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*VOR (version of record)
Geometric design of the limaçon-to-circular fluid processing machine

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Abstract
A limaçon machine is a rotary positive displacement device, in which the housing and rotor are constructed of limaçon of Pascal curves. Previous works have been published to investigate the working of these machines in two applications: gas expanders and compressors. This paper presents a theoretical investigation into the potential of modifying the rotor profile of the limaçon machines in order to simplify the machine’s manufacturing process and to reduce production cost. The proposed modification will produce new characteristics for the housing-rotor interaction. An outcome that motivates the need to obtain new mathematical models to investigate the housing-rotor interference, and describe the volumetric relationship of the new machine. This paper also employs an optimisation approach to design the best machine for a given set of operating conditions, i.e. expander, compressor, and pump. The outcome of this study confirms the validity of the proposed modification and its potential to produce a limaçon machine with favourable characteristics.

Keywords: Rotary machine; limaçon-to-circular machine; limaçon machine, limaçon motion; pump; expander; compressor; positive displacement machine; SPSA.

Article type: Research paper

Introduction
Positive displacement machines, the rotary type in particular, are increasingly gaining attention from the energy-efficient, energy-aware, sustainable development organisations as well as from the industry and the research sector. Such machines can operate at relatively low speed and are capable of handling small mass flow rate. Moreover, when utilised as expanders, these positive displacement machines can produce work from low-grade thermal energy sources. In addition to these, positive displacement machines are able to operate in multiphase flow conditions, which can damage non-positive displacement machines’ parts and components. With all the above-mentioned benefits and the advantage of having the fewest components compared to other classes of machines, positive displacement machines can be manufactured to suit compact power generation systems at relatively low cost.

Over the years, different types of rotary positive displacement machines have been developed and tested; these include compressors, expanders, and pumps. Some examples of these works are: scroll expanders and compressors by Kim et al. [1], Oralli et al. [2] and Georges et al. [3]; vane expanders by Jia et al. [4]; revolving vane expanders by Subiantoro et al. [5]; and limaçon machines by Sultan et al. [6, 8–10] to name only a few. Among those diverse types, the limaçon technology seems to...
receive the least focus from researchers despite the fact that this technology has been developed over a century ago [11]. Recently, however, Sultan [7, 9] has drawn the attention to this limaçon technology by developing mathematical and geometric models to describe the working principle of this technology. Sultan [9] argues that the unavailability of the technology needed to machine accurate limaçon profiles for the rotor and housing had previously inhibited the progress in this area.

Figure 1: (a) Limaçon-to-limaçon machine, (b) Circolimaçon machine, and (c) Limaçon-to-circular machine (investigated in this paper). Note the difference in the rotors’ lenticular curvatures of these embodiments and the gap between the rotor lobe, the rotor apex and the housing.
Manufacturing technology has now progressed to a point where it is possible to manufacture products with complex geometries albeit at a cost. Hence, Sultan \cite{9} proposed modifications of both the machine housing and rotor profiles in which the limaçon curves are replaced by circular ones to reduce the manufacturing cost. In that design, referred to as the ‘circolimaçon’ (Figure 1b), the distance separating the rotor apices from the housing varies with the rotor’s angular rotation. This variation reduces the effectiveness of the apex seals and adversely impacts the machine performance. In fact, the apex-housing gap variation is a consequence of replacing the limaçon housing by a circular housing; the rotor circular flanks do not contribute to this gap variation. This understanding motivated the work presented in this paper that aims at maintaining the circular rotor profile while reverting the housing profile back to its limaçon original (Figure 1c). Thus, maintaining the favourable sealing characteristic of the limaçon-to-limaçon machine while reducing the cost of manufacturing the rotor flanks. With this concept in mind, this paper attempts to study the new proposed arrangement, which is now referred to as limaçon-to-circular machine, with the aim of eliminating housing-rotor interference and determining the volumetric characteristic in relation to various machine dimensions and the rotor angular arrangements. Of note is the limaçon-to-circular machine can be considered a hybrid of limaçon-to-limaçon and circolimaçon machines.

