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The linkedness of cubical polytopes: Beyond the cube ☆

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ABSTRACT

A cubical polytope is a polytope with all its facets being combinatorially equivalent to cubes. The paper is concerned with the linkedness of the graphs of cubical polytopes.

A graph with at least $2k$ vertices is *k-linked* if, for every set of k disjoint pairs of vertices, there are k vertex-disjoint paths joining the vertices in the pairs. We say that a polytope is *k-linked* if its graph is k-linked. In a previous paper [3] we proved that every cubical d-polytope is $\lfloor d/2 \rfloor$-linked. Here we strengthen this result by establishing the $\lfloor (d+1)/2 \rfloor$-linkedness of cubical d-polytopes, for every $d \neq 3$.

A graph G is *strongly k-linked* if it has at least $2k+1$ vertices and, for every vertex v of G, the subgraph $G - v$ is k-linked. We say that a polytope is (strongly) *k-linked* if its graph is (strongly) k-linked. In this paper, we also prove that every cubical d-polytope is strongly $\lfloor d/2 \rfloor$-linked, for every $d \neq 3$.

These results are best possible for this class of polytopes.

1. Introduction

The *graph* $G(P)$ of a polytope P is the undirected graph formed by the vertices and edges of the polytope. This paper studies the linkedness of *cubical d-polytopes*, d-dimensional polytopes with all their facets being cubes. A *d-dimensional cube* is the convex hull in $\mathbb{R}^d$ of the 2^d vectors $(\pm 1, \ldots, \pm 1)$. By a cube we mean any polytope whose face lattice is isomorphic to the face lattice of a cube.

Denote by $V(X)$ the vertex set of a graph or a polytope X. Given sets A, B of vertices in a graph, a path from A to B, called an *$A - B$ path*, is a (vertex-edge) path $L := u_0 \ldots u_n$ in the graph such that $V(L) \cap A = \{u_0\}$ and $V(L) \cap B = \{u_n\}$. We write $a - B$ path instead of $\{a\} - B$ path, and likewise, write $A - b$ path instead of $A - \{b\}$ path.

Let G be a graph and X a subset of $2k$ distinct vertices of G. The elements of X are called *terminals*. Let $Y := \{\{s_1, t_1\}, \ldots, \{s_k, t_k\}\}$ be an arbitrary labelling and (unordered) pairing of all the vertices in X. We say that Y is *linked* in G if we can find disjoint $s_i - t_i$ paths for all $i \in [1, k]$, where $[1, k]$ denotes the interval $1, \ldots, k$. The set X is *linked* in G if every such pairing of its vertices is linked in G. Throughout this paper, by a set of disjoint paths, we mean a set of vertex-disjoint paths. If G has at least $2k$ vertices and every set of exactly $2k$ vertices is linked in G, we say that G is *k-linked*. If the graph of a polytope is k-linked, we say that the polytope is also *k-linked*.

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Linkedness is a stronger property than connectivity: let G be a graph with at least $2k$ vertices, and let $S := \{s_1, \dots, s_k\}$ and $T := \{t_1, \dots, t_k\}$ be two disjoint k-element sets of vertices in G. It follows from Menger's theorem that, if G is k-connected then the sets S and T can be joined **setwise** by disjoint paths (namely, by k disjoint $S - T$ paths). By contrast, if G is k-linked then the sets can be joined **pointwise** by disjoint paths.

A closely related problem to linkedness is the classical *disjoint paths problem* [9]: given a graph G and a set $Y := \{\{s_1, t_1\}, \dots, \{s_k, t_k\}\}$ of k pairs of terminals in G, decide whether or not Y is linked in G. A natural optimisation version of this problem is to find the largest subset of the pairs so that there exist disjoint paths connecting the selected pairs.

There is a linear function $f(k)$ such that every $f(k)$-connected graph is k-linked, which follows from works of Bollobás and Thomason [1]; Kawarabayashi, Kostochka, and Yu [6]; and Thomas and Wollan [11]. In the case of polytopes, Larman and Mani [7, Thm. 2] proved that every d-polytope is $\lfloor (d+1)/3 \rfloor$-linked, a result that was slightly improved to $\lfloor (d+2)/3 \rfloor$ in [12, Thm. 2.2]. Gallivan [5] proved that not every polytope is $\lfloor d/2 \rfloor$-linked. In view of this negative result, researchers have focused efforts on finding families of d-polytopes that are $\lfloor d/2 \rfloor$-linked. In his PhD thesis [13, Question 5.4.12], Wotzlaw asked whether every cubical d-polytope is $\lfloor d/2 \rfloor$-linked. In [3] we answer his question in the affirmative by establishing the following theorem.

Theorem 1. *For every $d \geq 1$, a cubical d-polytope is $\lfloor d/2 \rfloor$-linked.*

The paper [3] also established the linkedness of the d-cube.

Theorem 2 *(Linkedness of the cube). For every $d \neq 3$, a d-cube is $\lfloor (d+1)/2 \rfloor$-linked.*

In this paper, we extend these two results as follows:

Theorem 3 *(Linkedness of cubical polytopes). For every $d \neq 3$, a cubical d-polytope is $\lfloor (d+1)/2 \rfloor$-linked.*

Our methodology relies on results on the connectivity of strongly connected subcomplexes of cubical polytopes, whose proof ideas were first developed in [2], and a number of new insights into the structure of d-cube exposed in [3]. One obstacle that forces some tedious analysis is the fact that the 3-cube is not 2-linked.

Let X be a set of vertices in a graph G. Denote by $G[X]$ the subgraph of G induced by X, the subgraph of G that contains all the edges of G with vertices in X. Write $G - X$ for $G[V(G) \setminus X]$. If $X = \{v\}$, then we write $G - v$ instead of $G - \{v\}$.

In our paper [3], we introduce the notion of strong linkedness. We say that a graph G with at least $2k + 1$ vertices is *strongly k-linked* if for every vertex v of G, the subgraph $G - v$ is k-linked. A polytope is *strongly k-linked* if its graph is strongly k-linked. We proved the strong-linkedness of the cube as follows:

Theorem 4 *(Strong linkedness of the cube [3, Thm. 25]). For every $d \geq 1$, a d-cube is strongly $\lfloor d/2 \rfloor$-linked.*

In this paper, we extend this result to cubical polytopes:

Theorem 5 *(Strong linkedness of cubical polytopes). For every $d \neq 3$, a cubical d-polytope is strongly $\lfloor d/2 \rfloor$-linked.*

Unless otherwise stated, the graph theoretical notation and terminology follow from [4] and the polytope theoretical notation and terminology from [14]. Moreover, when referring to graph-theoretical properties of a polytope such as minimum degree, linkedness and connectivity, we mean properties of its graph.

2. Connectivity of cubical polytopes

The aim of this section is to present a couple of results related to the connectivity of strongly connected complexes in cubical polytopes. A pure polytopal complex $\mathcal{C}$ is *strongly connected* if every pair of facets F and F' is connected by a path $F_1 \dots F_n$ of facets in $\mathcal{C}$ such that $F_i \cap F_{i+1}$ is a ridge of $\mathcal{C}$ for each $i \in [1, n-1]$, $F_1 = F$ and $F_n = F'$; we say that such a path is a $(d-1, d-2)$-*path* or a *facet-ridge path* if the dimensions of the faces can be deduced from the context. Two basic examples of strongly connected complexes are given by the complex of all faces of a polytope P, called the *complex* of P and denoted by $\mathcal{C}(P)$, and the complex of all proper faces of P, called the *boundary complex* of P and denoted by $\mathcal{B}(P)$. For the definitions of polytopal complexes and pure polytopal complexes, refer to [14, Section 5.1].

Given a polytopal complex $\mathcal{C}$ with vertex set V and a subset X of V, the subcomplex of $\mathcal{C}$ formed by all the faces of $\mathcal{C}$ containing only vertices from X is said to be *induced by* X and is denoted by $\mathcal{C}[X]$. Removing from $\mathcal{C}$ all the vertices in a subset $X \subset V(\mathcal{C})$ results in the subcomplex $\mathcal{C}[V(\mathcal{C}) \setminus X]$, which we write as $\mathcal{C} - X$. If $X = \{x\}$ we write $\mathcal{C} - x$ rather than $\mathcal{C} - \{x\}$. We say that a subcomplex $\mathcal{C}'$ of a complex $\mathcal{C}$ is a *spanning* subcomplex of $\mathcal{C}$ if $V(\mathcal{C}') = V(\mathcal{C})$. The *graph* of a complex is the undirected graph formed by the vertices and edges of the complex; as in the case of polytopes, we denote the graph of a complex $\mathcal{C}$ by $G(\mathcal{C})$.

For a polytopal complex $\mathcal{C}$, the *star* of a face F of $\mathcal{C}$, denoted $\operatorname{star}(F,\mathcal{C})$, is the subcomplex of $\mathcal{C}$ formed by all the faces containing F, and their faces; the *antistar* of a face F of $\mathcal{C}$, denoted $\operatorname{astar}(F,\mathcal{C})$, is the subcomplex of $\mathcal{C}$ formed by all the faces disjoint from F; and the *link* of a face F, denoted $\operatorname{link}(F,\mathcal{C})$, is the subcomplex of $\mathcal{C}$ formed by all the faces of $\operatorname{star}(F,\mathcal{C})$ that are disjoint from F. That is, $\operatorname{astar}(F,\mathcal{C})=\mathcal{C}-V(F)$ and $\operatorname{link}(F,\mathcal{C})=\operatorname{star}(F,\mathcal{C})-V(F)$. Unless otherwise stated, when defining stars, antistars and links in a polytope, we always assume that the underlying complex is the boundary complex of the polytope.

The first results are from [2].

Lemma 6 *([2, Lem. 8]). Let F be a proper face in the d-cube Q_d. Then the antistar of F is a strongly connected $(d-1)$-complex.*

Proposition 7 *([2, Prop. 13]). Let F be a facet in the star $\mathcal{S}$ of a vertex in a cubical d-polytope. Then the antistar of F in $\mathcal{S}$ is a strongly connected $(d-2)$-subcomplex of $\mathcal{S}$.*

Let v be a vertex in a d-cube Q_d and let v^o denote the vertex at distance d from v, called the vertex *opposite* to v in Q_d; by distance in a cube, we mean the graph-theoretical distance in the cube. In the d-cube Q_d, the facet disjoint from a facet F is denoted by F^o, and we say that F and F^o are a pair of *opposite* facets.

We proceed with a simple but useful remark.

Remark 8. Let P be a cubical d-polytope. Let v be a vertex of P and let F be a face of P containing v, which is a cube. In addition, let v^o be the vertex of F opposite to v in F. The smallest face in the polytope containing both v and v^o is precisely F.

The proof idea in Proposition 7 can be pushed a bit further to obtain a rather technical result that we prove next. Two vertex-edge paths are *independent* if they share no inner vertex.

Lemma 9. *Let P be a cubical d-polytope with $d\ge 4$. Let s_1 be any vertex in P and let $\mathcal{S}_1$ be the star of s_1 in the boundary complex of P. Let s_2 be any vertex in $\mathcal{S}_1$, other than s_1. Define the following sets:*

- *F_1 in $\mathcal{S}_1$, a facet containing s_1 but not s_2;*
- *F_{12} in $\mathcal{S}_1$, a facet containing s_1 and s_2;*
- *$\mathcal{S}_{12}$, the star of s_2 in $\mathcal{S}_1$ (that is, the subcomplex of $\mathcal{S}_1$ formed by the facets of P in $\mathcal{S}_1$ containing s_2);*
- *$\mathcal{A}_1$, the antistar of F_1 in $\mathcal{S}_1$; and*
- *$\mathcal{A}_{12}$, the subcomplex of $\mathcal{S}_{12}$ induced by $V(\mathcal{S}_{12})\setminus(V(F_1)\cup V(F_{12}))$.*

Then the following assertions hold.

(i) The complex $\mathcal{S}_{12}$ is a strongly connected $(d-1)$-subcomplex of $\mathcal{S}_1$.
(ii) If there are more than two facets in $\mathcal{S}_{12}$, then, between any two facets of $\mathcal{S}_{12}$ that are different from F_{12}, there exists a $(d-1,d-2)$-path in $\mathcal{S}_{12}$ that does not contain the facet F_{12}.
(iii) If $\mathcal{S}_{12}$ contains more than one facet, then the subcomplex $\mathcal{A}_{12}$ of $\mathcal{S}_{12}$ contains a spanning strongly connected $(d-3)$-subcomplex.

Proof. Let us prove (i). Let ψ define the natural anti-isomorphism from the face lattice of P to the face lattice of its dual P^*. The facets in $\mathcal{S}_1$ correspond to the vertices in the facet $\psi(s_1)$ in P^* corresponding to s_1; likewise for the facets in $\operatorname{star}(s_2,\mathcal{B}(P))$ and the vertices in $\psi(s_2)$. The facets in $\mathcal{S}_{12}$ correspond to the vertices in the nonempty face $\psi(s_1)\cap\psi(s_2)$ of P^*. The existence of a facet-ridge path in $\mathcal{S}_{12}$ between any two facets J_1 and J_2 of $\mathcal{S}_{12}$ amounts to the existence of a vertex-edge path in $\psi(s_1)\cap\psi(s_2)$ between $\psi(J_1)$ and $\psi(J_2)$. That $\mathcal{S}_{12}$ is a strongly connected $(d-1)$-complex now follows from the connectivity of the graph of $\psi(s_1)\cap\psi(s_2)$ (Balinski's theorem), as desired.

We proceed with the proof of (ii). Let J_1 and J_2 be two facets of $\mathcal{S}_{12}$, other than F_{12}. If there are more than two facets in $\mathcal{S}_{12}$, then the face $\psi(s_1)\cap\psi(s_2)$ is at least bidimensional. As a result, the graph of $\psi(s_1)\cap\psi(s_2)$ is at least 2-connected by Balinski's theorem. By Menger's theorem, there are at least two independent vertex-edge paths in $\psi(s_1)\cap\psi(s_2)$ between $\psi(J_1)$ and $\psi(J_2)$. Pick one such path L^* that avoids the vertex $\psi(F_{12})$ of $\psi(s_1)\cap\psi(s_2)$. Dualising this path L^* gives a $(d-1,d-2)$-path between J_1 and J_2 in $\mathcal{S}_{12}$ that does not contain the facet F_{12}.

