *rep*. To prove it, we will make use of the following lemma.

Lemma 2.1 *Let C_1 and C_2 be two distinct cycles of length $2D-2$ in Γ. Let $x \in V(C_1)$ and $y \in V(C_2)$ such that xy is an edge of Γ. Then $rep(x)$ and $rep(y)$ are adjacent in Γ.*

Proof. Let us use the standard decomposition for a graph of even girth with respect to an edge xy [3]. For $0 \leq i \leq D-1$, let X_i be the set of vertices at distance i from x and distance $i+1$ from y, and let Y_i be the set of vertices at distance i from y and distance $i+1$ from x.

Since Γ has defect 2 and the partite sets are of equal size, $|X_i| = |Y_i| = (\Delta-1)^i$, for $i \leq D-2$ and $|X_{D-1}| = |Y_{D-1}| = (\Delta-1)^{D-1} - 1$.

We may assume (as $\Delta \geq 3$) that the edge xy is not on a cycle of length $2D-2$. Then there exist vertices x' and y' such that $x' = rep(x) \in X_{D-1}$ and x' has two neighbors in X_{D-2}, and $y' = rep(y) \in Y_{D-1}$ and y' has two neighbors in Y_{D-2}. A vertex $z \in X_1$ has distance $D-1$ from y'. Therefore, y' has a neighbor in X_{D-1} at distance $D-2$ from z. Since $|X_1| = \Delta - 1$ and y' has only $\Delta - 2$ neighbors in X_{D-1}, it follows that y' is adjacent to x'. $\square$

Proposition 2.1 *The function rep is an automorphism of Γ (an involution of Γ) that preserves each partite set when the diameter is odd and swaps the partite sets when the diameter is even.*

Proof. Let x and y be two vertices in Γ. If x and y are adjacent vertices such that the edge xy is not on a cycle of length $2D-2$ then, by Lemma 2.1, $rep(x)$ and $rep(y)$ are adjacent. If in addition x and y are adjacent vertices of a $(2D-2)$-cycle C then their repeats (vertices at distance $D-1$ from x and y on C, respectively) are also adjacent. Therefore, the proposition follows. □

Next we define two combinatorial objects: normally regular digraphs and symmetric group divisible designs.

Normally regular digraphs were introduced in [11], while a definition of symmetric group divisible designs can be found in [5].

A *normally regular digraph*, denoted by $\mathrm{NRD}(v,k,\lambda,\mu)$, is a k-regular digraph of order v, with the property that two adjacent vertices have exactly λ common out-neighbors and λ common in-neighbors, and two non-adjacent vertices have exactly μ common out-neighbors and μ common in-neighbors.

A *symmetric group divisible design* with m points and m blocks is an incidence structure with the following properties:

(i) Each block is incident with exactly k points.

(ii) The $m = rp$ points are partitioned into r groups, each of p points.

(iii) Any pair of points in the same group is incident with exactly μ_1 blocks.

(iv) Any pair of points not in the same group is incident with exactly μ_2 blocks.

We further assume that there is a one-to-one mapping of the points onto the blocks, and that there is a one-to-one mapping of the blocks onto the points, such that a point is incident with a block if, and only if, the image of the point is incident with the image of the block.

The existence of the two mappings implies that in the definition we may interchange the word "point" with the word "block".

Interesting properties of symmetric group divisible designs can be found in [5].

Henceforth, we concentrate on the case of diameter 3. By Γ, we denote a bipartite $(\Delta, 3, -2)$-graph, $\Delta \geq 3$.

3 Correspondence between normally regular digraphs and bipartite $(\Delta, 3, -2)$-graphs

In this section, we first establish a one-to-one correspondence between normally regular digraphs and bipartite $(\Delta, 3, -2)$-graphs (Theorem 3.1). Then we use this correspondence to prove the uniqueness of the known bipartite $(3, 3, -2)$-graph and bipartite $(4, 3, -2)$-graph (Theorem 3.2).

Let us now define a directed quotient digraph Λ with respect to a partition of the vertex set of Γ into 4-cycles. Denoting by V_1 and V_2 the partite sets of Γ, Λ is the digraph obtained from Γ by directing all edges from V_1 to V_2, except those edges contained in a 4-cycle, and then contracting each (undirected) 4-cycle to a vertex and replacing multiple directed edges (*arcs*) by single arcs.

Lemma 3.1 *Let Λ be a digraph described as above. Then Λ has $\frac{\Delta^2-\Delta}{2}$ vertices, each having in- and out-degree $\Delta - 2$. Any pair of non-adjacent vertices has exactly 2 common out-neighbors and 2 common in-neighbors, and a pair of adjacent vertices has no common out-neighbors or in-neighbors.*

Proof. Clearly, Λ has $\frac{\Delta^2-\Delta}{2}$ vertices.