To design a limaçon-to-circular machine to work as either a pump, a compressor, or an expander, the working conditions of such a machine, i.e. inlet and outlet pressures and temperatures, the type of working fluid, flow rate, lubricating and sealing to name only a few, need to be taken into account. These conditions will reflect on the dimension and geometry of the machine. Due to the iterative nature of the design process, an objective function with weighting factors and an optimisation procedure are employed in this paper. The design and optimisation processes will be developed in the following sections of this paper.

Background on limaçon technology

The limaçon motion, relative motion between the machine’s rotor and housing, can be produced by a class of mechanisms as described by Sultan \cite{6}, who also presented a number of positive displacement machines driven by these mechanisms. In further publications, Sultan suggests that rotors and housings of these machines could be manufactured to either the limaçon curves or circular ones, referred to as limaçon-to-limaçon (Figure 1a) and circolimaçon (Figure 1b) machines, respectively \cite{6,8}. In this paper, the rotor of the limaçon machine is modified in a way that instead of using the same limaçon profile as the housing, circular segments are utilised to achieve favourable thermodynamic induction of saturated liquid.

As shown in Figure 1c, the chord $p_1p_2$ of length $2L$ ($L$ is referred to as half the chord length) rotates and slides about the limaçon pole $o$, the housing of the limaçon machine is a curve formed by the traces of point $p_1$ (or $p_2$). The centre point $m$ of the chord $p_1p_2$ is restricted to move on a stationary circle of radius $r$, this circle is referred to as the base circle of limaçon. Two coordinate systems $XY$ and $X_rY_r$ are introduced; $XY$ is a stationary Cartesian frame attached to the pole $o$, while $X_rY_r$ is attached to the chord $p_1p_2$ at it centre point $m$, as shown in Figure 1c, and rotates and slides with the chord. The angle measured from $X$ to $X_r$ axis is the angle rotated by the chord $p_1p_2$ as the rotor performs its rotational motion. When the chord is sliding, its centre point $m$ slides
a distance $s$ measured from the pole $o$ to the centre of chord $m$, shown in Figure 1c, this distance, $s$, can be written as follows:

$$s = 2r \sin \theta$$  \hspace{1cm} (1)

where $r$ is the radius of the limaçon base circle. The radial distance of the housing, $R_h$, is measured along the chord $p_1p_2$ from the pole $o$ to $p_1$ when the chord is in motion. This distance, $R_h$, can be calculated as follow:

$$R_h = 2r \sin \theta + L$$  \hspace{1cm} (2)

Hence, the Cartesian position of point $p_1$ with respect to the $XY$ coordinates can be expressed as:

$$\begin{align*}
  x &= R_h \cos \theta \\
  y &= R_h \sin \theta
\end{align*}$$  \hspace{1cm} (3)

Costa et al. [12] have mentioned that in order for the limaçon curve to be looping and dimple free, the value of the limaçon aspect ratio, $b = \frac{r}{L}$, has to be less than 0.25 ($b < 0.25$). This limit will be employed in the design of this limaçon-to-circular machine. As such, Equation 2 and 3 can be manipulated so that the position of point $p_1$ can be expressed as functions of variables $b$ and $L$ as:

$$\begin{align*}
  x &= L[b \sin 2\theta + \cos \theta] \\
  y &= L[b(1 - \cos 2\theta) + \sin \theta]
\end{align*}$$  \hspace{1cm} (4)

The limaçon housing of the machine proposed in this paper can be manufactured following Equation 4; the machine rotor consists of two circular segments of radius $R$, which take the chord, $p_1p_2$, as the mirror line. The designer can introduce a small clearance $C$ between the housing and the rotor to prevent housing-rotor interference; considerations for the value of $C$ has been suggested by Sultan [8]. Therefore, half the rotor chord length, $L_r$, can be introduced as:

$$L_r = L - C$$  \hspace{1cm} (5)

where $C$ has to be calculated using a suitable optimisation procedure.