We finally prove (iii). Assume that $\mathcal{S}_{12}$ contains more than one facet. We need some additional notation.

- Let F be a facet in $\mathcal{S}_{12}$ other than F_{12}; it exists by our assumption on $\mathcal{S}_{12}$.
- For a facet J in $\mathcal{S}_{12}$, let $\mathcal{A}_1^J$ denote the subcomplex $J-V(F_1)$; that is, $\mathcal{A}_1^J$ is the antistar of $J\cap F_1$ in J.
- For a facet J in $\mathcal{S}_{12}$ other than F_{12}, let $\mathcal{A}_{12}^J$ denote the subcomplex $J-(V(F_1)\cup V(F_{12}))$, the subcomplex of J induced by $V(J)\setminus(V(F_1)\cup V(F_{12}))$.

We require the following claim.

Claim 1. $\mathcal{A}_{12}^F$ contains a spanning strongly connected $(d-3)$-subcomplex $\mathcal{C}^F$.

Proof. We first show that $\mathcal{A}_{12}^F \neq \emptyset$. Denoting by s_1^o the vertex in F opposite to s_1, we have that s_1^o is not in F_1 or in F_{12} by Remark 8. So s_1^o is in $\mathcal{A}_{12}^F$.

Notice that $s_1 \notin \mathcal{A}_1^F$. From Lemma 6 it follows that $\mathcal{A}_1^F$ is a strongly connected $(d-2)$-subcomplex of F. Write

$$\mathcal{A}_1^F = \mathcal{C}(R_1) \cup \cdots \cup \mathcal{C}(R_m),$$

where R_i is a $(d-2)$-face of F for each $i \in [1,m]$. Every $(d-2)$-face in F contains either s_1 or s_1^o, and since we have $s_1 \notin R_i$ for every $R_i \in \mathcal{A}_1^F$, it follows that $s_1^o \in R_i$. Consequently no ridge R_i is contained in F_{12}.

Let

$$\mathcal{C}_i := \mathcal{B}(R_i) - V(F_{12}).$$

As $R_i \not\subset F_{12}$, we have $\dim R_i \cap F_{12} \le d-3$. Furthermore, since $s_1^o \in \mathcal{C}_i$, $\mathcal{C}_i$ is nonempty. If $R_i \cap F_{12} \neq \emptyset$, then $\mathcal{C}_i$ is the antistar of $R_i \cap F_{12}$ in R_i, a spanning strongly connected $(d-3)$-subcomplex of R_i by Lemma 6. If $R_i \cap F_{12} = \emptyset$, then $\mathcal{C}_i$ is the boundary complex of R_i, again a spanning strongly connected $(d-3)$-subcomplex of R_i.

Let

$$\mathcal{C}^F := \bigcup \mathcal{C}_i.$$

Then the complex $\mathcal{C}^F$ is a spanning $(d-3)$-subcomplex of $\mathcal{A}_{12}^F$; we show it is strongly connected.

Take any two $(d-3)$-faces W and W' in $\mathcal{C}^F$. We find a $(d-3,d-4)$-path L in $\mathcal{C}^F$ between W and W'. There exist ridges R and R' in $\mathcal{A}_1^F$ with $W \subset R$ and $W' \subset R'$. Since $\mathcal{A}_1^F$ is a strongly connected $(d-2)$-complex, there is a $(d-2,d-3)$-path $R_{i_1} \dots R_{i_p}$ in $\mathcal{A}_1^F$ between $R_{i_1} = R$ and $R_{i_p} = R'$, with $R_{i_j} \in \mathcal{A}_1^F$ for each $j \in [1,p]$. We will show by induction on the length p of the $(d-2,d-3)$-path $R_{i_1} \dots R_{i_p}$ that there is a $(d-3,d-4)$-path in $\mathcal{C}^F$ between W and W'.

If $p=1$, then $R_{i_1} = R_{i_p} = R = R'$. The existence of the path follows from the strong connectivity of $\mathcal{C}_{i_1}$.

Suppose that the claim is true when the length of the path is $p-1$. We already established that $s_1^o \in R_{i_j}$ for every $j \in [1,p]$ and that $s_1^o \notin F_{12}$. Consequently, we get that $R_{i_{p-1}} \cap R_{i_p} \not\subset F_{12}$, and therefore, $R_{i_{p-1}} \cap R_{i_p} \cap F_{12}$ is a proper face of $R_{i_{p-1}} \cap R_{i_p}$. Hence the subcomplex $\mathcal{B}_{i_{p-1}} := \mathcal{B}(R_{i_{p-1}} \cap R_{i_p}) - V(F_{12})$ of $\mathcal{B}(R_{i_{p-1}} \cap R_{i_p})$ is a nonempty, strongly connected $(d-4)$-complex by Lemma 6; in particular, it contains a $(d-4)$-face U_{i_p}. Furthermore, $\mathcal{B}_{i_{p-1}} \subset \mathcal{C}_{i_{p-1}} \cap \mathcal{C}_{i_p}$.

Let $W_{i_{p-1}}$ and W_{i_p} be $(d-3)$-faces in $\mathcal{C}_{i_{p-1}}$ and $\mathcal{C}_{i_p}$ containing U_{i_p} respectively. By the induction hypothesis, the existence of the $(d-2,d-3)$-path $R_{i_1} \dots R_{i_{p-1}}$ implies the existence of a $(d-3,d-4)$-path L_{p-1} in $\mathcal{C}^F$ from W to $W_{i_{p-1}}$. The strong connectivity of $\mathcal{C}_{i_p}$ gives the existence of a path L_p from W_{i_p} to W'. Finally, the desired $(d-3,d-4)$-path L is the concatenation of these two paths: $L = L_{p-1} W_{i_{p-1}} U_{i_p} W_{i_p} L_p$. The existence of the path L between W and W' completes the proof of Claim 1. □

We are now ready to complete the proof of (iii). The proof goes along the lines of the proof of Claim 1. We let

$$\mathcal{S}_{12} = \bigcup_{i=1}^{m} \mathcal{C}(J_i),$$

where the facets $J_1, \dots, J_m$ are all the facets in P containing s_1 and s_2.

For every $i \in [1,m]$ we let $\mathcal{C}^{J_i}$ be the spanning strongly connected $(d-3)$-subcomplex in $\mathcal{A}_{12}^{J_i}$ given by Claim 1. And we let

$$\mathcal{C} := \bigcup \mathcal{C}^{J_i}.$$

Then $\mathcal{C}$ is a spanning $(d-3)$-subcomplex of $\mathcal{A}_{12}$; we show it is strongly connected.

If there are exactly two facets in $\mathcal{S}_{12}$, namely F_{12} and some other facet F, then the complex $\mathcal{A}_{12}$ coincides with the complex $\mathcal{A}_{12}^F$. The strong $(d-3)$-connectivity of $\mathcal{C}$ is then settled by Claim 1. Hence assume that there are more than two facets in $\mathcal{S}_{12}$; this implies that the smallest face containing s_1 and s_2 in $\mathcal{S}_{12}$ is at most $(d-3)$-dimensional.

Take any two $(d-3)$-faces W and W' in $\mathcal{C}$. Let $J \neq F_{12}$ and $J' \neq F_{12}$ be facets of $\mathcal{S}_{12}$ such that $W \subset J$ and $W' \subset J'$. By (ii), we can find a $(d-1,d-2)$-path $J_{i_1} \dots J_{i_q}$ in $\mathcal{S}_{12}$ between $J_{i_1} = J$ and $J_{i_q} = J'$ such that $J_{i_j} \neq F_{12}$ for any $j \in [1,q]$. We will show that a $(d-3,d-4)$-path L exists between W and W' in $\mathcal{C}$, using an induction on the length q of the path $J_{i_1} \dots J_{i_q}$.

If $q=1$, then W and W' belong to the same facet F in $\mathcal{S}_{12}$, which is different from F_{12}. In this case, W and W' are both in $\mathcal{A}_{12}^F$, and consequently, Claim 1 gives the desired $(d-3,d-4)$-path between W and W' in $\mathcal{A}_{12}^F \subseteq \mathcal{C}$.

Suppose that the induction hypothesis holds when the length of the path is $q-1$. First, we show that there exists a $(d-4)$-face U_q in $C^{J_{i_{q-1}}} \cap C^{J_{i_q}}$. As $J_{i_{q-1}}, J_{i_q} \neq F_{12}$, we obtain that $\mathcal{B}(J_{i_{q-1}} \cap J_{i_q}) - V(F_{12})$ is a nonempty, strongly connected $(d-3)$-subcomplex (Lemma 6); in particular, it contains a $(d-3)$-face K_q. The complex $\mathcal{B}(K_q) - V(F_1)$ is nonempty because

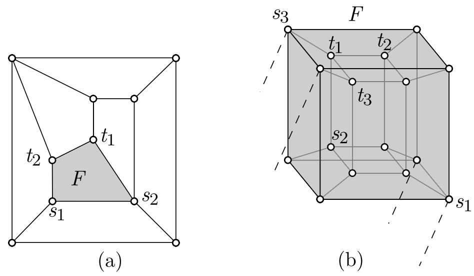

Fig. 1. Examples of Configuration dF. (a) A cubical 3-polytope where s_1 is in Configuration 3F. (b) A facet of a cubical 5-polytope where s_1 is in Configuration 5F.

$s_1 \in F_1$ and $s_1 \notin K_q$ (since K_q does not contain any vertex from F_{12}). Therefore $\mathcal{B}(K_q) - V(F_1)$ is a strongly connected $(d-4)$-subcomplex by Lemma 6. In particular, $\mathcal{B}(K_q) - V(F_1)$ contains a $(d-4)$-face U_q.

Pick $(d-3)$-faces $W_{q-1} \in \mathcal{C}^{J_{i_{q-1}}}$ and $W_q \in \mathcal{C}^{J_{i_q}}$ such that both contain the $(d-4)$ face U_q. The induction hypothesis tells us that there exists a $(d-3, d-4)$-path L_{q-1} from W to W_{q-1} in $\mathcal{C}$. And the strong $(d-3)$-connectivity of $\mathcal{C}^{J_{i_q}}$ ensures that there exists a $(d-3, d-4)$-path L_q from W_q to W'. By concatenating these two paths, we can obtain the path $L = WL_{q-1}W_{q-1}U_qW_qL_qW'$. This completes the proof of the lemma. $\square$

3. Linkedness of cubical polytopes

The aim of this section is to prove that, for every $d \neq 3$, a cubical d-polytope is $\lfloor (d+1)/2 \rfloor$-linked (Theorem 3). It suffices to prove Theorem 3 for odd $d \geq 5$; since $\lfloor d/2 \rfloor = \lfloor (d+1)/2 \rfloor$ for even d, Theorem 1 trivially establishes Theorem 3 in this case.

The proof of Theorem 3 heavily relies on Lemma 11. To state the lemma we require the following definition.

Definition 10 *(Configuration dF).* Let $d \geq 3$ be odd and let X be a set of at least $d+1$ terminals in a cubical d-polytope P. In addition, let Y be a labelling and pairing of the vertices in X. A terminal of X, say s_1, is in *Configuration dF* if the following conditions are satisfied:

(i) at least $d+1$ vertices of X appear in a facet F of P;
(ii) the terminals in the pair $\{s_1, t_1\} \in Y$ are at distance $d-1$ in F (that is, $\mathrm{dist}_F(s_1, t_1) = d-1$); and
(iii) the neighbours of t_1 in F are all vertices of X.

Fig. 1 illustrates examples of Configuration dF.

Lemma 11. *Let $d \geq 5$ be odd and let $k := (d+1)/2$. Let s_1 be a vertex in a cubical d-polytope and let $\mathcal{S}_1$ be the star of s_1 in the polytope. Moreover, let $Y := \{\{s_1, t_1\}, \ldots, \{s_k, t_k\}\}$ be a labelling and pairing of $2k$ distinct vertices of $\mathcal{S}_1$. Then the set Y is linked in $\mathcal{S}_1$ if the vertex s_1 is not in Configuration dF.*

Remark 12. It is easy to see that when the vertex s_1 is in Configuration dF, the set Y is not linked in $\mathcal{S}_1$. Indeed in this case, since $\mathrm{dist}_{F_1}(s_1, t_1) = d-1$ there is only one facet F_1 in $\mathcal{S}_1$ that contains t_1. Then all the neighbours of t_1 in F_1, and thus, in $\mathcal{S}_1$ are in X. As a consequence, every $s_1 - t_1$ path in $\mathcal{S}_1$ must touch X. Hence Y is not linked.

We defer the proof of Lemma 11 for $d \geq 7$ to Subsection 3.1, while the case $d = 5$ is proved in Appendix A. We are now ready to prove our main result, assuming Lemma 11. For a set $Y := \{\{s_1, t_1\}, \ldots, \{s_k, t_k\}\}$ of pairs of vertices in a graph, a *Y-linkage* $\{L_1, \ldots, L_k\}$ is a set of disjoint paths with the path L_i joining the pair $\{s_i, t_i\}$ for each $i \in [1, k]$. For a path $L := u_0 \ldots u_n$ we often write u_iLu_j for $0 \leq i \leq j \leq n$ to denote the subpath $u_i \ldots u_j$. We will rely on the following definition.

Definition 13 *(Projection π).* For a pair of opposite facets $\{F, F^o\}$ of Q_d, define a projection $\pi_{F^o}^{Q_d}$ from Q_d to F^o by sending a vertex $x \in F$ to the unique neighbour $x_{F^o}^p$ of x in F^o, and a vertex $x \in F^o$ to itself (that is, $\pi_{F^o}^{Q_d}(x) = x$); write $\pi_{F^o}^{Q_d}(x) = x_{F^o}^p$ to be precise, or write $\pi(x)$ or x^p if the cube Q_d and the facet F^o are understood from the context.

We extend this projection to sets of vertices: given a pair $\{F, F^o\}$ of opposite facets and a set $X \subseteq V(F)$, the projection $X_{F^o}^p$ or $\pi_{F^o}^{Q_d}(X)$ of X onto F^o is the set of the projections of the vertices in X onto F^o. For an i-face $J \subseteq F$, the projection $J_{F^o}^p$ or $\pi_{F^o}^{Q_d}(J)$ of J onto F^o is the i-face consisting of the projections of all the vertices of J onto F^o. For a pair $\{F, F^o\}$ of opposite facets in Q^d, the restrictions of the projection π_{F^o} to F and the projection π_F to F^o are bijections.

