Necessary and Sufficient Conditions for Stable Conjugate Duality*

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Abstract

The conjugate duality, which states that $\inf_{x \in X} \phi(x, 0) = \max_{v \in Y'} -\phi^*(0, v)$, whenever a regularity condition on ϕ is satisfied, is a key result in convex analysis and optimization, where $\phi : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ is a convex function, X and Y are Banach spaces, Y' is the continuous dual space of Y and ϕ^* is the Fenchel-Moreau conjugate of ϕ . In this paper, we establish a necessary and sufficient condition for the stable conjugate duality,

$$\inf_{x \in X} \{\phi(x, 0) + x^*(x)\} = \max_{v \in Y'} \{-\phi^*(-x^*, v)\}, \quad \forall x^* \in X',$$

and obtain a new global dual regularity condition, which is much more general than the popularly known interior-point type conditions, for the conjugate duality. As a consequence we present an epigraph closure condition which is necessary and sufficient for a stable Fenchel-Rockafellar duality theorem. In the case where one of the functions involved in the duality is a polyhedral convex function, we also provide generalized interior-point conditions for the epigraph closure condition. Moreover, we show that a stable Fenchel's duality for sublinear functions holds whenever a subdifferential sum formula for the functions holds. As applications, we give general sufficient conditions for a minimax theorem, a subdifferential composition formula and for duality results of convex programming problems.

Keywords: conjugate duality, constraint qualifications, convex programming, polyhedral functions, sublinear functions.

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1 Introduction

A fundamental duality scheme for studying the convex minimization problem

$$\min f(x), \quad x \in X, \quad (1.1)$$

where $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$, is by a *representation function* $\phi : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ such that $\phi(x, 0) = f(x)$. In which case the dual problem associated to (1.1) is given by

$$\max -\phi^*(0, v), \quad v \in Y', \quad (1.2)$$

where X and Y are Banach spaces with the duals X' and Y' respectively, and ϕ^* is the Fenchel-Moreau conjugate of ϕ . A conjugate duality states that

$$\inf_{x \in X} \phi(x, 0) = \max_{v \in Y'} -\phi^*(0, v) \quad (1.3)$$

whenever a regularity condition on ϕ is satisfied. In other words, under the regularity condition, there is no duality gap and the dual problem has a solution. The conjugate duality, which is a key to the study of convex optimization, constrained best approximation and interpolation (see [7, 8, 9]), enables, for instance, one to find the optimal solution by solving the corresponding dual optimization problem. A central question in convex analysis and optimization is to find a general regularity condition for the conjugate duality. From the point of view of applications, it is vital to find conditions on ϕ , which characterize the stable conjugate duality:

$$\inf_{x \in X} \{\phi(x, 0) + x^*(x)\} = \max_{v \in Y'} \{-\phi^*(-x^*, v)\}, \quad \forall x^* \in X'.$$

Various (primal) regularity conditions for the duality have been given in the literature (see [14, 20] and other reference therein). However, these regularity conditions are either (global) interior-point type conditions [1, 16, 18] which frequently restrict applications or are based on local conditions. In recent years, it has been shown in [5, 6, 4] that Fenchel's duality holds under a dual epigraph condition, which is strictly weaker than the usual interior-point conditions [1, 12], and the bounded linear regularity condition [2], which have played important roles in convex analysis and optimization.

The purpose of this paper is to establish a necessary and sufficient condition for the stable conjugate duality and derive new global dual regularity conditions, which are much more general than the popularly known interior-point type conditions, for conjugate duality results. We also present an epigraph closure condition that is necessary and sufficient for a stable Fenchel-Rockafellar duality theorem. In the case where one of the functions involved in the duality is a polyhedral convex function, we provide generalized interior-point conditions for the epigraph closure condition. Moreover, we show that a stable Fenchel duality for sublinear functions holds whenever a subdifferential sum formula for the functions holds. As applications, we give general sufficient conditions for a minimax theorem, a subdifferential composition formula and for duality results of convex programming problems involving polyhedral constraints.

2 Preliminaries: Epigraphs of Conjugate Functions

We begin by fixing some definitions and notations. We assume throughout that X and Y are Banach spaces. The continuous dual space of X will be denoted by X' and will be endowed with the weak* topology. For the set $D \subset X$ the **closure** of D will be denoted $\text{cl } D$. If a set $A \subset X'$, the expression $\text{cl } A$ will stand for the weak* closure. The **indicator function** δ_D is defined as $\delta_D(x) = 0$ if $x \in D$ and $\delta_D(x) = +\infty$ if $x \notin D$. The **support function** σ_D is defined by $\sigma_D(u) = \sup_{x \in D} u(x)$. The **normal cone** of D is given by $N_D(x) := \{v \in X' : \sigma_D(v) = v(x)\} = \{v \in X' : v(y - x) \leq 0, \forall y \in D\}$ when $x \in D$, and $N_D(x) := \emptyset$ when $x \notin D$. Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. Then, the **conjugate** function of f , $f^* : X' \rightarrow \mathbb{R} \cup \{+\infty\}$, is defined by

$$f^*(v) = \sup\{v(x) - f(x) \mid x \in \text{dom } f\}$$

where the *domain* of f , $\text{dom } f$, is given by $\text{dom } f = \{x \in X \mid f(x) < +\infty\}$. The *epigraph* of f , $\text{Epi } f$, is defined by

$$\text{Epi } f = \{(x, r) \in X \times \mathbb{R} \mid x \in \text{dom } f, f(x) \leq r\}.$$