The housing of the fluid processing machine can be manufactured by utilising Equation 1 to 4. The rotor is comprised of two circular segments with a radius

$$R = \sqrt{(L - C)^2 + (2rk)^2}$$  \hspace{1cm} (6)

mirrored about the rotor axis $X_r$. The length of rotor chord is defined in Equation 5. The centre of the circular segments lie on a line that is perpendicular to the rotor chord at its mid-point; the distance from the centre of the lower circular segment to the rotor chord is determined by the expression $2rk$. The factor $k$ is employed to facilitate the design process as detailed in the following sections. This is a major change in terms of housing-rotor arrangement. Motion of the rotor itself inside the limaçon housing, nevertheless, follows the limaçon curve (rotating and sliding motion). In order for the housing-rotor interference not to take place during the machine operation, design procedures should be incorporated such as the tangent approach and the radial clearance approach used by Sultan [6].

The procedure proposed in this paper to design limaçon-to-circular machines by applying the tangent approach and the radial clearance approach will be detailed below.
Housing-rotor interference: the tangent method

It is essential for the design of the limaçon-to-circular machine to employ a procedure that can prevent housing-rotor interference while the machine is operating. The tangent method is simple enough to be initially applied. This approach considers the slopes of tangent vectors on the rotor at the apices and the corresponding contact points on the housing. The lower part of the limaçon housing where $\theta \in [\pi, 2\pi]$ and the corresponding lower lobe of the rotor are subject to housing-rotor interference [6]. Figure 2 indicates the unit vector $\hat{T}_r$ tangent to the lower lobe of rotor at the apex $p_1$ and the corresponding unit vector $\hat{T}_h$ tangent to the housing. The condition for the interference to be prevented is defined as:

$$\left(\hat{T}_r \wedge \hat{T}_h\right) \cdot \hat{z} > 0 \quad (7)$$

which can also be written in terms of the two angles $\rho_h$ and $\rho_r$ shown in Figure 2. These two angles can be calculated as in Equation 8 and 9 below:

$$\tan \rho_h = \frac{2r \cos \alpha}{L - 2r \sin \alpha} \quad (8)$$

and

$$\tan \rho_r = \frac{2rk}{L - C} \quad (9)$$

where $\alpha = \theta - \pi$ as shown in Figure 2. Consequently, the condition for housing-rotor interference prevention can be expressed as:

$$\frac{2rk}{L - C} \geq \frac{2r \cos \alpha}{L - 2r \sin \alpha} \quad (10)$$
It is possible to prove that value of $\alpha$ which maximise the right hand side of the expression 10 above is given by $\sin \alpha = 2b$ and $\cos \alpha = \sqrt{1 - 4b^2}$. Expression 10 can now be manipulated to produce the following forms with clearance, $C$, half chord length, $L$, and the aspect ratio $b$ as variables:

$$k > \frac{1 - \frac{C}{L}}{\sqrt{1 - 4b^2}}$$

(11)

The expression 11 above can be rewritten as

$$k = a \frac{1 - C_L}{\sqrt{1 - 4b^2}}$$

(12)

where $a$ is a design factor and $C_L = \frac{C}{L}$. The value of the design factor, $a$, in Equation 12 has to be greater than one ($a > 1$). Also, from Equation 6 and 12, the rotor radius $R$ can be manipulated and expressed as

$$R = (1 - C_L) \sqrt{L^2 + \frac{4r^2a^2}{1 - 4b^2}}$$

(13)

In the next section, the radial clearance problem will be presented.

**The housing-rotor radial clearance method**

A sufficiently high value of factor $a$ (e.g. $a = 1.3$) can be employed to ensure that the interference between the housing and rotor will not occur. However, a high value of $a$ may adversely affect the volumetric efficiency of the machine. Consequently, a small value of clearance, $C$, is introduced to the rotor geometry with the objective of ensuring that a minimum housing-rotor distance, $\Delta_{min}$, is accomplished. As such the problem can be represented as an optimisation endeavour undertaken to calculate the value of $C$ and $a$ to meet certain design requirements. The mathematical formulation of the optimisation problem will feature the calculation of the radial distance, $\Delta$, between any point on the lobe of the rotor and the corresponding point on the housing on the same radial line $R_h$ as shown in Figure 3. Also in Figure 3, the distance from the pole, $o$, to the lower rotor lobe can be defined as $z$; $\beta = [0, \pi]$ is the angle between the rotor chord and $z$; $o_o = \sqrt{s^2 + (2rk)^2}$ is the distance from the centre of rotor lobe to the pole $o$; $\gamma$ represents the angle between $o_o$ and the rotor chord. Hence, the rotor lobe radius $R$ can be expressed as:

$$R^2 = s^2 + (2rk)^2 + z^2 - 2z\sqrt{s^2 + (2rk)^2} \cos (\gamma + \pi - \beta)$$