Proof of Theorem 3 (Linkedness of cubical polytopes). Theorem 1 settled the case of even d, so we assume d is odd.

Let d be odd and $d \geq 5$ and let $k := (d+1)/2$. Let X be any set of $2k$ vertices in the graph G of a cubical d-polytope P. Recall the vertices in X are called terminals. Also let $Y := \{\{s_1, t_1\}, \dots, \{s_k, t_k\}\}$ be a labelling and pairing of the vertices of X. We aim to find a Y-linkage $\{L_1, \dots, L_k\}$ in G where L_i joins the pair $\{s_i, t_i\}$ for $i = 1, \dots, k$.

For a set of vertices X of a graph G, a path in G is called *X-valid* if no inner vertex of the path is in X. The *distance* between two vertices s and t in G, denoted $\mathrm{dist}_G(s,t)$, is the length of a shortest path between the vertices.

The first step of the proof is to reduce the analysis space from the whole polytope to a more manageable space, the star $\mathcal{S}_1$ of a terminal vertex in the boundary complex of P, say that of s_1. We do so by considering $d = 2k-1$ disjoint paths $S_i := s_i - \mathcal{S}_1$ (for each $i \in [2,k]$) and $T_j := t_j - \mathcal{S}_1$ (for each $j \in [1,k]$) from the terminals into $\mathcal{S}_1$. Here we resort to the d-connectivity of G. In addition, let $S_1 := s_1$. We then denote by $\bar{s}_i$ and $\bar{t}_j$ the intersection of the paths S_i and T_j with $\mathcal{S}_1$. Using the vertices $\bar{s}_i$ and $\bar{t}_i$ for $i \in [1,k]$, define sets $\bar{X}$ and $\bar{Y}$ in $\mathcal{S}_1$, counterparts to the sets X and Y of G. In an abuse of terminology, we also say that the vertices $\bar{s}_i$ and $\bar{t}_i$ are terminals. In this way, the existence of a $\bar{Y}$-linkage $\{\bar{L}_1, \dots, \bar{L}_k\}$ with $\bar{L}_i := \bar{s}_i - \bar{t}_i$ in $G(\mathcal{S}_1)$ implies the existence of a Y-linkage $\{L_1, \dots, L_k\}$ in $G(P)$, since each path $\bar{L}_i$ ($i \in [1,k]$) can be extended with the paths S_i and T_i to obtain the corresponding path $L_i = s_i S_i \bar{s}_i \bar{L}_i \bar{t}_i T_i t_i$.

The second step of the proof is to find a $\bar{Y}$-linkage $\{\bar{L}_1, \dots, \bar{L}_k\}$ in $G(\mathcal{S}_1)$, whenever possible. According to Lemma 11, there is a $\bar{Y}$-linkage in $G(\mathcal{S}_1)$ provided that the vertex s_1 is not in Configuration dF. The existence of a $\bar{Y}$-linkage in turn gives the existence of a Y-linkage, and completes the proof of the theorem in this case.

The third and final step is to deal with Configuration dF for s_1. Hence assume that the vertex s_1 is in Configuration dF. This implies that

(i) there exists a unique facet F_1 of $\mathcal{S}_1$ containing $\bar{t}_1$; that
(ii) $|\bar{X} \cap V(F_1)| = d+1$; and that
(iii) $\mathrm{dist}_{F_1}(\bar{s}_1, \bar{t}_1) = d-1$ and all the $d-1$ neighbours of $\bar{t}_1$ in F_1, and thus in $\mathcal{S}_1$, belong to $\bar{X}$.

Let R be a $(d-2)$-face of F_1 containing the vertex s_1^o opposite to s_1 in F_1, then $s_1 \notin R$, and $\bar{t}_1 = s_1^o \in R$. Denote by R_{F_1} the $(d-2)$-face of F_1 disjoint from R. Let J be the other facet of P containing R and let R_J denote the $(d-2)$-face of J disjoint from R. Then R_J is disjoint from F_1. Partition the vertex set $V(R_J)$ of R_J into the vertex sets of two induced subgraphs G_{bad} and G_{good} such that G_{bad} contains the neighbours of the terminals in R, namely $V(G_{\mathrm{bad}}) = \pi^J_{R_J}(\bar{X} \cap V(R))$ and $V(G_{\mathrm{good}}) = V(R_J) \setminus V(G_{\mathrm{bad}})$. Then $\pi^J_R(V(G_{\mathrm{bad}})) \subseteq \bar{X}$ and $\pi^J_R(V(G_{\mathrm{good}})) \cap \bar{X} = \emptyset$. See Fig. 2(a).

Consider again the paths S_i and T_j that bring the vertices s_i ($i \in [2,k]$) and t_j ($j \in [1,k]$) into $\mathcal{S}_1$. Also recall that the paths S_i and T_j intersect $\mathcal{S}_1$ at $\bar{s}_i$ and $\bar{t}_j$, respectively. We distinguish two cases: either at least one path S_i or T_j touches R_J or no path S_i or T_j touches R_J. In the former case we redirect one aforementioned path S_i or T_j to break Configuration dF for s_1 and use Lemma 11, while in the latter case we find the $\bar{Y}$-linkage using the antistar of s_1.

Case 1. Suppose at least one path S_i or T_j touches R_J.

If possible, pick one such path, say S_ℓ, for which it holds that $V(S_\ell) \cap V(G_{\mathrm{good}}) \neq \emptyset$. Otherwise, pick one such path, say S_ℓ, that does not contain $\pi^J_{R_J}(t_1)$, if it is possible. If none of these two selections are possible, then there is exactly one path S_i or T_j touching R_J, say S_ℓ, in which case $\pi^J_{R_J}(t_1) \in V(S_\ell)$.

We replace the path S_ℓ by a new path $s_\ell - \mathcal{S}_1$ that is disjoint from the other paths S_i and T_j and we replace the old terminal $\bar{s}_\ell$ by a new terminal that causes s_1 not to be in Configuration dF. First suppose that there exists s'_ℓ in $V(S_\ell) \cap V(G_{\mathrm{good}})$. Then the old path S_ℓ is replaced by the path $s_\ell S_\ell s'_\ell \pi^J_R(s'_\ell)$, and the old terminal $\bar{s}_\ell$ is replaced by $\pi^J_R(s'_\ell)$. Now suppose that $V(S_\ell) \cap V(G_{\mathrm{good}}) = \emptyset$. Then every path S_i and T_j that touches R_J is disjoint from G_{good}. Denote by s'_ℓ the first intersection of S_ℓ with R_J. Let M_ℓ be a shortest path in R_J from $s'_\ell \in V(G_{\mathrm{bad}})$ to a vertex $s''_\ell \in V(G_{\mathrm{good}})$. By our selection of S_ℓ this path M_ℓ always exists and is disjoint from any S_i for $i \neq \ell$. If $s''_\ell \in V(G_{\mathrm{good}}) \setminus V(\mathcal{S}_1)$ then the old path S_ℓ is replaced by the path $s_\ell S_\ell s'_\ell M_\ell s''_\ell \pi^J_R(s''_\ell)$, and the old terminal $\bar{s}_\ell$ is replaced by $\pi^J_R(s''_\ell)$. If instead $s''_\ell \in V(G_{\mathrm{good}}) \cap V(\mathcal{S}_1)$ then the old path S_ℓ is replaced by the path $s_\ell S_\ell s'_\ell M_\ell s''_\ell$, and the old terminal $\bar{s}_\ell$ is replaced by s''_ℓ. Refer to Fig. 2(b) for a depiction of this case.

In any case, the replacement of the old vertex $\bar{s}_\ell$ with the new $\bar{s}_\ell$ forces s_1 out of Configuration dF, and we can apply Lemma 11 to find a $\bar{Y}$-linkage. The case of S_ℓ being equal to T_1 requires a bit more explanation in order to make sure that the vertex s_1 does not end up in a new configuration dF. Let $\mathcal{A}_1$ be the antistar of F_1 in $\mathcal{S}_1$. The new vertex $\bar{t}_1$ is either in F_1 or in $\mathcal{A}_1$. If the new $\bar{t}_1$ is in F_1 then it is plain that s_1 is not in Configuration dF. If the new vertex $\bar{t}_1$ is in $\mathcal{A}_1$, then a new facet F_1 containing s_1 and the new $\bar{t}_1$ cannot contain all the $d-1$ neighbours of the old $\bar{t}_1$ in the old F_1, since the intersection between the new and the old F_1 is at most $(d-2)$-dimensional and no $(d-2)$-dimensional face of the old F_1 contains all the $d-1$ neighbours of the old $\bar{t}_1$. This completes the proof of the case.

Case 2. For any $(d-2)$-face R in F_1 that contains $\bar{t}_1$, the aforementioned ridge R_J in the facet J is disjoint from all the paths S_i and T_j.

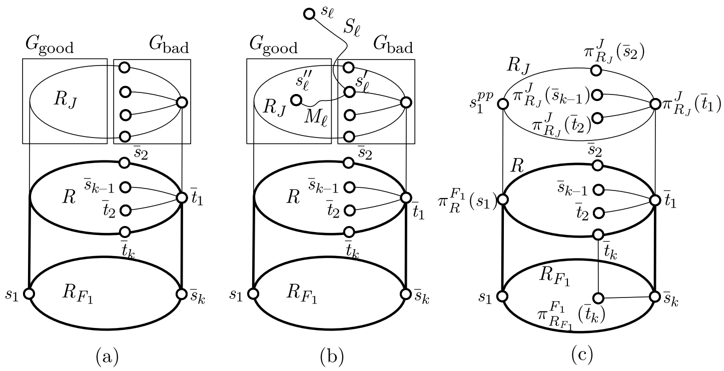

Fig. 2. Auxiliary figure for Theorem 3, where the facet F_1 is highlighted in bold. (a) A depiction of the subgraphs $G_{\rm good}$ and $G_{\rm bad}$ of R_J. (b) A configuration where a path S_i or T_j touches R_J. (c) A configuration where no path S_i or T_j touches R_J.

There is a unique neighbour of $\bar{t}_1$ in R_{F_1}, say $\bar{s}_k$, while every other neighbour of $\bar{t}_1$ in F_1 is in R. Let $\bar{X}^p := \pi^J_{R_J}(\bar{X} \setminus \{s_1, \bar{s}_k, \bar{t}_k\})$ and let $s_1^{pp} := \pi^J_{R_J}(\pi^{F_1}_R(s_1))$. See Fig. 2(c). The $d-1$ vertices in $\bar{X}^p \cup \{s_1^{pp}\}$ can be linked in R_J (Theorem 2) by a linkage $\{\bar{L}'_1, \ldots, \bar{L}'_{k-1}\}$. Observe that, for the special case of $d=5$ where R_J is a 3-cube, the sequence $s_1^{pp}, \pi^J_{R_J}(\bar{s}_2), \pi^J_{R_J}(\bar{t}_1), \pi^J_{R_J}(\bar{t}_2)$ cannot be in a 2-face in cyclic order, since $\mathrm{dist}_{R_J}(s_1^{pp}, \pi^J_{R_J}(\bar{t}_1)) = 3$. The linkage $\{\bar{L}'_1, \ldots, \bar{L}'_{k-1}\}$ together with the two-path $\bar{L}_k := \bar{s}_k \pi^{F_1}_{R_{F_1}}(\bar{t}_k)\bar{t}_k$ can be extended to a linkage $\{\bar{L}_1, \ldots, \bar{L}_k\}$ given by

$$\bar{L}_i := \begin{cases} s_1\pi^{F_1}_R(s_1)s_1^{pp}\bar{L}'_1\pi^J_{R_J}(\bar{t}_1)\bar{t}_1, & \text{for } i=1; \\ \bar{s}_i\pi^J_{R_J}(\bar{s}_i)\bar{L}'_i\pi^J_{R_J}(\bar{t}_i)\bar{t}_i, & \text{for } i \in [2, k-1]; \\ \bar{s}_k\pi^{F_1}_{R_{F_1}}(\bar{t}_k)\bar{t}_k, & \text{for } i=k. \end{cases}$$

Concatenating the paths S_i (for all $i \in [2,k]$) and T_j (for all $j \in [1,k]$) with the linkage $\{\bar{L}_1, \ldots, \bar{L}_k\}$ gives the desired Y-linkage. This completes the proof of the case, and with it the proof of the theorem. □

3.1. *Proof of Lemma 11 for $d \geq 7$*

Before starting the proof, we require several results.

Proposition 14 *([10, Sec. 2]). For every $d \geq 1$, the graph of a strongly connected d-complex is d-connected.*

Proposition 15 *([3, Prop. 27]). For every $d \geq 2$ such that $d \neq 3$, the link of a vertex in a $(d+1)$-cube Q_{d+1} is $\lfloor (d+1)/2 \rfloor$-linked.*

Let Z be a set of vertices in the graph of a d-cube Q_d. If, for some pair of opposite facets $\{F, F^o\}$, the set Z contains both a vertex $z \in V(F) \cap Z$ and its projection $z^p_{F^o} \in V(F^o) \cap Z$, we say that the pair $\{F, F^o\}$ is *associated* with the set Z in Q_d and that $\{z, z^p\}$ is an *associating pair*. Note that an associating pair can associate only one pair of opposite facets.

The next lemma lies at the core of our methodology.

Lemma 16 *([3, Lemma 8]). Let Z be a nonempty subset of $V(Q_d)$. Then the number of pairs $\{F, F^o\}$ of opposite facets associated with Z is at most $|Z|-1$.*

The relevance of the lemma stems from the fact that a pair of opposite facets $\{F, F^o\}$ not associated with a given set of vertices Z allows each vertex z in Z to have "free projection"; that is, for every $z \in Z \cap V(F)$ the projection $\pi_{F^o}(z)$ is not in Z, and for $z \in Z \cap V(F^o)$ the projection $\pi_F(z)$ is not in Z.

Lemma 17 *([12, Sec. 3]). Let G be a $2k$-connected graph and let G' be a k-linked subgraph of G. Then G is k-linked.*

Proposition 18. *Let F be a facet in the star $\mathcal{S}$ of a vertex in a cubical d-polytope. Then, for every $d \geq 2$, the antistar of F in $\mathcal{S}$ is $\lfloor (d-2)/2 \rfloor$-linked.*

Proof. Let $\mathcal{S}$ be the star of a vertex s in a cubical d-polytope and let F be a facet in the star $\mathcal{S}$. Let $\mathcal{A}$ denote the antistar of F in $\mathcal{S}$.