For any 4-cycle C in Γ, a vertex in $V(C) \cap V_1$ has neighbors in $\Delta - 2$ other 4-cycles. Since both vertices in $V(C) \cap V_1$ have neighbors in the same 4-cycles (by Lemma 2.1), C has out-degree $\Delta - 2$ in Λ. Analogously, it can be seen that C has in-degree $\Delta - 2$ in Λ.

Let C and C' be 4-cycles in Γ. If there is an edge between C and C' in Γ then, by Lemma 2.1, there are exactly two edges between C and C', the subgraph spanned by $C \cup C'$ has diameter 3, and no vertex outside $C \cup C'$ has a neighbor in both C and C'. Thus, in Λ, there is an arc joining C and C' but no vertex is dominated by both C and C'.

Suppose that no edge joins C and C' in Γ (and thus, no arc joins C and C' in Λ). Let $V(C) \cap V_1 = \{C_{1,x}, C_{1,y}\}$ and $V(C') \cap V_1 = \{C'_{1,x}, C'_{1,y}\}$. Since the distance between $C_{1,x}$

and $C'_{1,x}$ is 2 and by Lemma 2.1, there is a 4-cycle C'' with $V(C'') \cap V_2 = \{C''_{2,x}, C''_{2,y}\}$ so that $C_{1,x}C''_{2,x}, C'_{1,x}C''_{2,x}, C_{1,y}C''_{2,y}, C'_{1,y}C''_{2,y} \in E(\Gamma)$. Moreover, since the distance between $C_{1,x}$ and $C'_{1,y}$ is 2, there is a 4-cycle C''' with $V(C''') \cap V_2 = \{C'''_{2,x}, C'''_{2,y}\}$ so that $C_{1,x}C'''_{2,x}, C'_{1,y}C'''_{2,x}, C_{1,y}C'''_{2,y}, C'_{1,x}C'''_{2,y} \in E(\Gamma)$. No other vertex has two neighbors in $(V(C) \cup V(C')) \cap V_1$. It follows that in Λ, C and C' have two common out-neighbors, namely, C'' and C'''. Common in-neighbors are counted in a similar way. □

Thus, we see that the digraph defined above from a bipartite $(\Delta, 3, -2)$-graph is an NRD($\frac{\Delta^2-\Delta}{2}, \Delta-2, 0, 2$).

We now give each arc of a directed graph Λ a sign "+" or "−". We assign the signs as follows: let V_1 and V_2 be the partite sets of a bipartite $(\Delta, 3, -2)$-graph Γ, and let Λ be a digraph defined as above. For each 4-cycle C in Γ, we choose a labeling of its vertices so that $V(C) \cap V_1 = \{C_{1,x}, C_{1,y}\}$ and $V(C) \cap V_2 = \{C_{2,x}, C_{2,y}\}$. Now let C and C' be 4-cycles in Γ so that there is an arc directed from C to C' in Λ. Then, in Γ the set of edges between C and C' is either $\{C_{1,x}C'_{2,x}, C_{1,y}C'_{2,y}\}$ or $\{C_{1,x}C'_{2,y}, C_{1,y}C'_{2,x}\}$. In the first case, we give the sign "+" to the arc directed from C to C' in Λ, whereas in the second case, the sign is "−".

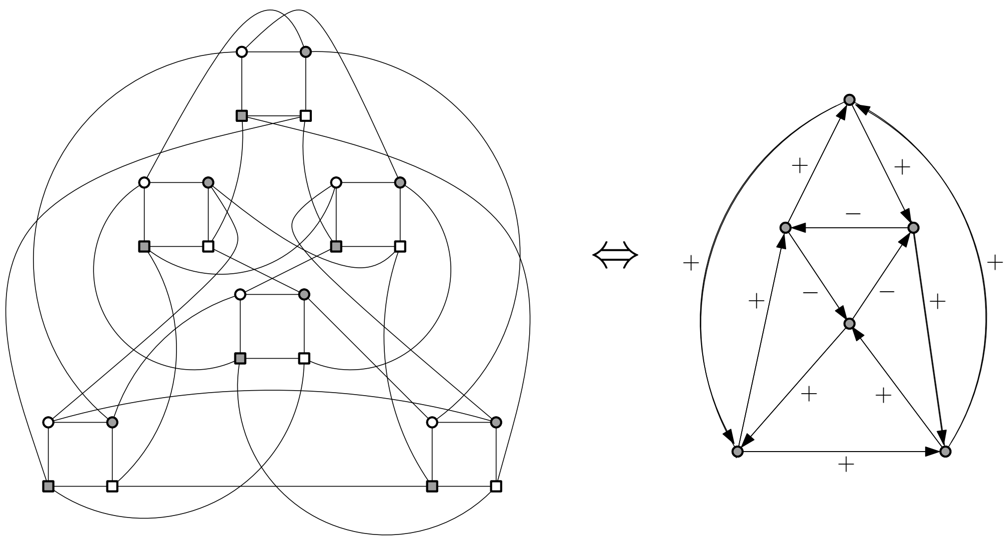
An *antidirected* 4-cycle in Λ is a set of four vertices x_1, x_2, x_3, x_4, where x_2 and x_4 are out-neighbors of x_1 and x_3. We say that an antidirected 4-cycle is *negative* if it has either three positive arcs and one negative arc or if it has three negative arcs and one positive arc, otherwise the antidirected cycle is *positive.*