(14)

which can also be written as

$$R^2 = s^2 + (2rk)^2 + z^2 + 2z(s \cos \beta + 2rk \sin \beta)$$

(15)

From Equation 15 the rotor radial distance, $z$, can be calculated as below, note that only the positive root is selected:

$$z = -(s \cos \beta + 2rk \sin \beta) + \sqrt{R^2 - (s \sin \beta - 2rk \cos \beta)^2}$$

(16)

The value of $R_h$ given in Equation 2 can also be expressed as:

$$R_h = L + 2r \sin(\theta + \beta)$$

(17)
which may be re-written in terms of \(s, r, \alpha = \theta - \pi\), and \(\beta\) as follows:

\[
R_h = L - s \cos \beta - 2r \cos \alpha \sin \beta
\]  

(18)

The radial distance from the pole, \(o\), to any point on the lower rotor lobe is expressed in Equation 16; hence, the housing–rotor radial clearance, \(\Delta\), can be evaluated as:

\[
\Delta = R_h - z
\]  

(19)

With \(R = \sqrt{(L - C)^2 + (2rk)^2}\), it is possible to express the radial clearance, \(\Delta\), as below:

\[
\Delta = L - 2r \cos \alpha \sin \beta + 2rk \sin \beta - \sqrt{(L - C)^2 + (2rk)^2 - (s \sin \beta - 2rk \cos \beta)^2}
\]  

(20)

The aspect ratio \(b\) can also be applied to Equation 20 to yield:

\[
\Delta = L \left[ 1 - 2b \cos \alpha \sin \beta - 2bk \sin \beta - \sqrt{(1 - C_L)^2 + (2bk)^2 - (2b \sin \alpha \sin \beta - 2bk \cos \beta)^2} \right]
\]  

(21)

In order to calculate \(C_L, \alpha\), and \(\beta\) needed to achieve the minimum radial clearance \(\Delta_{\text{min}}\) assigned by the designer, the following system of non-linear equations has to be solved simultaneously:

\[
\begin{cases}
\Delta - \Delta_{\text{min}} = 0 \\
\partial \Delta / \partial \alpha = 0 \\
\partial \Delta / \partial \beta = 0
\end{cases}
\]  

(22)

The system of equations above can be solved for angles \(\alpha\) and \(\beta\) at which the radial clearance, \(\Delta\), is at its minimum value. This system of equations has been manipulated to produce the following outcome:

\[
\Delta_{\text{min}} = L \left[ 1 - 2b + 2bk - \sqrt{(1 - C_L)^2 + (2bk)^2} \right]
\]  

(23)

where \(k\) is a function of factor \(a\) shown in Equation 12.

The volumetric relations of the limaçon-to-circular machine are conveyed in the following section.
The volumetric relations

The geometry described in Figure 1c can be used to obtain the volumetric relationship of the limaçon-to-circular machine. The cross-sectional area of the housing above the rotor chord, $A_h$, suggested by Sultan [6] is as follow:

$$A_h = \left( r^2 + \frac{L^2}{2} \right) \pi + 4rL \cos \theta$$

which can be written in terms of half chord length, $L$, and aspect ratio $b$ as

$$A_h = L^2 \left[ \left( b^2 + \frac{1}{2} \right) + 4b \cos \theta \right]$$

The rotor area, $A_r$, can be calculated by the following expressions:

$$A_r = [(L - C)^2 + (2rk)^2] \varphi - (L - C)2rk$$

The volumetric ratio of net chamber volume/area when the angle $\theta$ is $\pi$ to the net chamber volume/area when the angle $\theta = 0$:

$$V_{net} = A_{net}H$$

The volumetric ratio $R_{vol}$, defined as the minimum chamber volume to the maximum chamber volume, has a critical impact on the efficiency of the positive displacement machine, particularly when gas is used as a working fluid. This volumetric ratio, $R_{vol}$, can be expressed in terms of the ratio of net chamber volume/area when the angle $\theta = \pi$ to the net chamber volume/area when $\theta = 0$:

$$R_{vol} = \frac{V_{net|\theta=\pi}}{V_{net|\theta=0}} = \frac{A_{net|\theta=\pi}}{A_{net|\theta=0}}$$