The case of $d = 2, 3$ imposes no demand on $\mathcal{A}$, while the case $d = 4, 5$ amounts to establishing that the graph of $\mathcal{A}$ is connected. The graph of $\mathcal{A}$ is in fact $(d-2)$-connected, since $\mathcal{A}$ is a strongly connected $(d-2)$-complex (Proposition 7). So assume $d \geq 6$.

There is a $(d-2)$-face R in $\mathcal{A}$. Indeed, take a $(d-2)$-face R' in F containing s and consider the other facet F' in $\mathcal{S}$ containing R'; the $(d-2)$-face of F' disjoint from R' is the desired R. By Theorem 2 the ridge R is $\lfloor (d-1)/2 \rfloor$-linked but we only require it to be $\lfloor (d-2)/2 \rfloor$-linked. By Propositions 7 and 14 the graph of $\mathcal{A}$ is $(d-2)$-connected. Combining the linkedness of R and the connectivity of the graph of $\mathcal{A}$ settles the proposition by virtue of Lemma 17. $\square$

For a pair of opposite facets $\{F, F^o\}$ in a cube, the restriction of the projection $\pi_{F^o} : Q_d \to F^o$ (Definition 13) to F is a bijection from $V(F)$ to $V(F^o)$. With the help of π, given the star $\mathcal{S}$ of a vertex s in a cubical polytope and a facet F in $\mathcal{S}$, we can define an injection from the vertices in F, except the vertex opposite to s, to the antistar of F in $\mathcal{S}$. Defining this injection is the purpose of Lemma 19.

Lemma 19. *Let F be a facet in the star $\mathcal{S}$ of a vertex s in a cubical d-polytope. Then there is an injective function, defined on the vertices of F except the vertex s^o opposite to s, that maps each such vertex in F to a neighbour in $V(\mathcal{S}) \setminus V(F)$.*

Proof. We construct the aforementioned injection f between $V(F) \setminus \{s^o\}$ and $V(\mathcal{S}) \setminus V(F)$ as follows. Let $R_1, \ldots, R_{d-1}$ be the $(d-2)$-faces of F containing s, and let $J_1, \ldots, J_{d-1}$ be the other facets of $\mathcal{S}$ containing $R_1, \ldots, R_{d-1}$, respectively. Every vertex in F other than s^o lies in $R_1 \cup \cdots \cup R_{d-1}$. Let R_i^o be the $(d-2)$-face in J_i that is opposite to R_i for each $i \in [1, d-1]$. For every vertex v in $V(R_j) \setminus (V(R_1) \cup \cdots \cup V(R_{j-1}))$ define $f(v)$ as the projection π in J_j of v onto $V(R_j^o)$, namely $f(v) := \pi_{R_j^o}(v)$; observe that $\pi_{R_j^o}(v) \in V(R_j^o) \setminus (V(R_1^o) \cup \cdots \cup V(R_{j-1}^o))$. Here R_{-1} and R_{-1}^o are empty sets. The function f is well defined as R_i and R_i^o are opposite $(d-2)$-cubes in the $(d-1)$-cube J_i.

To see that f is an injection, take distinct vertices $v_1, v_2 \in V(F) \setminus \{s^o\}$, where $v_1 \in V(R_i) \setminus (V(R_1) \cup \cdots \cup V(R_{i-1}))$ and $v_2 \in V(R_j) \setminus (V(R_1) \cup \cdots \cup V(R_{j-1}))$ for $i \leq j$. If $i = j$ then $f(v_1) = \pi_{R_i^o}(v_1) \neq \pi_{R_i^o}(v_2) = f(v_2)$. If instead $i < j$ then $f(v_1) \in V(R_i^o) \subseteq V(R_1^o) \cup \cdots \cup V(R_{j-1}^o)$, while $f(v_2) \notin V(R_1^o) \cup \cdots \cup V(R_{j-1}^o)$. $\square$

Proof of Lemma 11 for $d \geq 7$. The proof of the case $d = 5$ follows a similar pattern to this one, but includes additional technical considerations due to the fact that the 3-cube is not 2-linked. These technical considerations will be presented in a separate proof in Appendix A. In this proof, we identify the arguments that fail for $d = 5$ with a dagger sign †. This will make it easier for the reader to follow the proof for $d = 5$ in the appendix.

Let $d \geq 7$ be odd and let $k := (d+1)/2$. Let s_1 be a vertex in a cubical d-polytope P such that s_1 is not in Configuration dF, and let $\mathcal{S}_1$ denote the star of s_1 in $\mathcal{B}(P)$. Let X be any set of $2k$ vertices in the graph $G(\mathcal{S}_1)$ of $\mathcal{S}_1$. The vertices in X are our terminals. Also let $Y := \{\{s_1, t_1\}, \ldots, \{s_k, t_k\}\}$ be a labelling and pairing of the vertices of X. We aim to find a Y-linkage $\{L_1, \ldots, L_k\}$ in G where L_i joins the pair $\{s_i, t_i\}$ for $i = 1, \ldots, k$. Recall that a path is X-valid if it contains no inner vertex from X.

We consider a facet F_1 of $\mathcal{S}_1$ containing t_1 and having the largest possible number of terminals. We decompose the proof into four cases based on the number of terminals in F_1, proceeding from the more manageable case to the more involved one.

Case 1. $|X \cap V(F_1)| = d$.
Case 2. $3 \leq |X \cap V(F_1)| \leq d-1$.
Case 3. $|X \cap V(F_1)| = 2$.
Case 4. $|X \cap V(F_1)| = d+1$.

The proof of Lemma 11 is long, so we outline the main ideas. We let $\mathcal{A}_1$ be the antistar of F_1 in $\mathcal{S}_1$ and let $\mathcal{L}_1$ be the link of s_1 in F_1. Using the $(k-1)$-linkedness of F_1 (Theorem 2), we link as many pairs of terminals in F_1 as possible through disjoint X-valid paths $L_i := s_i - t_i$. For those terminals that cannot be linked in F_1, if possible we use the injection from $V(F_1)$ to $V(\mathcal{A}_1)$ granted by Lemma 19 to find a set $N_{\mathcal{A}_1}$ of pairwise distinct neighbours in $V(\mathcal{A}_1) \setminus X$ of those terminals. Then, using the $(k-2)$-linkedness of $\mathcal{A}_1$ (Proposition 18), we link the corresponding pairs of terminals in $\mathcal{A}_1$ and vertices in $N_{\mathcal{A}_1}$ accordingly†. This general scheme does not always work, as the vertex s_1^o opposite to s_1 in F_1 does not have an image in $\mathcal{A}_1$ under the aforementioned injection or the image of a vertex in F_1 under the injection may be a terminal. In those scenarios we resort to ad hoc methods, including linking corresponding pairs in the link of s_1 in F_1, which is $(k-1)$-linked by Proposition 15†and does not contain s_1 or s_1^o, or linking corresponding pairs in $(d-2)$-faces disjoint from F_1, which are $(k-1)$-linked by Theorem 2†.

To aid the reader, each case is broken down into subcases highlighted in bold.

Recall that, given a pair $\{F, F^o\}$ of opposite facets in a cube Q, for every vertex $z \in V(F)$ we denote by $z^p_{F^o}$ or $\pi^Q_{F^o}(z)$ the unique neighbour of z in F^o.

Case 1. $|X \cap V(F_1)| = d$.

Without loss of generality, assume that $t_2 \notin V(F_1)$.

Suppose first that $\mathrm{dist}_{F_1}(s_2, s_1) < d-1$. There exists a neighbour s_2' of s_2 in $\mathcal{A}_1$. With the use of the strong $(k-1)$-linkedness of F_1 (Theorem 4), find disjoint paths $L_1 := s_1 - t_1$ and $L_i := s_i - t_i$ (for each $i \in [3, k]$) in F_1, each avoiding s_2. Find a path L_2 in $\mathcal{S}_1$ between s_2 and t_2 that consists of the edge $s_2 s_2'$ and a subpath in $\mathcal{A}_1$ between s_2' and t_2, using the connectivity of $\mathcal{A}_1$ (see Proposition 7). The paths L_i $(i \in [1, k])$ give the desired Y-linkage.

Now assume $\mathrm{dist}_{F_1}(s_2, s_1) = d-1$. Since $2k-1 = d$ and there are $d-1$ pairs of opposite $(d-2)$-faces in F_1, by Lemma 16 there exists a pair $\{R, R^o\}$ of opposite $(d-2)$-faces in F_1 that is not associated with the set $X_{s_2} := (X \cap V(F_1)) \setminus \{s_2\}$, whose cardinality is $d-1$. Assume $s_2 \in R$. Then $s_1 \in R^o$.

Suppose all the neighbours of s_2 in R are in X; that is, $N_R(s_2) = X \setminus \{s_1, s_2, t_2\}$. The projection $\pi_{R^o}^{F_1}(s_2)$ of s_2 onto R^o is not in X since s_1 is the only terminal in R^o and $\mathrm{dist}_{F_1}(s_2, s_1) = d-1 \geq 2$. Next find disjoint paths $L_i := s_i - t_i$ for all $i \in [3, k]$ in R that do not touch s_2 or t_1, using the $(k-1)$-linkedness of R (*the argument also applies for* $d = 5$ *due to the 3-connectivity of* R *in this case*). With the help of Lemma 19, find a neighbour s_2' of $\pi_{R^o}^{F_1}(s_2)$ in $\mathcal{A}_1$, and with the connectivity of $\mathcal{A}_1$, a path L_2 between s_2 and t_2 that consists of the length-two path $s_2 \pi_{R^o}^{F_1}(s_2) s_2'$ and a subpath in $\mathcal{A}_1$ between s_2' and t_2. Finally, find a path L_1 in F_1 between s_1 and t_1 that consists of the edge $t_1 \pi_{R^o}^{F_1}(t_1)$ and a subpath in R^o disjoint from $\pi_{R^o}^{F_1}(s_2)$ (here use the 2-connectivity of R^o). The paths L_i $(i \in [1, k])$ give the desired Y-linkage.

Thus assume there exists a neighbour $\bar{s}_2$ of s_2 in $V(R) \setminus X$. Let $X_{R^o} := \pi_{R^o}^{F_1}(X \setminus \{s_2, t_2\})$. Find a path L_2' in $\mathcal{A}_1$ between a neighbour s_2' of $\bar{s}_2$ in $\mathcal{A}_1$ and t_2 using the connectivity of $\mathcal{A}_1$. Then let $L_2 := s_2 \bar{s}_2 s_2' L_2' t_2$. Find disjoint paths $L_i := \pi_{R^o}^{F_1}(s_i) - \pi_{R^o}^{F_1}(t_i)$ $(i \in [1, k]$ and $i \neq 2)$ in R^o linking the $d-1$ vertices in X_{R^o} using the $(k-1)$-linkedness of $R^{o\dagger}$; add the edge $\pi_{R^o}^{F_1}(t_i) t_i$ to L_i if $t_i \in R$ or the edge $\pi_{R^o}^{F_1}(s_i) s_i$ to L_i if $s_i \in R$. The disjoint paths L_i $(i \in [1, k])$ give the desired Y-linkage.

Case 2. $3 \leq |X \cap V(F_1)| \leq d-1$.

The number of terminals in $\mathcal{A}_1$ is at most $d+1-3 = d-2$. Since $2k-1 = d$ and there are $d-1$ pairs of opposite $(d-2)$-faces in F_1, by Lemma 16 there exists a pair $\{R, R^o\}$ of opposite $(d-2)$-faces in F_1 that is not associated with $X \cap V(F_1)$. Assume $s_1 \in R$. We consider two subcases according to whether $t_1 \in R$ or $t_1 \in R^o$.

Suppose first that $t_1 \in R$. The $(d-2)$-connectivity of R ensures the existence of an X-valid path $L_1 := s_1 - t_1$ in R. Let

$$X_{R^o} := \pi_{R^o}^{F_1}((X \setminus \{s_1, t_1\}) \cap V(F_1)).$$

Then $1 \leq |X_{R^o}| \leq d-3$. Let s_1^o be the vertex opposite to s_1 in F_1; the vertex s_1^o has no neighbour in $\mathcal{A}_1$.

Let $\bar{Z}$ be a set of $|V(\mathcal{A}_1) \cap X|$ distinct vertices in $V(R^o) \setminus (X_{R^o} \cup \{s_1^o\})$. To see that $|\bar{Z}| \leq |V(R^o) \setminus (X_{R^o} \cup \{s_1^o\})|$, observe that, for $d \geq 5$ and $|X_{R^o}| \leq d-3$, we get

$$|V(R^o) \setminus (X_{R^o} \cup \{s_1^o\})| \geq 2^{d-2} - (d-3) - 1 \geq d-2 \geq |V(\mathcal{A}_1) \cap X| = |\bar{Z}|.$$

Use Lemma 19 to obtain a set Z in $\mathcal{A}_1$ of $|\bar{Z}|$ distinct vertices adjacent to vertices in $\bar{Z}$. Then $|Z| = |V(\mathcal{A}_1) \cap X| \leq d-2$.