Theorem 3.1 *There exists a bipartite* $(\Delta, 3, -2)$*-graph if, and only if, there exists an NRD(*$\frac{\Delta^2-\Delta}{2}, \Delta-2, 0, 2$*) with arcs signed so that every antidirected 4-cycle is negative.*

Proof. From the previous assignment of signs, we see that every antidirected 4-cycle in Λ is negative, as every vertex in Γ is in just one 4-cycle.

Conversely, if Λ is a NRD($\frac{\Delta^2-\Delta}{2}, \Delta-2, 0, 2$), where the arcs have signs so that every antidirected 4-cycle is negative, then we construct an undirected graph Γ by replacing each vertex C of Λ by a 4-cycle $C_{1,x}, C_{2,x}, C_{1,y}, C_{2,y}$, and by replacing each positive arc directed from C to C' by edges $C_{1,x}C'_{2,x}, C_{1,y}C'_{2,y}$ and a negative arc directed from C to C' by edges $C_{1,x}C'_{2,y}, C_{1,y}C'_{2,x}$. Then

Γ is a bipartite Δ-regular graph with $2(\Delta^2 - \Delta)$ vertices such that every vertex is contained in exactly one 4-cycle. Thus, Γ is a bipartite $(\Delta, 3, -2)$-graph. $\square$



(a) (b)

Figure 2: Bipartite $(4, 3, -2)$-graph (a) and the corresponding $\mathrm{NRD}(6, 2, 0, 2)$ (b).

To exemplify this one-to-one correspondence, we depict the unique bipartite $(4, 3, -2)$-graph and the corresponding $\mathrm{NRD}(6, 2, 0, 2)$ in Fig. 2. In the proof of Theorem 3.1, we denoted the vertices of a 4-cycle C by $C_{1,x}$, $C_{1,y}$, $C_{2,x}$ and $C_{2,y}$, whereas in Fig. 2, these vertices are depicted by a white circle, a white square, a black circle and a black square, respectively. Note that, in the unique $\mathrm{NRD}(6, 2, 0, 2)$, all the antidirected 4-cycles are negative.

Note also that, for a 4-cycle C in Γ, the labeling of vertices in $V(C) \cap V_1$ is arbitrary. We may interchange the labels, and thus change the sign of the arcs directed out from C in Λ. Similarly, we may interchange the labels of $V(C) \cap V_2$, and thus change the sign of the arcs directed into C in Λ. We say that two assignments of signs to arcs are *equivalent* if one assignment can be obtained from the other by a series of changes of signs of arcs directed into or out of a vertex.

Figure 3 shows two equivalent assignments of signs to arcs of the unique $\mathrm{NRD}(6, 2, 0, 2)$.

Note that any triangle in a $\mathrm{NRD}(\frac{\Delta^2-\Delta}{2}, \Delta - 2, 0, 2)$ is a directed triangle.

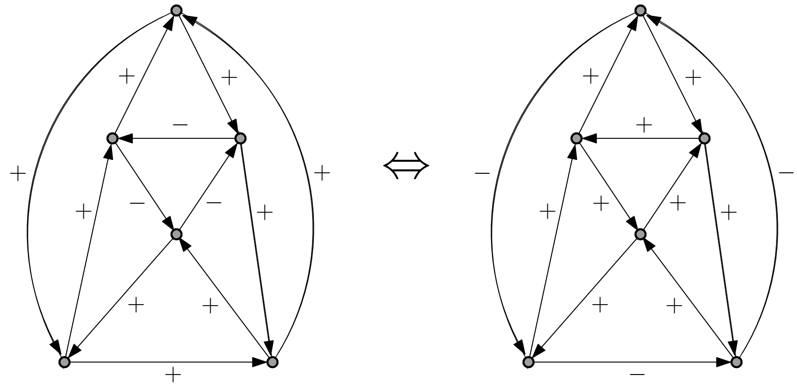


Figure 3: Two equivalent assignments of signs to arcs of the unique NRD$(6, 2, 0, 2)$.

Uniqueness of the known bipartite $(\Delta, 3, -2)$-graphs

Theorem 3.2 *There is a unique bipartite* $(3, 3, -2)$*-graph and a unique bipartite* $(4, 3, -2)$*-graph.*

Proof. Let us consider a bipartite $(\Delta, 3, -2)$-graph. For $\Delta = 3$, the directed triangle is the unique NRD$(3, 1, 0, 2)$. All assignments of signs are equivalent.