From Equation 28 to 30, $R_{vol}$ can be expanded as follow:

$$R_{vol} = \frac{\left[ \left( b^2 + \frac{1}{2} \right) \pi - 4b \right] - (1 - \frac{C}{L}) \left[ tan^{-1}\frac{\sqrt{1 - 4b^2}}{2ab} \left( 1 + \frac{4a^2b^2}{1 - 4b^2} \right) - \frac{2ab}{\sqrt{1 - 4b^2}} \right]}{\left[ \left( b^2 + \frac{1}{2} \right) \pi + 4b \right] - (1 - \frac{C}{L}) \left[ tan^{-1}\frac{\sqrt{1 - 4b^2}}{2ab} \left( 1 + \frac{4a^2b^2}{1 - 4b^2} \right) - \frac{2ab}{\sqrt{1 - 4b^2}} \right]}$$

An important design criterion for a positive displacement machine is the volume of the working fluid expected to be induced into the working chamber, $V_{ind}$, during the suction stroke. This induced volume, $V_{ind}$, can be calculated as:

$$V_{ind} = V_{net|\theta=\pi} - \mu V_{net|\theta=0}$$
where $\theta_{\text{cut}, \text{off}} \in [0, \pi]$ is the angle at which the inlet port of the machine is closed. The fluid left in the working chamber from the previous cycle will have its effect onto the intake volume of the machine, this effect is taken care of by the clearance volume factor $\mu$ in Equation 32. The value of $\mu$ can be assigned by the designer to match the expected compressibility, valve losses, and leakages. In less conservative designs, $\mu$ can be defined as the ratio between the average fluid density of the discharge and induction strokes [9] as below:

$$\mu = \frac{\rho_{\text{out}}}{\rho_{\text{in}}}$$

To design a limaçon-to-circular machine for a particular application, the working conditions of such a machine, i.e. flow rate, volume of the fluid expected to be induced into the working chamber, need to be taken into account. These conditions will affect the machine’s geometry which can be presented as a design vector $Y_d$. An optimisation procedure is then employed to produce a suitable machine. The design vector, $Y_d$, optimisation approach, and some case studies will be detailed in the following sections.

The optimisation method

The optimisation process in this paper employs the use of the simultaneous perturbation stochastic approximation (SPSA) method introduced and detailed by Spall [13–16], and since being refined [17–19]. This method has been used in a wide range of applications such as tidal modelling [20], automation, computing and control [21,22], and design [8]. The goal of this approach is to minimise a scalar-valued loss function, $\Lambda(\Phi)$, in relation to a design vector, $\Phi$. The measured value of the loss function available at any value of $\Phi$ may be expressed as follow:

$$F(\Phi) = \Lambda(\Phi) + \text{noise}$$

From an initial guess value of $\Phi$, an iteration process takes place following the approximation of the gradient $G(\Phi) = \frac{\partial \Lambda(\Phi)}{\partial \Phi}$. Additionally, it is assumed that $\Lambda(\Phi)$ is a differentiable function of $\Phi$ and the minimum point $\Phi_0$ corresponds to the zero point of the gradient $G(\Phi) = 0$.