Using the $(d-2)$-connectivity of $\mathcal{A}_1$ (Proposition 7) and Menger's theorem, find disjoint paths $\bar{S}_i$ and $\bar{T}_j$ (for all $i, j \neq 1$) in $\mathcal{A}_1$ between $V(\mathcal{A}_1) \cap X$ and Z. Then produce disjoint paths S_i and T_j (for all $i, j \neq 1$) from terminals s_i and t_j in $\mathcal{A}_1$, respectively, to R^o by adding edges $z_\ell \bar{z}_\ell$ with $z_\ell \in Z$ and $\bar{z}_\ell \in \bar{Z}$ to the corresponding paths $\bar{S}_i$ and $\bar{T}_j$. If s_i or t_j is already in R^o, let $S_i := s_i$ or $T_j := t_j$, accordingly. If instead s_i or t_j is in R, let S_i be the edge $s_i \pi_{R^o}^{F_1}(s_i)$ or let T_j be the edge $t_j \pi_{R^o}^{F_1}(t_j)$. It follows that the paths S_i and T_i for $i \in [2, k]$ are all pairwise disjoint. Let $X_{R^o}^+$ be the intersections of R^o and the paths S_i and T_j $(i, j \neq 1)$. Then $|X_{R^o}^+| = d-1$. Suppose that $X_{R^o}^+ = \{\bar{s}_2, \bar{t}_2, \ldots, \bar{s}_k, \bar{t}_k\}$. The corresponding pairing $Y_{R^o}^+$ of the vertices in $X_{R^o}^+$ can be linked through paths $\bar{L}_i := \bar{s}_i - \bar{t}_i$ (for all $i \in [2, k]$) in R^o using the $(k-1)$-linkedness of R^o (Theorem 2)†. See Fig. 3(a) for a depiction of this configuration. In this case, the desired Y-linkage is given by the following paths.

$$L_i := \begin{cases} s_1 L_1 t_1, & \text{for } i = 1; \\ s_i S_i \bar{s}_i \bar{L}_i \bar{t}_i T_i t_i, & \text{otherwise.} \end{cases}$$

Suppose now that $t_1 \in R^o$. Let

$$X_R := \pi_R^{F_1}((X \setminus \{t_1\}) \cap V(F_1)).$$

There are at most $d-2$ terminal vertices in R^o. Therefore, the $(d-2)$-connectivity of R^o ensures the existence of an X-valid $\pi_{R^o}^{F_1}(s_1) - t_1$ path $\bar{L}_1$ in R^o. Then let $L_1 := s_1 \pi_{R^o}^{F_1}(s_1) \bar{L}_1 t_1$. Let J be the other facet in $\mathcal{S}_1$ containing R and let R_J be the $(d-2)$-face of J disjoint from R. Then $R_J \subset \mathcal{A}_1$. Since there are at most $d-2$ terminals in $\mathcal{A}_1$ and since $\mathcal{A}_1$ is $(d-2)$-connected (Proposition 7), we can find corresponding disjoint paths S_i and T_j from the terminals in $\mathcal{A}_1$ to R_J by Menger's theorem [4, Theorem 3.3.1]. For terminals s_i and t_j in $X \cap V(R)$, let $S_i := s_i$ and $T_j := t_j$ for all $i, j \neq 1$, while for terminals s_i and t_j in $X \cap V(R^o)$, let $S_i := s_i \pi_R^{F_1}(s_i)$ and $T_j := t_j \pi_R^{F_1}(t_j)$ for all $i, j \neq 1$. Let X_J be the set of the intersections of the paths S_i and T_j with J plus the vertex s_1. Then $X_J \subset V(J)$ and $|X_J| = d$ (since $t_1 \in R^o$). Suppose that $X_J = \{s_1, \bar{s}_2, \bar{t}_2, \ldots, \bar{s}_k, \bar{t}_k\}$ and let $Y_J = \{\{\bar{s}_2, \bar{t}_2\}, \ldots, \{\bar{s}_k, \bar{t}_k\}\}$ be a pairing of $X_J \setminus \{s_1\}$.

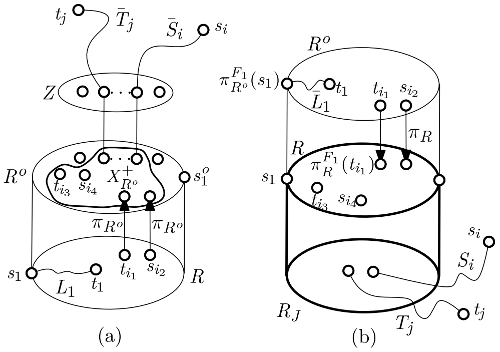

Fig. 3. Auxiliary figure for Case 2 of Lemma 11. (a) A configuration where $t_1 \in R$ and the subset $X^+_{R^o}$ of R^o is highlighted in bold. (b) A configuration where $t_1 \in R^o$ and the facet J is highlighted in bold.

Resorting to the strong $(k-1)$-linkedness of the facet J (Theorem 4), we obtain $k-1$ disjoint paths $\bar{L}_i := \bar{s}_i - \bar{t}_i$ for all $i \neq 1$ that correspondingly link Y_J in J, with all the paths avoiding s_1. See Fig. 3(b) for a depiction of this configuration. In this case, the desired Y-linkage is given by the following paths.

$$L_i := \begin{cases} s_1 L_1 t_1, & \text{for } i = 1; \\ s_i S_i \bar{L}_i T_i t_i, & \text{otherwise.} \end{cases}$$

Case 3. $|X \cap V(F_1)| = 2$.

In this case, we have that $V(F_1) \cap X = \{s_1, t_1\}$ and $|V(\mathcal{A}_1) \cap X| = d-1$. The proof of this case requires the definition of several sets. For quick reference, we place most of these definitions in itemised lists. We begin with the following sets:

- $\mathcal{S}_{12}$, the star of s_2 in $\mathcal{S}_1$ (that is, the complex formed by the facets of P containing s_1 and s_2);
- $G(\mathcal{S}_{12})$, the graph of $\mathcal{S}_{12}$; and
- Γ_{12}, the subgraph of $G(\mathcal{S}_{12})$ and $G(\mathcal{A}_1)$ that is induced by $V(\mathcal{S}_{12}) \setminus V(F_1)$.

It follows that every neighbour in $G(\mathcal{A}_1)$ of s_2 is in Γ_{12}:

$$N_{\Gamma_{12}}(s_2) = N_{G(\mathcal{A}_1)}(s_2). \tag{1}$$

Note that when $d \geq 5$, $|V(\Gamma_{12})| \geq 2^{d-2} \geq d-2$, since $\mathcal{S}_{12}$ contains at least one facet (other than F_1), and that facet contains at least one $(d-2)$-face disjoint from F_1. The vertices of that $(d-2)$-face are in Γ_{12}.

The first step for this case is to bring the terminals in $\mathcal{A}_1$ into Γ_{12}. The $(d-2)$-connectivity of the graph $G(\mathcal{A}_1)$ (Proposition 7) ensures the existence of pairwise disjoint paths from $(V(\mathcal{A}_1) \cap X) \setminus \{s_2\}$ to Γ_{12}. Among these paths, denote by S_i the path from the terminal $s_i \in \mathcal{A}_1$ to Γ_{12} and let $V(S_i) \cap V(\Gamma_{12}) = \{\hat{s}_i\}$. Similarly, define T_j and $\hat{t}_j$. By (1) each path S_i or T_j touches Γ_{12} at a vertex other than s_2; this is so because each such path will need to reach the neighbourhood of s_2 in Γ_{12} before reaching s_2. We also let $\hat{s}_2$ denote s_2. The set of vertices $\hat{x}$ is accordingly denoted by $\hat{X}$. Then $|\hat{X}| = d-1$. Abusing terminology, since there is no potential for confusion, we call the vertices in $\hat{X}$ terminals as well. Fig. 4(a) depicts this configuration.

Pick a facet F_{12} in $\mathcal{S}_{12}$ that contains $\hat{t}_2$. An important point is that t_1 is not in F_{12}; otherwise F_{12} would contain s_1, s_2 and t_1, and it should have been chosen instead of F_1.

The second step is to find a path L_1 in F_1 between s_1 and t_1 such that $V(L_1) \cap V(F_{12}) = \{s_1\}$.

Remark 20. For any two faces F, J of a polytope, with F not contained in J, there is a facet containing J but not F. In particular, for any two distinct vertices of a polytope, there is a facet containing one but not the other.

To see the existence of such a path, note that the intersection of F_{12} and F_1 is a face that does not contain t_1 and therefore is contained in a $(d-2)$-face R of F_1 containing s_1 but not t_1 (Remark 20). Find a path L'_1 in R^o, the $(d-2)$-face in F_1 disjoint from R (R^o contains t_1), between $\pi^{F_1}_{R^o}(s_1)$ and t_1 and let $L_1 := s_1 \pi^{F_1}_{R^o}(s_1) L'_1 t_1$.

The third step is to bring the $d-1$ terminal vertices $\hat{x} \in \Gamma_{12}$ into the facet F_{12} so that they can be linked there, avoiding s_1. We consider two cases depending on the number of facets in $\mathcal{S}_{12}$.

Suppose $\mathcal{S}_{12}$ only consists of F_{12}. Then

$$\hat{X} = \{\hat{s}_2, \ldots, \hat{s}_k, \hat{t}_2, \ldots, \hat{t}_k\} \subset V(\Gamma_{12}) \subset V(F_{12}).$$

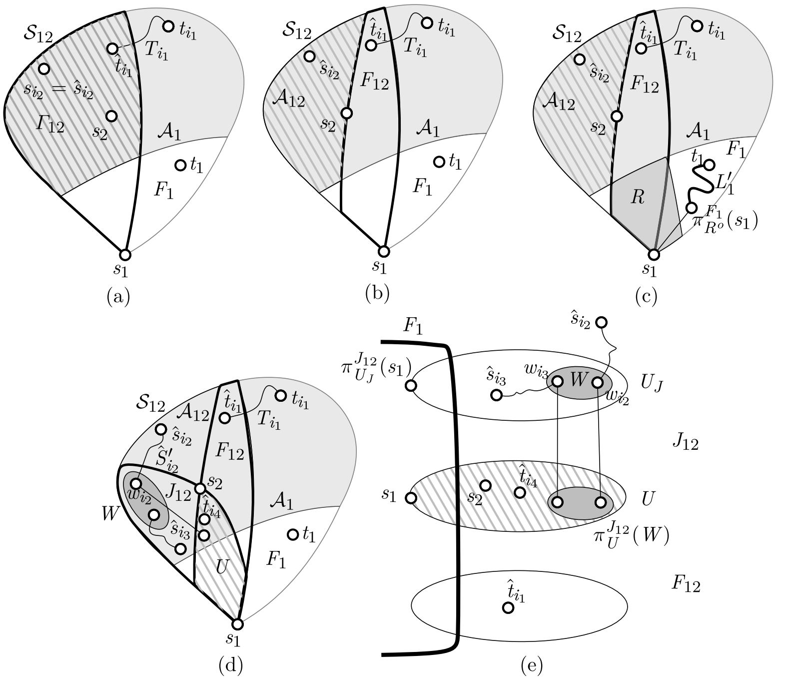

Fig. 4. Auxiliary figure for Case 3 of Lemma 11. A representation of $\mathcal{S}_1$. (a) A configuration where the subgraph Γ_{12} is tiled in falling pattern and the complex $\mathcal{A}_1$ is coloured in grey. (b) A depiction of $\mathcal{S}_{12}$ with more than one facet; the facet F_{12} is highlighted in bold, the complex $\mathcal{A}_1$ is coloured in grey and the complex $\mathcal{A}_{12}$ is highlighted in falling pattern. (c) The construction of the path $L_1 := s_1\pi^{F_1}_{R^o}(s_1)L'_1t_1$ from s_1 to t_1 in F_1 such that $L_1 \cap V(\Gamma_{12}) = \{s_1\}$. (d) A depiction of $\mathcal{S}_{12}$ with more than one facet; the facets F_{12} and J_{12} are highlighted in bold and their intersection U is highlighted in falling pattern; the set W in J_{12} is coloured in dark grey. (e) A depiction of a portion of $\mathcal{S}_{12}$, zooming in on the facets F_{12} and J_{12}; each facet is represented as the convex hull of two disjoint $(d-2)$-faces, and their intersection U is highlighted in falling pattern. The sets W and $\pi^{J_{12}}_U(W)$ in J_{12} are coloured in dark grey.

With the help of the strong $(k-1)$-linkedness of F_{12} (Theorem 4), we can link the pair $\{\hat{s}_i, \hat{t}_i\}$ for each $i \in [2, k]$ in F_{12} through disjoint paths $\hat{L}_i$, all avoiding s_1. For each $i \in [2, k]$, we concatenate the path $\hat{L}_i$ with the paths S_i and T_i in this order, resulting in the path L_i. These new $k-1$ paths give a $(Y \setminus \{s_1, t_1\})$-linkage $\{L_2, \ldots, L_k\}$. Hence the desired Y-linkage is as follows.

$$L_i := \begin{cases} s_1\pi^{F_1}_{R^o}(s_1)L'_1t_1, & \text{for } i = 1; \\ s_iS_i\hat{s}_i\hat{L}_i\hat{t}_iT_it_i, & \text{otherwise.} \end{cases}$$

Assume $\mathcal{S}_{12}$ has more than one facet. We have that

$$\hat{X} = \{\hat{s}_2, \ldots, \hat{s}_k, \hat{t}_2, \ldots, \hat{t}_k\} \subset V(\Gamma_{12}).$$

Define

- $\mathcal{A}_{12}$ as the complex of $\mathcal{S}_{12}$ induced by $V(\mathcal{S}_{12}) \setminus (V(F_1) \cup V(F_{12}))$.

Then the graph $G(\mathcal{A}_{12})$ of $\mathcal{A}_{12}$ coincides with the subgraph of Γ_{12} induced by $V(\Gamma_{12}) \setminus V(F_{12})$. Fig. 4(b) depicts this configuration.

Our strategy is first to bring the $d-3$ terminal vertices $\hat{x}$ in Γ_{12} other than $\hat{s}_2$ and $\hat{t}_2$ into $F_{12} \setminus F_1$ through disjoint paths $\hat{S}_i$ and $\hat{T}_j$, without touching $\hat{s}_2$ and $\hat{t}_2$. Second, denoting by $\tilde{s}_i$ and $\tilde{t}_j$ the intersection of $\hat{S}_i$ and $\hat{T}_j$ with $V(F_{12}) \setminus V(F_1)$, respectively, we link the pairs $\{\tilde{s}_i, \tilde{t}_i\}$ for all $i \in [2, k]$ in F_{12} through disjoint paths $\tilde{L}_i$, without touching s_1; here we resort to the strong $(k-1)$-linkedness of F_{12}. We develop these ideas below.

From Lemma 9(iii), it follows that $\mathcal{A}_{12}$ is nonempty and contains a spanning strongly connected $(d-3)$-subcomplex, thereby implying, by Proposition 14, that

$$G(\mathcal{A}_{12}) \text{ is } (d-3)\text{-connected.}$$

Since $\mathcal{S}_{12}$ contains more than one facet, the following sets exist:

- U, a $(d-2)$-face in F_{12} that contains s_1 and $\hat{s}_2$ $(= s_2)$ (since several facets in $\mathcal{S}_{12}$ contain both s_1 and s_2);

- J_{12}, the other facet in $\mathcal{S}_{12}$ containing U;
- U_J, the $(d-2)$-face in J_{12} disjoint from U, and as a consequence, disjoint from F_{12};
- $\mathcal{C}_U$, the subcomplex of $\mathcal{B}(U)$ induced by $V(U) \setminus V(F_1)$, namely the antistar of $U \cap F_1$ in U; and
- $\mathcal{C}_{U_J}$, the subcomplex of $\mathcal{B}(U_J)$ induced by $V(U_J) \setminus V(F_1)$.