For $\Delta = 4$, the unique NRD$(6, 2, 0, 2)$ is the directed graph in Figure 3. All assignments of signs to arcs with all antidirected 4-cycles negative are equivalent.

This implies that the bipartite $(3, 3, -2)$-graph and the bipartite $(4, 3, -2)$-graph, both depicted in Fig. 1, are unique. $\square$

4 Correspondence between symmetric group divisible designs and bipartite $(\Delta, 3, -2)$-graphs

In this section, we first establish that the existence of a bipartite $(\Delta, 3, -2)$-graph Γ is equivalent to the existence of a symmetric group divisible design with m points and m blocks, where $m = \frac{n}{2} = \Delta(\Delta - 1)$.

We use this correspondence to prove that Γ exists only if Δ or $\Delta - 2$ is a perfect square. Then, we make use of the correspondence involving normally regular digraphs to rule out the existence of bipartite $(\Delta, 3, -2)$-graphs, for $5 \leq \Delta \leq 10$.

To see a bipartite $(\Delta, 3, -2)$-graph as a symmetric group divisible design, let the vertices in one partite set of the bipartite graph be represented by the blocks of the symmetric group divisible design, and let the vertices in the other partite set of the bipartite graph be represented by the points of the symmetric group divisible design. Furthermore, verify that, in a bipartite $(\Delta, 3, -2)$-graph, the following assertions hold.

- (i) Each block is incident with exactly Δ points.
- (ii) The m points are partitioned into $\frac{m}{2}$ groups, each of 2 points.
- (iii) Any pair of points in the same group is incident with exactly 2 blocks.
- (iv) Any pair of points not in the same group is incident with exactly 1 block.

As an illustration of the aforementioned one-to-one correspondence, we depict the unique bipartite $(3, 3, -2)$-graph in the form of a symmetric group divisible design in Fig. 4.

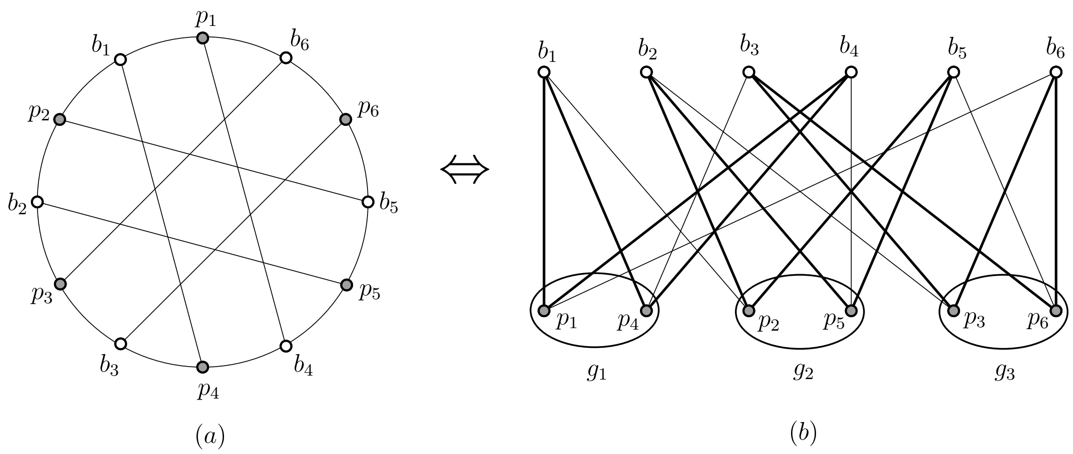


Figure 4: Bipartite $(3, 3, -2)$-graph (a) depicted in the form of a symmetric group divisible design (b), where $b_1, \ldots, b_6$, $p_1, \ldots, p_6$ and g_1, g_2, g_3 are the blocks, points and groups, respectively, of the corresponding design.

We know that the adjacency matrix A of Γ takes the form

$$A = \begin{pmatrix} 0 & M \\ M^T & 0 \end{pmatrix}$$

where M is called the *reduced adjacency matrix* of Γ, and M^T stands for the transpose of M.

With a suitable labeling of the vertices of Γ, the defect matrix B of Γ can be considered as the direct sum of $\frac{n}{2}$ 2×2 matrices of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and, consequently, takes the form $\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$. We call the matrix R the *reduced defect matrix.* Then

$$M^T M = (\Delta - 1)I_m + J_m + R \tag{1}$$

As M^TM, R and J_m are symmetric matrices, they are diagonalizable. We have that R commutes with J_m (every row and column of R has one 1 and $m - 1$ 0's, so $RJ_m = J_mR = J_m$), and obviously with I_m and itself. Therefore, R commutes with M^TM. We also have that M^TM commutes with J_m. Hence, all the three matrices are simultaneously diagonalizable, that is, there is an orthogonal matrix P for which $P^{-1}(M^TM)P$, $P^{-1}RP$ and $P^{-1}J_mP$ are diagonal, and the columns of P are the corresponding eigenvectors for each of these matrices.