The SPSA procedure starts at step $k = 0$ with a guess design vector $\Phi_0$ (the symbol $\hat{\cdot}$ represents estimates) and a given set of five non-negative parameters $\Psi$, $\Omega$, $A$, $\psi_s$, and $\omega_s$. The values of $\psi_s$ and $\omega_s$ have been suggested by Spall [23] as 0.602 and 0.101, respectively, and the value of $A$ is one-tenth the number of iterations allowed by the user. The remaining two parameters, $\Psi$ and $\Omega$, are selected to match a given application. The gain sequences $\psi_k$ and $\omega_k$ are calculated at iteration step number $k$ using the following equations:

$$\psi_k = \frac{\Psi}{(A + k + 1)^{\psi_s}}$$

and

$$\omega_k = \frac{\Omega}{(k + 1)^{\omega_s}}$$

At each iteration, an $n$-dimensional perturbation vector $D_k$ is generated where each of its components is randomly assigned a value of $\pm 1$ using a binary Bernoulli distribution. As such, vector $D_k$ can be expressed as:

$$D_k = \pm 1 \text{ with probability of } \frac{1}{2}$$
To this end, two measurements of the loss function at $\hat{\Phi}_k$ can be obtained as $F(\hat{\Phi}_k + \omega_k D_k)$ and $F(\hat{\Phi}_k - \omega_k D_k)$. Hence, the gradient of the loss function with respect to the vector $\hat{\Phi}$ can be calculated as:

$$G(\hat{\Phi}_k) = \frac{F(\hat{\Phi}_k + \omega_k D_k) - F(\hat{\Phi}_k - \omega_k D_k)}{2\omega_k}$$

(37)

where $D_k$ is the $i^{th}$ component of the $D_k$ vector. At the end of iteration $k$, the estimated value $\hat{\Phi}_k$ is replaced by $\hat{\Phi}_{k+1}$ by using the following equation:

$$\hat{\Phi}_{k+1} = \hat{\Phi}_k - \psi_k \hat{G}_k(\hat{\Phi}_k)$$

(38)

The SPSA approach can be used to optimise the design process of the limaçon-to-circular machine in which an objective function will be established to encompass the desired performance characteristics. The optimisation procedure is undertaken to ensure that the designed machine will meet certain geometric requirements. These requirements are minimum clearance required by the designer to suit a specific application, $\Delta_{\text{min.req}}$, a required volumetric ratio, $R_{\text{vol.req}}$, and a required induced volume, $V_{\text{ind.req}}$. The design vector which will realise these requirements include $L$, $a$, $b$ and $C_L$. The expected outcome of the optimisation procedure is that the achieved values of $\Delta_{\text{min}}$, $R_{\text{vol}}$, and $V_{\text{ind}}$ are related to their corresponding required values as follows:

$$\begin{align*}
\Delta_{\text{min}} - \Delta_{\text{min.req}} &> 0 \\
R_{\text{vol.req}} - R_{\text{vol}} &> 0 \\
V_{\text{ind}} - V_{\text{ind.req}} &> 0
\end{align*}$$

(39)

The manner in which the variable design parameters affect $\Delta_{\text{min}}$ and $R_{\text{vol}}$ is shown in Figure 4. It is obvious that the settings of factors $a$ and $b$ effect the values of $R_{\text{vol}}$ and $\Delta_{\text{min}}$ considerably.

The integration of the above inequalities, shown in 39, into the optimisation process will introduce slack variables $d$, $R_v$ and $v$, as follows:

$$\begin{align*}
d &= \Delta_{\text{min}} - \Delta_{\text{min.req}} \\
R_v &= R_{\text{vol.req}} - R_{\text{vol}} \\
v &= V_{\text{ind}} - V_{\text{ind.req}}
\end{align*}$$

(40)

The expressions 40 can also be expressed in terms of zero-valued functions as follows:

$$\begin{align*}
F_{\Delta} &= (\Delta_{\text{min}} - \Delta_{\text{min.req}} - d)^2 \\
F_R &= (R_{\text{vol}} - R_{\text{vol.req}} + R_v)^2 \\
F_V &= (V_{\text{ind}} - V_{\text{ind.req}} - v)^2
\end{align*}$$

(41)

To this end, the design vector $Y_d$ can be constructed as $Y_d = [L \ a \ b \ C_L \ d \ R_v]^T$. Functions of $\Delta_{\text{min}}$, $R_{\text{vol}}$, and $V_{\text{ind}}$ in the expressions 41 can now be incorporated into an overall zero-valued objective function $E$ as given below:

$$E = w_1 F_{\Delta} + w_2 F_R + w_3 F_V$$

(42)

where $w_1$, $w_2$, and $w_3$ are the weighting factors assigned to highlight the significance of the various terms in the expression $E$ (Equation 42). Of note is that the significance of each zero-value
function in expressions 41 and the objective function, $E$, can be decided by the designer, based on specific applications of the limaçon-to-circular machine, by assigning an appropriate value for the corresponding weighting factor.