The subcomplex $\mathcal{C}_U$ is nonempty, since $\hat{s}_2 \in V(U) \setminus V(F_1)$, and so, thanks to Lemma 6, it is a strongly connected $(d-3)$-complex. Then, from C_U containing a $(d-3)$-face it follows that

$$|V(\mathcal{C}_U)| = |V(U) \setminus V(F_1))| \geq 2^{d-3} \geq d-1 \text{ for } d \geq 5. \tag{2}$$

The subcomplex $\mathcal{C}_{U_J}$ is nonempty: the vertex in J_{12} opposite to s_1 is not in U, since $s_1 \in U$, nor is it in F_1 (Remark 8), and so it must be in $\mathcal{C}_{U_J}$. If $U_J \cap F_1 = \emptyset$ then $\mathcal{C}_{U_J} = \mathcal{B}(U_J)$; otherwise $\mathcal{C}_{U_J}$ is the antistar of $U_J \cap F_1$ in U_J, and since $U \cap F_1 \neq \emptyset$ (s_1 is in both), it follows that $U_J \nsubseteq F_1$. Therefore, according to Lemma 6, $\mathcal{C}_{U_J}$ is or contains a strongly connected $(d-3)$-complex. Hence, in both instances,

$$|V(\mathcal{C}_{U_J})| = |V(U_J) \setminus V(F_1))| \geq 2^{d-3} \geq d-1 \text{ for } d \geq 5. \tag{3}$$

Recall that we want to bring every vertex in the set $\hat{X}$, which is contained in Γ_{12}, into $F_{12} \setminus F_1$. We construct $|\hat{X} \cap V(\mathcal{A}_{12})|$ pairwise disjoint paths $\hat{S}_i$ and $\hat{T}_j$ from $\hat{s}_i \in \mathcal{A}_{12}$ and $\hat{t}_j \in \mathcal{A}_{12}$, respectively, to $V(F_{12}) \setminus V(F_1)$ as follows. Pick a set

$$W \subset V(\mathcal{C}_{U_J}) \setminus \pi_{U_J}^{J_{12}}\left((\hat{X} \cup \{s_1\}) \cap U\right)$$

of $|\hat{X} \cap V(\mathcal{A}_{12})|$ vertices in $\mathcal{C}_{U_J}$. Then $\pi_U^{J^{12}}(W)$ is disjoint from $(\hat{X} \cup \{s_1\}) \cap U$. In other words, the vertices in W are in $\mathcal{C}_{U_J}$ and are not projections of the vertices in $(\hat{X} \cup \{s_1\}) \cap U$ onto U_J. We show that the set W exists, which amounts to showing that $\mathcal{C}_{U_J}$ has enough vertices to accommodate W.

First note that

$$\begin{aligned} &|\hat{X} \cap V(\mathcal{A}_{12})| + |(\hat{X} \cup \{s_1\}) \cap V(F_{12})| = |\hat{X} \cup \{s_1\}| = d, \\ &\qquad (\hat{X} \cup \{s_1\}) \cap V(U) \subseteq (\hat{X} \cup \{s_1\}) \cap V(F_{12}). \end{aligned} \tag{4}$$

If $U_J \cap F_1 = \emptyset$ then $\mathcal{C}_{U_J} = \mathcal{B}(U_J)$. And (4) together with $|V(U_J)| = 2^{d-2} \geq d$ for $d \geq 7$ (indeed, for $d \geq 5$) gives the following chain of inequalities

$$\begin{aligned} \left|V(C_{U_J}) \setminus \pi_{U_J}^{J_{12}}\left((\hat{X} \cup \{s_1\}) \cap V(U)\right)\right| &\geq \left|V(U_J)\right| - \left|(\hat{X} \cup \{s_1\}) \cap V(U)\right| \\ &\geq d - \left|(\hat{X} \cup \{s_1\}) \cap V(U)\right| \geq \left|\hat{X} \cup \{s_1\}\right| - \left|(\hat{X} \cup \{s_1\}) \cap V(F_{12})\right| \\ &= \left|\hat{X} \cap V(\mathcal{A}_{12})\right| = |W|, \end{aligned}$$

as desired.

Suppose now $U_J \cap F_1 \neq \emptyset$. Since $s_1 \in U \cap F_1$ and $J_{12} = \operatorname{conv}\{U \cup U_J\}$, the cube $J_{12} \cap F_1$ has opposite facets $U_J \cap F_1$ and $U \cap F_1$. From $s_1 \in U \cap F_1$ it follows that $\pi_{U_J}^{J_{12}}(s_1) \in U_J \cap F_1$, and thus, that $\pi_{U_J}^{J_{12}}(s_1) \notin \mathcal{C}_{U_J}$; here we use the following remark.

Remark 21. Let (K, K^o) be opposite facets in a cube Q and let B be a proper face of Q such that $B \cap K \neq \emptyset$ and $B \cap K^o \neq \emptyset$. Then $\pi_{K^o}^{Q}(B \cap K) = B \cap K^o$.

Since $\pi_{U_J}^{J_{12}}(s_1) \notin \mathcal{C}_{U_J}$, using (3) and (4) we get

$$\begin{aligned} \left|V(C_{U_J}) \setminus \pi_{U_J}^{J_{12}}\left((\hat{X} \cup \{s_1\}) \cap V(U)\right)\right| &= \left|V(C_{U_J}) \setminus \pi_{U_J}^{J_{12}}\left(\hat{X} \cap V(U)\right)\right| \\ &\geq \left|V(C_{U_J})\right| - \left|\hat{X} \cap V(U)\right| \geq d-1-\left|\hat{X} \cap V(U)\right| \\ &\geq \left|\hat{X}\right| - \left|\hat{X} \cap V(F_{12})\right| = \left|\hat{X} \cap V(\mathcal{A}_{12})\right| = |W|. \end{aligned}$$

In this way, we have shown that $\mathcal{C}_{U_J}$ can accommodate the set W. We now finalise the case.

There are at most $d-3$ vertices $\hat{x}$ in $\hat{X} \cap V(\mathcal{A}_{12})$ because $\hat{s}_2$ and $\hat{t}_2$ are already in $V(F_{12}) \setminus V(F_1)$. Since $G(\mathcal{A}_{12})$ is $(d-3)$-connected, we can find $|W| = |\hat{X} \cap V(\mathcal{A}_{12})|$ pairwise disjoint paths $\hat{S}_i'$ and $\hat{T}_j'$ in $\mathcal{A}_{12}$ from the terminals $\hat{s}_i$ and $\hat{t}_j$ in $\hat{X} \cap V(\mathcal{A}_{12})$ to W. The $\hat{X}$-valid path $\hat{S}_i$ from $\hat{s}_i \in \mathcal{A}_{12}$ to $V(F_{12}) \setminus V(F_1)$ then consists of the subpath $\hat{S}_i' := \hat{s}_i - w_i$

with $w_i \in W$ plus the edge $w_i\pi_U^{J_{12}}(w_i)$; from the choice of W it follows that $\pi_U^{J_{12}}(w_i) \notin \hat{X} \cup \{s_1\}$. The paths $\hat{T}_j'$ and $\hat{T}_j$ are defined analogously. Fig. 4(d)-(e) depicts this configuration.

Denote by $\tilde{s}_i$ the intersection of $\hat{S}_i$ and $V(F_{12}) \setminus V(F_1)$; similarly, define $\tilde{t}_j$. Every terminal vertex $\hat{x}$ already in F_{12} is also denoted by $\tilde{x}$, and in this case we let $\hat{S}_i$ or $\hat{T}_j$ be the vertex $\tilde{x}$.

Now F_{12} contains the pairs $\{\tilde{s}_i, \tilde{t}_i\}$ for all $i \in [2,k]$ and the terminal s_1, as desired. Link these pairs in F_{12} through disjoint paths $\tilde{L}_i$, each avoiding s_1, with the use of the strong $(k-1)$-linkedness of F_{12} (Theorem 4). The paths $\tilde{L}_i$ concatenated with the paths S_i, $\hat{S}_i$, T_i and $\hat{T}_i$ for $i \in [2,k]$ give a $(Y \setminus \{s_1,t_1\})$-linkage $\{L_2,\dots,L_k\}$. Hence the desired Y-linkage is as follows.

$$L_i := \begin{cases} s_1\pi_{R^o}^{F_1}(s_1)L_1't_1, & \text{for } i=1;\\ s_iS_i\hat{s}_i\hat{S}_i\tilde{s}_i\tilde{L}_i\tilde{t}_i\hat{T}_i\hat{t}_iT_it_i, & \text{otherwise.}\end{cases}$$

Case 4. $|X \cap V(F_1)| = d+1$.

Remember that by assumption s_1 is not in configuration dF. Here we have that $V(\mathcal{A}_1) \cap X = \emptyset$. This case is decomposed into three main subcases A, B and C, based on the nature of the vertex s_1^o opposite to s_1 in F_1, which is the only vertex in F_1 that does not have an image under the injection from F_1 to $\mathcal{A}_1$ defined in Lemma 19.

SUBCASE A. The vertex s_1^o opposite to s_1 in F_1 does not belong to X

Let $X' := X \setminus \{t_1\}$ and let $Y' := Y \setminus \{\{s_1,t_1\}\}$. Since $|X'| = d$, the strong $(k-1)$-linkedness of F_1 (Theorem 4) gives a Y'-linkage $\{L_2,\dots,L_k\}$ in the facet F_1 with each path $L_i := s_i - t_i$ ($i \in [2,k]$) avoiding s_1. We find pairwise distinct neighbours s_1' and t_1' in $\mathcal{A}_1$ of s_1 and t_1, respectively. If none of the paths L_i touches t_1, we find a path $L_1 := s_1 - t_1$ in $\mathcal{S}_1$ that contains a subpath in $\mathcal{A}_1$ between s_1' and t_1' (here use the connectivity of $\mathcal{A}_1$, Proposition 7), and we are home. Otherwise, assume that the path L_j contains t_1. With the help of Lemma 19, find pairwise distinct neighbours s_j' and t_j' in $\mathcal{A}_1$ of s_j and t_j, respectively, such that the vertices s_1', t_1', s_j' and t_j' are pairwise distinct. According to Proposition 18, the complex $\mathcal{A}_1$ is 2-linked for $d \ge 7^\dagger$. Hence, we can find disjoint paths $L_1' := s_1' - t_1'$ and $L_j' := s_j' - t_j'$ in $\mathcal{A}_1$, respectively; these paths naturally give rise to paths $L_1 := s_1s_1'L_1't_1't_1$ in $\mathcal{S}_1$ and $L_j := s_js_j'L_j't_j't_j$ in $\mathcal{S}_1$. The paths $\{L_1,\dots,L_k\}$ give the desired Y-linkage.

SUBCASE B. The vertex s_1^o opposite to s_1 in F_1 belongs to X but is different from t_1, say $s_1^o = s_2$

Since F_1 is a cube, the link $\mathcal{L}_1$ of s_1 in F_1 contains all the vertices in F_1 except s_1 and s_2. First find a neighbour s_1' of s_1 and a neighbour t_1' of t_1 in $\mathcal{A}_1$. There is a neighbour $s_2^{F_1}$ of s_2 in F_1 that is either t_2 or a vertex not in X: $\{s_1,s_2\} \cap N_{F_1}(s_2) = \emptyset$ and $|N_{F_1}(s_2)| = d-1$.

Suppose $s_2^{F_1} = t_2$, and let $L_2 := s_2t_2$. Using the $(k-1)$-linkedness of $\mathcal{L}_1$ (Proposition 15), we find disjoint paths $t_1 - t_2$ and $L_i := s_i - t_i$ for each $i \in [3,k]$ in $\mathcal{L}_1{}^\dagger$. Then define a path $L_1 := s_1 - t_1$ in $\mathcal{S}_1$ that contains a subpath in $\mathcal{A}_1$ between s_1' and t_1'; here we use the connectivity of $\mathcal{A}_1$ (Proposition 7). The paths $\{L_1,\dots,L_k\}$ give the desired Y-linkage.

Assume $s_2^{F_1}$ is not in X. Observe that $|(X \setminus \{s_1,s_2\}) \cup \{s_2^{F_1}\}| = d$. Using the $(k-1)$-linkedness of $\mathcal{L}_1$ for $d \ge 7$ (Proposition 15), find in $\mathcal{L}_1$ disjoint paths $L_2' := s_2^{F_1} - t_2$ and $L_i' := s_i - t_i$ for $i \in [3,k]^\dagger$. Since t_1 is also in $\mathcal{L}_1$ it may happen that it lies in one of the paths L_i'. If t_1 does not belong to any of the paths L_i' for $i \in [2,k]$, then find a path $L_1 := s_1s_1'L_1't_1't_1$ in $\mathcal{S}_1$ where L_1' is a subpath in $\mathcal{A}_1$ between s_1' and t_1', using the connectivity of $\mathcal{A}_1$ (Proposition 7). In this scenario, let $L_2 := s_2s_2^{F_1}L_2't_2$ and $L_i := L_i'$ for each $i \in [3,k]$; the desired Y-linkage is given by the paths $\{L_1,\dots,L_k\}$.

If t_1 belongs to one of the paths L_i' with $i \in [2,k]$, say L_j', then consider in $\mathcal{A}_1$ a neighbour t_j' of t_j and, either a neighbour s_j' of s_j if $j \ne 2$ or a neighbour s_2' of $s_2^{F_1}$. From Lemma 19 it follows that the vertices s_1', t_1', s_j' and t_j' can be taken pairwise distinct. Since $\mathcal{A}_1$ is 2-linked for $d \ge 7$ (see Proposition 18), find in $\mathcal{A}_1$ a path L_1' between s_1' and t_1' and a path L_j'' between s_j' and $t_j'{}^\dagger$. As a consequence, we obtain in $\mathcal{S}_1$ a path $L_1 := s_1s_1'L_1't_1't_1$ and, either a path $L_j := s_js_j'L_j''t_j't_j$ if $j \ne 2$ or a path $L_2 := s_2s_2^{F_1}s_2'L_2''t_2't_2$. In addition, let $L_i := L_i'$ for each $i \in [3,k]$ and $i \ne j$. The paths $\{L_1,\dots,L_k\}$ give the desired Y-linkage.