If a matrix N has k distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ with corresponding multiplicities $m(\lambda_i)$, we write $\mathrm{Spec}(N) = \begin{pmatrix} \lambda_1 & \ldots & \lambda_k \\ m(\lambda_1) & \ldots & m(\lambda_k) \end{pmatrix}$

As $R^2 = I_m$ and the trace of R is 0, $\mathrm{Spec}(R) = \begin{pmatrix} 1 & -1 \\ \frac{m}{2} & \frac{m}{2} \end{pmatrix}$. It is also known that $\mathrm{Spec}(J_m) = \begin{pmatrix} m & 0 \\ 1 & m-1 \end{pmatrix}$. The eigenvalue 1 of R and the eigenvalue m of J_m are associated with the all 1 vector. Therefore, $\mathrm{Spec}(M^TM) = \begin{pmatrix} \Delta^2 & \Delta & \Delta - 2 \\ 1 & \frac{m}{2} - 1 & \frac{m}{2} \end{pmatrix}$.

Since $A^2 = MM^T \oplus M^TM$, we have $\mathrm{Spec}(A^2) = \begin{pmatrix} \Delta^2 & \Delta & \Delta - 2 \\ 2 & m-2 & m \end{pmatrix}$, where $\oplus$ denotes the direct sum of matrices. Thus,

$$\mathrm{Spec}(A) = \begin{pmatrix} \Delta & -\Delta & \sqrt{\Delta} & -\sqrt{\Delta} & \sqrt{\Delta-2} & -\sqrt{\Delta-2} \\ 1 & 1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2} & \frac{m}{2} \end{pmatrix}$$

Since the characteristic polynomial $f(x)$ of $M^T M$ is $(x-\Delta^2)(x-\Delta)^{\frac{m}{2}-1}[x-(\Delta-2)]^{\frac{m}{2}}$, we have $\det M^T M = \det M^T \det M = (\det M)^2 = \Delta^{\frac{m}{2}+1}(\Delta-2)^{\frac{m}{2}}$. Therefore, we are able to state the following theorem, which is a particular case of Theorem 9 from [5].

Theorem 4.1 *Let Γ be a bipartite $(\Delta, 3, -2)$-graph. If Γ exists then either Δ or $\Delta - 2$ is a perfect square.*

Proof. As $\Delta^{\frac{m}{2}+1}(\Delta-2)^{\frac{m}{2}}$ is a perfect square, for $\frac{m}{2}$ even, Δ must be a perfect square. If instead $\frac{m}{2}$ is odd, then $\Delta - 2$ must be a perfect square. $\square$

Corollary 4.1 *If Γ exists then $\Delta \not\equiv 5, 7 \pmod 8$.*

Non-existence of bipartite $(6, 3, -2)$-graphs and bipartite $(9, 3, -2)$-graphs

The first two values of Δ not ruled out by Theorem 4.1 are 6 and 9.

To prove the non-existence of bipartite $(6, 3, -2)$-graphs and bipartite $(9, 3, -2)$-graphs, we use the correspondence between a subclass of NRD$(\frac{\Delta^2-\Delta}{2}, \Delta-2, 0, 2)$ and bipartite $(\Delta, 3, -2)$-graphs.

Proposition 4.1 *There is no NRD$(15, 4, 0, 2)$.*

Proof. Suppose that Λ is an NRD$(15, 4, 0, 2)$. Let x be a vertex in Λ. Denote the out-neighbours and in-neighbours by $N^+(x)$ and $N^-(x)$, respectively, and the set of the remaining six vertices by U_x. Suppose that $y, z \in U_x$ and that there is an arc directed from y to z. By the definition of an NRD$(15, 4, 0, 2)$, $N^+(x) \cap N^+(y)$ and $N^+(x) \cap N^+(z)$ are disjoint, and both have cardinality 2. Thus they form a partition of $N^+(x)$ and since every triangle is directed, $N^-(z) \cap N^+(x) = \emptyset$. Since $|N^-(z) \cap N^-(x)| = 2$, z has one more in-neighbour, $y' \in U_x$. Since $N^+(z) \cap N^+(y') = \emptyset$, we have $N^+(x) \cap N^+(y') = N^+(x) \cap N^+(y)$. Thus, y and y' have three common out-neighbours, a contradiction. Thus, U_x is independent, implying that x (and, by symmetry, every other vertex in Λ) is contained in an independent set of seven vertices, and that x is adjacent to every other vertex. But this is impossible, as 7 does not divide 15. $\square$

Corollary 4.2 *There are no bipartite* $(6, 3, -2)$*-graphs.*

For the value $\Delta = 9$, we need to consider NRD$(36, 7, 0, 2)$.