![Graphs showing the effect of factors $a$ and $b$ on radial clearance, $\Delta_{\text{min}}$, and volumetric ratio, $R_{\text{vol}}$.](image)

Figure 4: The effect of factors $a$ and $b$ on radial clearance, $\Delta_{\text{min}}$, and volumetric ratio, $R_{\text{vol}}$.

Applying the SPSA approach to this specific problem, at iteration $k$ the updated values $Y_{k+1}$ can be expressed as:

$$Y_{k+1} = Y_k - \Psi_k \frac{E(Y_k + \omega_k D_k) - E(Y_k - \omega_k D_k)}{2\omega_k} D_k$$

(43)

To make sure that an entry number $j$, $Y_{k+1}^j$ of $Y_{k+1}$, does not fall outside its allowable domain, an adjustment process has been introduced as suggested by Kothandaraman et al. [24] to re-assign this entry to its nearest boundary value. The process can be expressed mathematically as follows:

$$Y_{k+1}^j = \begin{cases} Y_{\text{min}}^j & \text{if } Y_{k+1}^j < Y_{\text{min}}^j \\ Y_{\text{max}}^j & \text{if } Y_{k+1}^j > Y_{\text{max}}^j \end{cases}$$

(44)
Numerical illustrations

The design procedure presented above is employed in this section to calculate the limaçon-to-circular machine dimensions (i.e. half rotor length, \( L \), designing factor, \( a \), aspect ratio, \( b \), and clearance, \( C_L \)) which will correspond to the desired performance characteristics (\( \Delta_{\text{min}} \), \( R_{\text{vol}} \), and \( V_{\text{ind}} \)). A measured constrain impose in all designs is \( b \in (0, 0.25] \).

In order to validate the design model presented in this paper, case studies of the limaçon-to-circular machine in different operating conditions have been established and fed to the optimisation process.

**Case study 1.** This case study features a limaçon-to-circular compressor with \( H = hL \), where \( h = 1.5 \), \( \mu = 5 \), \( \theta_{\text{cut off}} = \pi \), \( \Delta_{\text{min req}} \geq 0.2 \, \text{mm} \), \( R_{\text{vol req}} \leq 0.03 \), and \( V_{\text{ind req}} \geq 1.056 \times 10^7 \, \text{mm}^3 \). The above data has been fed to the design and optimisation procedure to obtain the required results: half chord length \( L = 222.688 \, \text{mm} \), designing factor \( a = 1.052 \), aspect ratio \( b = 0.09 \), housing-rotor clearance \( C_L = 9.385 \times 10^{-3} \), induced volume \( V_{\text{ind}} = 1.032 \times 10^7 \, \text{mm}^3 \), volume metric ratio \( R_{\text{vol}} = 0.027 \), and minimum radial clearance \( \Delta_{\text{min}} = 0.395 \, \text{mm} \).

Figure 5 shows how the value of the objective function steadily diminished during the iterative procedure.

![The reduction of the objective function](image)

**Figure 5:** The reduction of the objective function \( E \) against the iterations.

**Case study 2.** This case study features a limaçon-to-circular expander with \( H = hL \), where \( h = 1.5 \), \( \mu = \frac{1}{6} \), \( \theta_{\text{cut off}} = \frac{5\pi}{6} \), \( \Delta_{\text{min req}} \geq 0.2 \, \text{mm} \), \( R_{\text{vol req}} \leq 0.03 \), and \( V_{\text{ind req}} \geq 1.056 \times 10^7 \, \text{mm}^3 \). The above data has been fed to the design and optimisation procedure to obtain the required results: half chord length \( L = 194.777 \, \text{mm} \), designing factor \( a = 1.142 \), aspect ratio \( b = 0.127 \), housing-rotor clearance \( C_L = 2.87 \times 10^{-3} \), induced volume \( V_{\text{ind}} = 0.9848 \times 10^7 \, \text{mm}^3 \), volume metric ratio \( R_{\text{vol}} = 0.024 \), and minimum radial clearance \( \Delta_{\text{min}} = 0.793 \, \text{mm} \).