SUBCASE C. The vertex opposite to s_1 in F_1 coincides with t_1

Then t_1 has no neighbour in $\mathcal{A}_1$. In fact, F_1 is the only facet in $\mathcal{S}_1$ containing t_1.

Because the vertex s_1 is not in Configuration dF, t_1 has a neighbour $t_1^{F_1}$ in F_1 that is not in X. Here we reason as in the scenario in which $s_2 = s_1^o$ and s_2 has a neighbour not in X.

First, using the $(k-1)$-linkedness of $\mathcal{L}_1$ (Proposition 15) find disjoint paths $L_i := s_i - t_i$ in $\mathcal{L}_1$ for all $i \in [2,k]^\dagger$. It may happen that $t_1^{F_1}$ is in one of the paths L_i for $i \in [2,k]$. Second, consider neighbours s_1' and t_1' in $\mathcal{A}_1$ of s_1 and $t_1^{F_1}$, respectively.

If $t_1^{F_1}$ doesn't belong to any path L_i, then find a path $L_1 := s_1 - t_1$ that contains the edge $t_1 t_1^{F_1}$ and a subpath L_1' in $\mathcal{A}_1$ between s_1' and t_1'; that is, $L_1 = s_1 s_1' L_1' t_1' t_1^{F_1} t_1$. The desired Y-linkage is given by $\{L_1, \ldots, L_k\}$.

If $t_1^{F_1}$ belongs to one of the paths L_i with $i \in [2, k]$, say L_j, then disregard this path L_j and consider in $\mathcal{A}_1$ a neighbour s_j' of s_j and a neighbour t_j' of t_j. From Lemma 19, it follows that the vertices s_1', t_1', s_j' and t_j' can be taken pairwise distinct. Using the 2-linkedness of $\mathcal{A}_1$ for $d \geq 7$, find a path L_1' in $\mathcal{A}_1$ between s_1' and t_1' and a path L_j' in $\mathcal{A}_1$ between s_j' and $t_j'^{\dagger}$. Let $L_1 := s_1 s_1' L_1' t_1' t_1^{F_1} t_1$ and let $L_j := s_j s_j' L_j' t_j' t_j$ be the new $s_j - t_j$ path. The paths $\{L_1, \ldots, L_k\}$ form the desired Y-linkage.

And finally, the proof of Lemma 11 is complete. $\square$

4. Strong linkedness of cubical polytopes

Proof of Theorem 5 (Strong linkedness of cubical polytopes). Let P be a cubical d-polytope. For odd d Theorem 5 is a consequence of Theorem 3. The result for $d = 4$ is given by [3, Theorem 16]. So assume $d = 2k \geq 6$. Let X be a set of $d+1$ vertices in P. Arbitrarily pair $2k$ vertices in X to obtain $Y := \{\{s_1, t_1\}, \ldots, \{s_k, t_k\}\}$. Let x be the vertex of X not paired in Y. We find a Y-linkage $\{L_1, \ldots, L_k\}$ where each path L_i joins the pair $\{s_i, t_i\}$ and avoids the vertex x.

Using the d-connectivity of $G(P)$ and Menger's theorem, bring the $d = 2k$ terminals in $X \setminus \{x\}$ to the link of x in the boundary complex of P through $2k$ disjoint paths L_{s_i} and L_{t_i} for $i \in [1, k]$. Let $s_i' := V(L_{s_i}) \cap \operatorname{link}(x)$ and $t_i' := V(L_{t_i}) \cap \operatorname{link}(x)$ for $i \in [1, k]$. Thanks to Theorem 3, when $d \geq 6$, the link of x is k-linked. Using the k-linkedness of link(x), find disjoint paths $L_i' := s_i' - t_i'$ in link(x). Observe that all these k paths $\{L_1', \ldots, L_k'\}$ avoid x. Extend each path L_i' with L_{s_i} and L_{t_i} to form a path $L_i := s_i - t_i$ for each $i \in [1, k]$. The paths $\{L_1, \ldots, L_k\}$ form the desired Y-linkage. $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A. Proof of Lemma 11 for the case $d = 5$

The proof of the lemma for the case $d = 5$ follows a similar structure as the case $d \geq 7$, but requires some technical adjustments. We rely on the following lemmas:

Lemma 22 *([3, Lemma 14]). Let P be a cubical d-polytope with $d \geq 4$. Let X be a set of $d+1$ vertices in P, all contained in a facet F. Let $k := \lfloor (d+1)/2 \rfloor$. Arbitrarily label and pair $2k$ vertices in X to obtain $Y := \{\{s_1, t_1\}, \ldots, \{s_k, t_k\}\}$. Then, for at least $k-1$ of these pairs $\{s_i, t_i\}$, there is an X-valid $s_i - t_i$ path in F.*

Proposition 23 *([3, Prop. 4 and Cor. 5]). Let G be the graph of a 3-polytope and let X be a set of four vertices of G. The set X is linked in G if and only if there is no facet of the polytope containing all the vertices of X. In particular, no nonsimplicial 3-polytope is 2-linked.*

Given sets A, B, X of vertices in a graph G, the set X *separates* A from B if every $A - B$ path in the graph contains a vertex from X. A set X separates two vertices a, b not in X if it separates $\{a\}$ from $\{b\}$. We call the set X a *separator* of the graph. A set of vertices in a graph is *independent* if no two of its elements are adjacent.

Corollary 24 *([3, Corollary 10]). A separator of cardinality d in a d-cube is an independent set.*

Proof of Lemma 11 for $d = 5$. We proceed as in the proof for $d \geq 7$, and consider the same four cases. We let $k := 3$ and let s_1 be a vertex in a cubical 5-polytope P such that s_1 is not in Configuration 5F. Recall that $\mathcal{S}_1$ denotes the star of s_1 in $\mathcal{B}(P)$. Let X be any set of 6 vertices in the graph $G(\mathcal{S}_1)$ of $\mathcal{S}_1$. The vertices in X are our terminals. Also let $Y := \{\{s_1, t_1\}, \{s_2, t_2\}, \{s_3, t_3\}\}$ be a labelling and pairing of the vertices of X. We aim to find a Y-linkage $\{L_1, L_2, L_3\}$ in G where L_i joins the pair $\{s_i, t_i\}$ for $i \in \{1, 2, 3\}$. Recall that a path is X-valid if it contains no inner vertex from X.

We consider a facet F_1 of $\mathcal{S}_1$ containing t_1 and having the largest possible number of terminals. The four cases we consider in the Proof for the case $d \geq 7$ are:

Case 1. $|X \cap V(F_1)| = 5$.
Case 2. $3 \leq |X \cap V(F_1)| \leq 4$.
Case 3. $|X \cap V(F_1)| = 2$.

Case 4. $|X \cap V(F_1)| = 6$.

Case 3 does not require any modification: all the arguments apply for $d \geq 5$. Let us consider the other three cases.

Case 1. $|X \cap V(F_1)| = 5$.

Without loss of generality, assume that $t_2 \notin V(F_1)$.

In this case we proceed as for the case $d \geq 7$ until the final part of the proof where we find disjoint paths $L_i := \pi^{F_1}_{R^o}(s_i) - \pi^{F_1}_{R^o}(t_i)$ ($i \in [1,k]$ and $i \neq 2$) in R^o linking the $d-1$ vertices in X_{R^o}. When $d = 5$ we can only do that when the terminals in R^o are not in cyclic order (in which case we proceed as in the proof for $d \geq 7$). Thus assume that the terminals are in cyclic order. This in turn implies that $\pi^{F_1}_R(s_3) \notin \{s_2, s'_2\}$ and $\pi^{F_1}_R(t_3) \notin \{s_2, s'_2\}$, since $\mathrm{dist}_{F_1}(s_1, s_2) = 4$.

Find a path L'_3 in R between $\pi^{F_1}_R(s_3)$ and $\pi^{F_1}_R(t_3)$ such that L'_3 is disjoint from both s_2 and s'_2 and disjoint from t_1 if $t_1 \in R$; here use Corollary 24, which ensures that the vertices s_2, s'_2 and t_1, if they are all in R, cannot separate $\pi^{F_1}_R(s_3)$ from $\pi^{F_1}_R(t_3)$ in R, since a separator of size three in R must be an independent set. Extend the path L'_3 in R to a path $L_3 := s_3\pi^{F_1}_R(s_3)L'_3\pi^{F_1}_R(t_3)t_3$ in F_1, if necessary. Find a path $L'_1 := s_1 - \pi^{F_1}_{R^o}(t_1)$ in R^o disjoint from $\pi^{F_1}_{R^o}(s_3)$ and $\pi^{F_1}_{R^o}(t_3)$, using the 3-connectivity of R^o. Extend L'_1 to a path $L_1 := s_1L'_1\pi^{F_1}_{R^o}(t_1)t_1$ in F_1, if necessary. The linkage $\{L_1, L_2, L_3\}$ is a Y-linkage. This completes the proof of Case 1.

Case 2. $3 \leq |X \cap V(F_1)| \leq 4$.

In this case we proceed as in the proof for $d \geq 7$, but some comments for $d = 5$ are in order. By virtue of Proposition 23, we need to make sure that the sequence $\bar{s}_2, \bar{s}_3, \bar{t}_2, \bar{t}_3$ in $X^+_{R^o}$ is not in a 2-face of R^o in cyclic order. To ensure this, we need to be a bit more careful when selecting the vertices in $\bar{Z}$. Indeed, if there are already two vertices in X_{R^o} at distance three in R^o, no care is needed when selecting $\bar{Z}$, so proceed as in the case of $d \geq 7$. Otherwise, pick a vertex $\bar{z} \in \bar{Z} \subseteq V(R^o) \setminus (X_{R^o} \cup \{s^o_1\})$ such that $\bar{z}$ is the unique vertex in R^o with $\mathrm{dist}_{R^o}(\bar{z}, x) = 3$ for some vertex $x \in X_{R^o}$; this vertex x exists because $|X \cap V(F_1)| \geq 3$. Selecting such a $\bar{z} \neq s^o_1$ is always possible because s^o_1 is not at distance three in R^o from *any* vertex in X_{R^o}: the unique vertex in R^o at distance three from s^o_1 is $\pi^{F_1}_{R^o}(s_1)$, and $\pi^{F_1}_{R^o}(s_1) \notin X$ because the pair $\{R, R^o\}$ is not associated with $X \cap V(F_1)$. Once $\bar{z}$ is selected, the set Z will contain a neighbour z of $\bar{z}$. In this way, some path S_i or T_j bringing terminals s_i or t_j in $\mathcal{A}_1$ into R^o through Z would use the vertex z, thereby ensuring that x and $\bar{z}$ would be both in $X^+_{R^o}$. This will cause the sequence $\bar{s}_2, \bar{s}_3, \bar{t}_2, \bar{t}_3$ not to be in a 2-face, and thus, not in cyclic order.

Case 4. $|X \cap V(F_1)| = 6$.

The difficulty with $d = 5$ stems from the 3-faces of the polytope not being 2-linked (Proposition 23). Recall that in this case, all the terminals are in the facet F_1. The proof is divided into subcases depending on the nature of the vertex opposite to s_1 in F_1. Either it is not in X (subcase A), or it is a terminal but not t_1 (subcase B), or it is t_1 (subcase C).

SUBCASES A AND B. The vertex s^o_1 opposite to s_1 in F_1 either does not belong to X or belongs to X but is different from t_1

Let $X := \{s_1, s_2, s_3, t_1, t_2, t_3\}$ be any set of six vertices in the graph G of a cubical 5-polytope P. Also let $Y := \{\{s_1, t_1\}, \{s_2, t_2\}, \{s_3, t_3\}\}$. We aim to find a Y-linkage $\{L_1, L_2, L_3\}$ in G where L_i joins the pair $\{s_i, t_i\}$ for $i = 1, 2, 3$.

In both subcases there is a 3-face R of F_1 containing both s_1 and t_1. Let J_1 be the other facet in $\mathcal{S}_1$ containing R. Denote by R_J and R_F the 3-faces in J_1 and F_1, respectively, that are disjoint from R. Then $s^o_1 \in R_F$. We need the following claim.

Claim 1. If a 3-cube contains three pairs of terminals, there must exist two pairs of terminals in the 3-cube, say $\{s_1, t_1\}$ and $\{s_2, t_2\}$, that are not arranged in the cyclic order s_1, s_2, t_1, t_2 in a 2-face of the cube.

Remark 25. If x and y are vertices of a cube, then they share at most two neighbours. In other words, the complete bipartite graph $K_{2,3}$ is not a subgraph of the cube; in fact, it is not an induced subgraph of any simple polytope [8, Cor. 1.12(iii)].