Lemma 4.1 *There exist exactly two non-isomorphic NRD*$(36, 7, 0, 2)$*.* $\square$

This lemma was proved by computer enumeration. We used the standard orderly search technique developed by Faradžev [8] and Read [14]. The computation took about 20 seconds on a Linux PC with Pentium 4, 3.2GHz CPU.

These two NRD$(36, 7, 0, 2)$ are described in Appendix A.

Proposition 4.2 *There are no bipartite* $(9, 3, -2)$*-graphs.*

Proof. In order to prove this, we need to show that the two directed graphs mentioned above do not have an assignment of signs to the arcs such that every antidirected 4-cycle is negative.

By the equivalence of assignments, we may assume that the first non-zero entry in each row of the adjacency matrix is an arc with the sign "+". We may also assume that in columns, where no arc has yet been given a sign, the first non-zero entry is an arc with the sign "+". This forces the sign of several other arcs. An easy computer search now proves that the requested assignment is not possible. $\square$

In view of Theorem 4.1, Corollary 4.2 and Proposition 4.2, we obtain

Theorem 4.2 *There is no bipartite* $(\Delta, 3, -2)$*-graph, for* $5 \leq \Delta \leq 10$*.*

By Theorem 4.2, the next possible degree is $\Delta = 11$. Lam *et al.* [12] has proved that a projective plane of order 10 does not exist, and so there is no bipartite Moore graph of degree 11 and diameter 3. Thus, if a bipartite $(11, 3, -2)$ graph exists then it would be the largest bipartite graph of degree 11 and diameter 3. We tried to apply the same computer search technique as in the case $\Delta = 9$, but we were unable to complete this search, as we estimate that the complete search, using current resources and techniques, would take about five years.

5 Necessary conditions obtained from quadratic forms

As $\det M^T M \neq 0$, for $\Delta \geq 3$, we can consider the matrix $M^T M$ as the matrix of a (non-degenerate) quadratic form over the field $\mathbb{Q}$ of rational numbers.

Matrices X and Y are said to be *congruent*, denoted by $X \sim Y$, if there is an invertible matrix Z with rational entries such that $Y = Z^T X Z$.

Let g and h be quadratic forms on a $\mathbb{Q}$-vector space of finite dimension. We say that g is *equivalent* to h ($g \sim h$) if there is an invertible matrix Z with rational entries such that $M_g = Z^T M_h Z$, where M_g and M_h are the symmetric matrices associated with g and h, respectively. Therefore, the equivalence of quadratic forms means that the associated symmetric matrices are congruent.

In this section, our reasoning resembles that used in [5, 15].

We need to state the following known assertions.

Theorem 5.1 (Witt's cancellation theorem, [16, p. 34]) *Let A_1, A_2, B and C be non-singular symmetric matrices over $\mathbb{Q}$. If $(A_1 \oplus B) \sim (A_2 \oplus C)$ and $A_1 \sim A_2$, then $B \sim C$.*

Theorem 5.2 (Lagrange four squares theorem, [16, p. 47]) *Every positive integer k can be written as a sum of four perfect squares.*

Corollary 5.1 (see [7]) *For every positive integer k, the matrices kI_r and I_r are congruent, for $r \equiv 0 \pmod 4$.*

We observe that

$$\begin{pmatrix} \Delta & -1_m \\ 1_m^T & I_m \end{pmatrix} \begin{pmatrix} 1 & 0_m \\ 0_m^T & M^T M - J_m \end{pmatrix} \begin{pmatrix} \Delta & 1_m \\ -1_m^T & I_m \end{pmatrix} = \begin{pmatrix} \Delta^3 & 0_m \\ 0_m^T & M^T M \end{pmatrix}$$

where 1_m (0_m) denotes the vector of order m having all coordinates equal to 1 (0).

As $\Delta^3 I_1 \sim \Delta I_1$ and $M^T M \sim I_m$, we have that $\begin{pmatrix} \Delta^3 & 0_m \\ 0_m^T & M^T M \end{pmatrix} \sim \begin{pmatrix} \Delta & 0_m \\ 0_m^T & I_m \end{pmatrix}$. Therefore, the existence of a bipartite $(\Delta, 3, -2)$-graph implies that the $(m+1) \times (m+1)$ matrices $I_1 \oplus M^T M - J_m$ and $\Delta I_1 \oplus I_m$ are congruent over $\mathbb{Q}$.