**Case study 3.** This case study features a limaçon-to-circular pump with \( H = hL \), where \( h = 1.112 \), \( \mu = 1.15 \), \( \theta_{\text{cut off}} = \pi \), \( \Delta_{\text{min req}} \geq 0.2 \, \text{mm} \), \( R_{\text{vol req}} \leq 0.03 \), and \( V_{\text{ind req}} \geq 1.056 \times 10^7 \, \text{mm}^3 \). The above
data has been fed to the design and optimisation procedure to obtain the required results: half chord length \( L = 204.169 \, mm \), designing factor \( a = 1.149 \), aspect ratio \( b = 0.141 \), housing-rotor clearance \( C_L = 3.674 \times 10^{-3} \), induced volume \( V_{ind} = 1.245 \times 10^7 \, mm^3 \), volume metric ratio \( R_{vol} = 0.024 \), and minimum radial clearance \( \Delta_{min} = 0.592 \, mm \).

<table>
<thead>
<tr>
<th>Results obtained for case studies</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta_{min, req} \geq 0.2 , mm, \ R_{vol, req} \leq 0.03, ) ( and \ V_{ind, req} \geq 1.056 \times 10^7 , mm^3 )</td>
</tr>
<tr>
<td>Case 1: Compressor</td>
</tr>
<tr>
<td>( h )</td>
</tr>
<tr>
<td>( \mu )</td>
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<tr>
<td>( \theta_{cut, off}(rad) )</td>
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<tr>
<td>( L(mm) )</td>
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<tr>
<td>( a )</td>
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<td>( b )</td>
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<tr>
<td>( C_L )</td>
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<tr>
<td>( V_{ind}(mm^3) )</td>
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<tr>
<td>( R_{vol} )</td>
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<tr>
<td>( \Delta_{min}(mm) )</td>
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</tbody>
</table>

### Conclusion

This paper presented a background on the rotary positive displacement machines, which are based on the limaçon technology. A modification has been proposed for the rotor design with the aim of simplifying the manufacturing process and reducing the production cost. This proposed modification results in new housing-rotor interactions, which require an in-depth geometric modelling in relation to housing-rotor interference and volumetric characteristics. The housing-rotor interference has been discussed using two mathematical approaches: a tangent based technique and a radial clearance based method. Moreover, an optimisation procedure has been employed to calculate the optimum dimensions for the machine to meet certain operational requirements. The outcome of the study confirms the validity of the proposed modification and the suitability to describe the geometric characteristics of the new machine. This was highlighted by the three case studies presented at the end of the paper. The next important work on this limaçon-to-circular machine will feature the inclusion of the thermodynamical behaviour of the working fluid and the effects of various geometric parameters on the machine performance.

### Nomenclature

- \( A_h \) Cross-sectional area of the housing
- \( A_{net} \) Area between the rotor and limaçon housing
Cross-sectional area of the rotor

The limaçon ratio \( \frac{r}{L} \)

Housing-rotor clearance

The ratio \( \frac{C}{L} \)

Radial clearance between the rotor and housing

Minimum radial clearance introduced by the designer

Housing and rotor depth

Half rotor chord length

Half the rotor chord length with clearance \( C \)

Clearance volume factor

Points on the rotor chord

Radius of the limaçon base circle

Radius of the rotor lobe

Radial distance from pole \( o \) to the limaçon housing

Volumetric ratio

Average intake density

Average discharge density

Angle between \( X_r \) and the radius of the rotor lobe

Angle between \( X_r \) and the radius of curvature at the contact point on the housing

Sliding distance measured along \( X_r \)

Trilateral Flash Cycle

Correspondent tangent to the limaçon housing

Tangent to the rotor at the apex

Angle rotated by the rotor, measured from \( X_r \) to \( X \)

Induced volume

The stationary coordinates located at the pole \( o \)

The moving coordinates attached to the rotor at mid point \( m \)

Radial distance from pole \( o \) to the rotor lobe

References