Proof. If no terminal in the cube is in Configuration 3F, we are done. So suppose that one is, say s_1, and that the sequence s_1, x_1, t_1, x_2 of vertices of X is present in cyclic order in a 2-face. Without loss of generality, assume that $s_2 \notin \{x_1, x_2\}$. Then s_2 cannot be adjacent to both s_1 and t_1, since the bipartite graph $K_{2,3}$ is not a subgraph of $G(Q_3)$ (Remark 25). Thus the sequence s_1, s_2, t_1, t_2 cannot be in a 2-face in cyclic order. □

Suppose all the six terminals are in the 3-face R**.** By virtue of Claim 1, we may assume that the pairs $\{s_1, t_1\}$ and $\{s_2, t_2\}$ are not arranged in the cyclic order s_1, s_2, t_1, t_2 in a 2-face of R. Proposition 23 ensures that the pairs $\{\pi^{J_1}_{R_J}(s_1), \pi^{J_1}_{R_J}(t_1)\}$ and $\{\pi^{J_1}_{R_J}(s_2), \pi^{J_1}_{R_J}(t_2)\}$ in R_J can be linked in R_J through disjoint paths L'_1 and L'_2, since the sequence $\pi^{J_1}_{R_J}(s_1), \pi^{J_1}_{R_J}(s_2), \pi^{J_1}_{R_J}(t_1), \pi^{J_1}_{R_J}(t_2)$ cannot be in a 2-face of R_J in cyclic order. Moreover, by the connectivity of R_F, there is a path L_3' in R_F linking the pair $\{\pi_{R_F}^{F_1}(s_3), \pi_{R_F}^{F_1}(t_3)\}$. The linkage $\{L_1', L_2', L_3'\}$ can naturally be extended to a Y-linkage $\{L_1, L_2, L_3\}$ as follows.

$$L_i := \begin{cases} s_i \pi_{R_J}^{J_1}(s_i) L_i' \pi_{R_J}^{J_1}(t_i) t_i, & \text{for } i = 1, 2; \\ s_3 \pi_{R_F}^{F_1}(s_3) L_3' \pi_{R_F}^{F_1}(t_3) t_3, & \text{otherwise.} \end{cases}$$

Suppose that R **contains a pair** $\{s_i, t_i\}$ **for** $i = 2, 3$**, say** $\{s_2, t_2\}$. There are at most five terminals in R, and consequently, applying Lemma 22 to the polytope F_1 and its facet R, we obtain an X-valid path $L_1 := s_1 - t_1$ in R or an X-valid path $L_2 := s_2 - t_2$ in R. For the sake of concreteness, say an X-valid path L_2 exists in R. From the connectivity of R_F and R_J follows the existence of a path L_3' in R_F between $\pi_{R_F}^{F_1}(s_3)$ and $\pi_{R_F}^{F_1}(t_3)$, and of a path L_1' in R_J between $\pi_{R_J}^{J_1}(s_1)$ and $\pi_{R_J}^{J_1}(t_1)$ (recall that $t_1 \in R \subset J_1$). The linkage $\{L_1', L_2', L_3'\}$ can be extended to a linkage $\{s_1 - t_1, s_2 - t_2, s_3 - t_3\}$ in $\mathcal{S}_1$.

Suppose that the ridge R **contains no other pair from** Y **and that the ridge** R_F **contains a pair** (s_i, t_i) **(**$i = 2, 3$**).** Without loss of generality, assume s_2 and t_2 are in R_F.

First suppose that $s_3 \in R$, which implies that $t_3 \in R_F$. Further suppose that there is a path T_3 of length at most two from t_3 to R that is disjoint from $X \setminus \{s_3, t_3\}$. Let $\{t_3'\} := V(T_3) \cap V(R)$. Use the 2-linkedness of the 4-polytope J_1 [3, Prop. 6] to find disjoint paths $L_1 := s_1 - t_1$ and $L_3' := s_3 - t_3'$ in J_1. Let $L_3 := s_3 L_3' t_3' T_3 t_3$. Use the 3-connectivity of R_F to find an X-valid path $L_2 := s_2 - t_2$ in R_F that is disjoint from $V(T_3)$; note that $|V(T_3) \cap V(R_F)| \le 2$. The paths $\{L_1, L_2, L_3\}$ give the desired Y-linkage. Now suppose there is no such path T_3 from t_3 to R. Then, the projection $\pi_R^{F_1}(t_3)$ is in $\{s_1, t_1\}$, say $\pi_R^{F_1}(t_3) = t_1$; the projection $\pi_{R_F}^{F_1}(s_1)$ is a neighbour of t_3 in R_F; and both s_2 and t_2 are neighbours of t_3 in R_F. This configuration implies that s_1 and t_1 are adjacent in R. Let $L_1 := s_1 t_1$. Find a path $L_2 := s_2 - t_2$ in R_F that is disjoint from t_3, using the 3-connectivity of R_F. Then using Lemma 19 find a neighbour s_3' in $\mathcal{A}_1$ of s_3 and a neighbour t_3' in $\mathcal{A}_1$ of t_3; note that, since $\operatorname{dist}_{F_1}(s_1, t_3) \le 2$, we have that $t_3 \ne s_1^o$, and since $\{s_1, s_3\} \in V(R)$, $s_3 \ne s_1^o$. Find a path L_3 in $\mathcal{S}_1$ between s_3 and t_3 that contains a subpath L_3' in $\mathcal{A}_1$ between s_3' and t_3'; here use the connectivity of $\mathcal{A}_1$ (Proposition 7): $L_3 := s_3 s_3' L_3' t_3' t_3$. The linkage $\{L_1, L_2, L_3\}$ is the desired Y-linkage.

Assume that $s_3 \in R_F$; by symmetry we can further assume that $t_3 \in R_F$. The connectivity of R ensures the existence of a path $L_1 := s_1 - t_1$ therein. In the case of $s_1^o \in X$, without loss of generality, assume $s_1^o = s_2$. The 3-connectivity of R_F ensures the existence of an X-valid path $L_2 := s_2 - t_2$ therein. Use Lemma 19 to find pairwise distinct neighbours s_3' of s_3 and t_3' of t_3 in $\mathcal{A}_1$; these exist since $s_3 \ne s_1^o$ and $t_3 \ne s_1^o$. Using the connectivity of $\mathcal{A}_1$ (Proposition 7), find a path $L_3 := s_3 - t_3$ in $\mathcal{S}_1$ that contains a subpath $s_3' - t_3'$ in $\mathcal{A}_1$. The linkage $\{L_1, L_2, L_3\}$ is the desired Y-linkage.

Assume neither R **nor** R_F **contains a pair** $\{s_i, t_i\}$ **(**$i = 2, 3$**).** Without loss of generality, assume that $s_2, s_3 \in R$, that $t_2, t_3 \in R_F$ and that $t_2 \ne s_1^o$.

First suppose that there exists a path S_3 in F_1 from s_3 to R_F that is of length at most two and is disjoint from $X \setminus \{s_3, t_3\}$. Let $\{\hat{s}_3\} := V(S_3) \cap V(R_F)$. Find pairwise distinct neighbours s_2' and t_2' of s_2 and t_2, respectively, in $\mathcal{A}_1$. And find a path $L_2 := s_2 - t_2$ in $\mathcal{S}_1$ that contains a subpath $s_2' - t_2'$ in $\mathcal{A}_1$ (using the connectivity of $\mathcal{A}_1$). Using the 3-connectivity of R_F link the pair $\{\hat{s}_3, t_3\}$ in R_F through a path L_3' that is disjoint from t_2. Let $L_3 := s_3 S_3 \hat{s}_3 L_3' t_3$. Since Corollary 24 ensures that any separator of size three in a 3-cube must be independent, we can find a path $L_1 := s_1 - t_1$ in R that is disjoint from s_2 and $V(S_3) \cap V(R)$; the set $V(S_3) \cap V(R)$ has either cardinality one or contains an edge. The paths $\{L_1, L_2, L_3\}$ form the desired Y-linkage.

Assume that there is no such path S_3. In this case, the neighbours of s_3 in F_1 are s_1, t_1, s_2 from R and t_2 from R_F. Use Lemma 19 to find a neighbour s_3' of s_3 in $\mathcal{A}_1$. Again use Lemma 19 either to find a neighbour t_3' of t_3 if $t_3 \ne s_1^o$ or to find a neighbour t_3' of a neighbour u of t_3 in R_F (with $u \ne t_2$) if $t_3 = s_1^o$. Let T_3 be the path of length at most two from t_3 to $\mathcal{A}_1$ defined as $T_3 = t_3 t_3'$ if $t_3 \ne s_1^o$ and $T_3 = t_3 u t_3'$ if $t_3 = s_1^o$. Find a path L_3 in $\mathcal{S}_1$ between s_3 and t_3 that contains a subpath in $\mathcal{A}_1$ between s_3' and t_3'; here use the connectivity of $\mathcal{A}_1$ (Proposition 7). We next find a path S_2 in F_1 from s_2 to R_F that is of length at most two and is disjoint from $V(T_3) \cup \{s_1, t_1, s_3\}$. There are exactly four disjoint $s_2 - R_F$ paths of length at most two, one through each of the neighbours of s_2 in F_1. One such path is $s_2 s_3 t_2$. Among the remaining three $s_2 - R_F$ paths, since none of them contains s_1 or t_1 and since $|V(T_3) \cap V(R_F)| \le 2$, we find the path S_2. Let $\hat{s}_2 := V(S_2) \cap V(R_F)$. Find a path $L_2' := \hat{s}_2 - t_2$ in R_F that is disjoint from $V(T_3)$, using the 3-connectivity of R_F. Let $L_2 := s_2 S_2 \hat{s}_2 L_2' t_2$. Since the vertices in $(V(S_2) \cap V(R)) \cup \{s_3\}$ cannot separate s_1 from t_1 in R (Corollary 24), find a path $L_1 := s_1 - t_1$ in R disjoint from $V(S_2) \cap V(R) \cup \{s_3\}$; the set $V(S_2)$ has cardinality one or contains one edge. The paths $\{L_1, L_2, L_3\}$ form the desired Y-linkage.

SUBCASE C. The vertex opposite to s_1 in F_1 coincides with t_1

Since s_1 is not in configuration $d3$ we may suppose that t_1 has a neighbour t_1' not in X. We reason as in Subcases A and B. We give the details for the sake of completeness.

Let R denote the 3-face in F_1 containing both s_1 and t_1'; $\operatorname{dist}_R(s_1, t_1') = 3$. Let R_F be the 3-face of F_1 disjoint from R. Let J_1 be the other facet in $\mathcal{S}_1$ containing R and let R_J be the 3-face of J_1 disjoint from R.

Suppose R **contains a pair** $\{s_i, t_i\}$ **(**$i = 2, 3$**), say** (s_2, t_2)**.** There are at most five terminals in R (as t_1 is in R_F). Since the smallest face in R containing s_1 and t_1' is 3-dimensional, the sequence $\pi_{R_J}^{J_1}(s_1), \pi_{R_J}^{J_1}(s_2), \pi_{R_J}^{J_1}(t_1'), \pi_{R_J}^{J_1}(t_2)$ cannot appear in a 2-face of R_J in cyclic order. As a consequence, the pairs $\{\pi_{R_J}^{J_1}(s_1), \pi_{R_J}^{J_1}(t_1')\}$ and $\{\pi_{R_J}^{J_1}(s_2), \pi_{R_J}^{J_1}(t_2)\}$ can be linked in R_J

through disjoint paths L_1' and L_2', thanks to Proposition 23. Let $L_1 := s_1\pi^{J_1}_{R_J}(s_1)L_1'\pi^{J_1}_{R_J}(t_1')t_1't_1$ and $L_2 := s_2\pi^{J_1}_{R_J}(s_2)L_2'\pi^{J_1}_{R_J}(t_2)t_2$. From the 3-connectivity of R_F follows the existence of a path L_3' in R_F between $\pi^{F_1}_{R_F}(s_3)$ and $\pi^{F_1}_{R_F}(t_3)$ that avoids t_1. Let $L_3 := s_3\pi^{F_1}_{R_F}(s_3)L_3'\pi^{F_1}_{R_F}(t_3)t_3$. The paths $\{L_1, L_2, L_3\}$ form the desired Y-linkage.

Suppose that the ridge R contains no pair $\{s_i, t_i\}$ ($i = 2, 3$) and that the ridge R_F contains a pair $\{s_i, t_i\}$ ($i = 2, 3$), say $\{s_2, t_2\}$. Then, there are at most five terminals in R_F. If there are at most four terminals in R_F, the 3-connectivity of R_F ensures the existence of an X-valid path $L_2 := s_2 - t_2$ in R_F; if there are exactly five terminals in R_F, applying Lemma 22 to the polytope F_1 and its facet R_F gives either an X-valid path $L_2 := s_2 - t_2$ or an X-valid path $L_3 := s_3 - t_3$ in R_F. As a result, regardless of the number of terminals in R_F, we can assume there is an X-valid path $L_2 := s_2 - t_2$ in R_F. Find pairwise distinct neighbours s_3' and t_3' in $\mathcal{A}_1$ of s_3 and t_3, respectively, and a path L_3 in $\mathcal{S}_1$ between s_3 and t_3 that contains a subpath in $\mathcal{A}_1$ between s_3' and t_3'; here use the connectivity of $\mathcal{A}_1$ (Proposition 7). In addition, let L_1' be a path in R between s_1 and t_1'; here use the 3-connectivity of R to avoid any terminal in R. Let $L_1 := s_1L_1't_1't_1$. The Y-linkage is given by the paths $\{L_1, L_2, L_3\}$.

Assume neither R nor R_{F_1} contains a pair $\{s_i, t_i\}$ ($i = 2, 3$). Without loss of generality, we can assume $s_2, s_3 \in R$ and $t_2, t_3 \in R_F$.

There exists a path S_3 from s_3 to R_F that is of length at most two and is disjoint from $\{s_1, t_1, t_1', s_2, t_2\}$. If $\pi_{R_F}(s_3) \neq t_2$, then $S_3 = s_3\pi_{R_F}(s_3)$. Otherwise, there are exactly three disjoint paths of length 2 from s_3 to R_F. At most two of them contain a vertex in $N_R(s_3) \cap (X \cup \{t_1'\})$ (since $\operatorname{dist}(s_1, t_1) = 3$, they cannot be both neighbours of s_3). Thus we can take S_3 as the path $s_3u\pi_{R_F}(u)$ through a neighbour u of s_3 in R such that $u \notin X \cup \{t_1'\}$ and $\pi_{R_F}(u) \notin \{t_1, t_2\} = \{\pi_{R_F}(s_3), \pi_{R_F}(t_1')\}$.

Let $\{\hat{s}_3\} := V(S_3) \cap V(R_F)$. Find an X-valid path $L_3' := \hat{s}_3 - t_3$ in R_F using its 3-connectivity. Let $L_3 := s_3S_3\hat{s}_3L_3't_3$. Then find neighbours s_2' and t_2' of s_2 and t_2, respectively, in $\mathcal{A}_1$, and a path $L_2 := s_2 - t_2$ in $\mathcal{S}_1$ that contains a subpath $s_2' - t_2'$ in $\mathcal{A}_1$ (using the connectivity of $\mathcal{A}_1$). Since Corollary 24 ensures that any separator of size three in a 3-cube must be independent, we can find an $L_1' := s_1 - t_1'$ in R that is disjoint from s_2 and $V(S_3) \cap V(R)$; the set $V(S_3) \cap V(R)$ has either cardinality one or contains an edge. Let $L_1 := s_1L_1't_1't_1$. The paths $\{L_1, L_2, L_3\}$ form the desired Y-linkage.

This concludes the proof of Lemma 11 for $d = 5$. □

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