We further observe that

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \Delta - 1 & 1 \\ 1 & \Delta - 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2\Delta & 0 \\ 0 & 2\Delta - 4 \end{pmatrix}$$

Let us set $P = \begin{pmatrix} 2\Delta & 0 \\ 0 & 2\Delta - 4 \end{pmatrix}$. Then $I_1 \oplus M^T M - J_m \sim I_1 \oplus \underbrace{P \oplus P \oplus \ldots \oplus P}_{\frac{m}{2}}$

Let $(e_0, e_1, \ldots, e_m)$ be a basis of a $\mathbb{Q}$-vector space such that the matrix $I_1 \oplus \underbrace{P \oplus P \oplus \ldots \oplus P}_{\frac{m}{2}}$ is the associated symmetric matrix of a quadratic form g with respect to that basis. Then, by a permutation of e_i, for $i = 0, \ldots m$, we obtain a basis $(e'_0, e'_1, \ldots, e'_m)$ in which $I_1 \oplus 2\Delta I_{\frac{m}{2}} \oplus (2\Delta - 4) I_{\frac{m}{2}}$ is the associated symmetric matrix of g. Therefore,

$$I_1 \oplus 2\Delta I_{\frac{m}{2}} \oplus (2\Delta - 4) I_{\frac{m}{2}} \sim \Delta I_1 \oplus I_m \tag{2}$$

This result and Corollary 5.1 suggest to us analyzing Equation (2), for $\frac{m}{2} \equiv 0, 1, 2, 3 \pmod 4$.

Case $\frac{m}{2} \equiv 1 \pmod 4$ ($\Delta \equiv 2 \pmod 8$ and $\Delta - 2$ must be a perfect square).

By Theorem 5.1 and Corollary 5.1, Equation (2) turns into $I_1 \oplus 2\Delta I_1 \oplus (2\Delta - 4) I_1 \sim \Delta I_1 \oplus I_2$.

As $I_2 \sim 2I_2$, $I_1 \oplus 2\Delta I_1 \oplus (2\Delta - 4) I_1 \sim \Delta I_1 \oplus 2I_2$. Also, as $(2\Delta - 4) I_1 \sim 2I_1$, we finally have that

$$I_1 \oplus 2\Delta I_1 \sim \Delta I_1 \oplus 2I_1$$

which implies that the diophantine equation $x_1^2 = \Delta x_2^2 + 2x_3^2$ has non-trivial integer solutions.

Case $\frac{m}{2} \equiv 2 \pmod 4$ ($\Delta \equiv 4 \pmod 8$ and Δ must be a perfect square).

By Theorem 5.1 and Corollary 5.1, Equation (2) turns into $I_1 \oplus 2\Delta I_2 \oplus (2\Delta - 4) I_2 \sim \Delta I_1 \oplus I_4$.

As Δ is a perfect square, $2\Delta I_2 \sim 2I_2 \sim I_2$. Therefore, we derive

$$(2\Delta - 4)I_2 \sim I_2 \sim 2I_2$$

which implies that $\Delta - 2$ is a sum of two perfect squares.

The other two cases do not provide new conditions.

Thus we have obtained

Theorem 5.3 *If a bipartite* $(\Delta, 3, -2)$*-graph exists then*

(i) for $\Delta - 2$ *a perfect square and* $\Delta \equiv 2 \pmod 8$*, the diophantine equation* $x_1^2 = \Delta x_2^2 + 2x_3^2$ *must have non-trivial integer solutions, and*

(ii) for Δ *a perfect square and* $\Delta \equiv 4 \pmod 8$*,* $\Delta - 2$ *must be a sum of two perfect squares.*

Some degrees ruled out by Theorem 5.3 are the following.

$$\Delta \bmod 8 \begin{cases} 4: 324 & 1444 & 2116 & 2916 & 4356 & 4900... \\ 2: \;\; 66 & 258 & 402 & 786 & 1026 & 1298... \end{cases}$$

6 Conclusion

In this paper, we have proved the following results for bipartite $(\Delta, 3, -2)$-graphs.

(i) For $\Delta = 3$ and 4, the bipartite $(\Delta, 3, -2)$-graph is unique.

(ii) If a bipartite $(\Delta, 3, -2)$-graph exists then either Δ or $\Delta - 2$ is a perfect square.

(iii) There are no bipartite $(\Delta, 3, -2)$-graphs, for $5 \le \Delta \le 10$.

(iv) For $\Delta - 2$ a perfect square and $\Delta \equiv 2 \pmod 8$, if a bipartite $(\Delta, 3, -2)$-graph exists then the diophantine equation $x_1^2 = \Delta x_2^2 + 2x_3^2$ has non-trivial integer solutions.

(v) For Δ a perfect square and $\Delta \equiv 4 \pmod 8$, if a bipartite $(\Delta, 3, -2)$-graph exists then $\Delta - 2$ is a sum of two perfect squares.

However, for other values of Δ, deciding the existence or otherwise of bipartite $(\Delta, 3, -2)$-graphs remains an open problem.

Contributions to the degree/diameter problem

In terms of the degree/diameter problem, our contribution can be outlined as follows.

It is known that the existence of a bipartite Moore graph of degree Δ and diameter 3 is equivalent to the existence of a projective plane of order $\Delta - 1$, that is, a symmetric $(v, \Delta, 1)$-design with $v = \Delta^2 - \Delta + 1$, see, for instance, [2, 17]. Projective planes of order $\Delta - 1$ are known to exist only when $\Delta - 1$ is a prime power.

The following theorem by Bruck and Ryser [6] is a special case of the classical theorem from Bruck, Ryser and Chowla ([7]) about symmetric designs (see also [15]).

Theorem 6.1 ([6]) *If there is projective plane of order* $\Delta - 1 \equiv 1, 2 \pmod 4$ *then* $\Delta - 1$ *is the sum of two perfect squares.*

By Theorem 6.1, we see that there is no projective plane of order 6. Also by Theorem 4.1, there is no bipartite $(7, 3, -2)$-graph either. Therefore, $N^b_{7,3} \leq M^b_{7,3}$-4.

The non-existence of a projective plane of order 10 was proved by using computer. Therefore, if a bipartite $(11, 3, -2)$-graph existed then we would have $N^b_{11,3} = M^b_{11,3} - 2$. At this moment, we have that $N^b_{11,3} \leq M^b_{11,3} - 2$.

Concerning the non-existence of projective planes of order $\Delta - 1$, when $\Delta - 1$ is not a prime power, nothing else is known. It has been conjectured that such a projective plane does not exist. If this conjecture was true then, for those values of Δ for which there is no bipartite $(\Delta, 3, -2)$-graph, we would have that $N^b_{\Delta,3} \leq M^b_{\Delta,3} - 4$.

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Appendix A. All the NRD$(36, 7, 0, 2)$

We next present the two non-isomorphic NRD$(36, 7, 0, 2)$, denoted by Λ_A and Λ_B. The adjacency matrix of a digraph Λ is defined as follows.

$$(A(\Lambda))_{\alpha,\beta} = \begin{cases} 1 & \text{if } \beta \in N^+(\alpha) \\ 0 & \text{otherwise} \end{cases}$$

For the automorphism group $Aut(\Lambda)$, we give a generating set, and other relevant information. Both Λ_A and Λ_B are vertex-transitive.

$$
A(\Lambda_A) = \begin{pmatrix}
0&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&1&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&1&0&0&0&1&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&1&0&0&0&0&0&0&0&0\\
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1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&1&0&0&0&1&1&0&0&0&1&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&1&0&0&1&1&0&0&0&0&0&0&1&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&1&1&0&0&0&0&0&0&0&0&0&1&0\\
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0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&1&1&0&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0&0&0&0
\end{pmatrix}
$$

$Aut(\Lambda_A) = \langle$(1,30)(2,25)(5,36)(6,35)(7,13)(8,12)(9,16)(14,34)(15,33) (19,20)(21,22)(27,28), (1,4,16,15,18,12,28,35,2,11,30,36,26,33,20,7,9,32, 25,8,24,5,21,14)(3,22,6,23,31,19,34,29,10,27,13,17)$\rangle$.

The order of $Aut(\Lambda_A)$ is 2160.

The digraph Λ_A has two arc-orbits, whose representatives are the arcs $(1, 2)$ and $(1, 3)$, respectively.

$$A(\Lambda_B) = \begin{pmatrix}
0&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
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0&0&0&1&0&1&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&1&0&0\\
0&0&0&0&1&0&1&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&1&0\\
0&0&1&0&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0\\
0&0&0&0&1&0&0&1&0&1&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0\\
0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&1&0&0&1&0\\
0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0\\
0&0&0&1&0&0&0&1&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&0\\
0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&1\\
0&0&1&1&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&1\\
0&0&0&0&0&0&1&1&1&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&1\\
0&0&1&0&1&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&1\\
0&0&0&0&0&1&0&1&1&0&0&0&0&1&0&0&0&0&0&1&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0\\
0&0&0&1&1&0&0&0&0&1&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0\\
0&0&0&0&0&1&1&0&1&0&0&0&1&0&0&0&0&1&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

The digraph Λ_B is arc-transitive. It has automorphism group $\mathrm{PSU}(3, 3^2)$ and was constructed from this group by Iwasaki [9].

$Aut(\Lambda_B) = \langle$(1,34)(4,30)(5,33)(6,27)(7,24)(8,21)(11,15)(12,14)(16,19)(17,20)(22,25)(29,32), (1,5,25,2,12,24,14,18,8,17,32,11)(3,27,6,9,20,10,28,29,34,23,36,33)(4,21,35, 13,22,19)(7,26,16,15,30,31)$\rangle$.