The *subdifferential* of f , $\partial f : X \rightrightarrows X'$ is defined as

$$\partial f(x) = \{v \in X' \mid f(y) \geq f(x) + v(y - x), \forall y \in X\}.$$

Note also that $\partial \delta_D = N_D$. If $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper lower semi-continuous *sublinear* function, i.e., f is convex and positively homogeneous ($f(0) = 0$, and $f(\lambda x) = \lambda f(x), \forall x \in X, \forall \lambda \in (0, \infty)$), then $\partial f(0)$ is non-empty and for each $x \in \text{dom } f$,

$$\partial f(x) = \{v \in \partial f(0) \mid v(x) = f(x)\}.$$

Applying the well-known Moreau-Rockafellar Theorem (see e.g. [18, Theorem 3.2]) we obtain the conclusion in the lemma below.

Lemma 2.1 [18] *Let X be a Banach space. Let $f, g : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous convex functions such that $\text{dom } f \cap \text{dom } g \neq \emptyset$. Then $\text{Epi } (f + g)^* = \text{cl } (\text{Epi } f^* + \text{Epi } g^*)$.*

If both f and g are proper lower semi-continuous and sublinear functions then it easily follows from Lemma 2.1 that

$$\partial(f + g)(0) = \text{cl } (\partial f(0) + \partial g(0)),$$

For the details see [20].

3 Stable Conjugate Duality

In this section we establish characterizations of the conjugate duality and derive a global dual condition for conjugate duality. We begin by establishing a necessary and sufficient condition for the conjugate duality.

Proposition 3.1 *Let $\phi : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper and lower semi-continuous convex function. Let $\alpha := \inf_{x \in X} \phi(x, 0)$ be finite. Then the following statements are equivalent.*

- (i) $\inf_{x \in X} \phi(x, 0) = \max_{v \in Y'} -\phi^*(0, v)$.
- (ii) $(0, 0, -\alpha) \in \text{Epi } \phi^* + (\{0\} \times Y' \times \mathbb{R}_+)$.

Proof. [(i) $\implies$ (ii)] By assumption (i), there exists $v \in Y'$ such that $-\phi^*(0, v) = \alpha \in \mathbb{R}$. Since $(0, v, \phi^*(0, v)) \in \text{Epi } \phi^*$ and $(0, -v, 0) \in \{0\} \times Y' \times \mathbb{R}_+$ we can write

$$(0, 0, -\alpha) = (0, 0, \phi^*(0, v)) = (0, v, \phi^*(0, v)) + (0, -v, 0) \in \text{Epi } \phi^* + (\{0\} \times Y' \times \mathbb{R}_+).$$

[(ii) $\implies$ (i)] From (ii), there exist $(u, v, \delta) \in \text{Epi } \phi^*$, $w \in Y'$ and $\varepsilon \in \mathbb{R}_+$ such that

$$(0, 0, -\alpha) = (u, v, \delta) + (0, w, \varepsilon) = (u, v + w, \delta + \varepsilon).$$

Then $u = 0, v = -w$ and $\delta + \varepsilon + \alpha = 0$. Now, by Fenchel's inequality, for each $x \in X$ and $v' \in Y'$, $\phi(x, 0) + \phi^*(0, v') \geq 0$. Using this for $v' := v$ we get

$$\begin{aligned} \inf_{x \in X} \phi(x, 0) &\geq -\phi^*(0, v) \\ &\geq -\delta \\ &= \varepsilon + \alpha \\ &\geq \alpha = \inf_{x \in X} \phi(x, 0). \end{aligned}$$

Thus, $\inf_{x \in X} \phi(x, 0) = \max_{y' \in Y'} -\phi^*(0, y') = -\phi^*(0, v)$, and hence (i) holds. $\square$

We now establish a necessary and sufficient condition for the stable conjugate duality.

Theorem 3.1 (Stable Conjugate Duality) *Let $\phi : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper and lower semi-continuous convex function. Suppose that $\alpha := \inf_{x \in X} \{\phi(x, 0)\} < +\infty$. Then the following statements are equivalent.*

- (i) $\inf_{x \in X} \{\phi(x, 0) + x^*(x)\} = \max_{v \in Y'} \{-\phi^*(-x^*, v)\}, \quad \forall x^* \in X'.$
- (ii) $\text{Epi } \phi^* + (\{0\} \times Y' \times \mathbb{R}_+)$ is weak* closed.

Proof. [(i) $\implies$ (ii)]. Let $M = \{(x, y) \in X \times Y \mid y = 0\}$. Then, $\text{Epi } \delta_M^* = \{0\} \times Y' \times \mathbb{R}_+$. Since $\alpha = \inf_{x \in X} \phi(x, 0) < +\infty$ we have that $\text{dom } \phi \cap M \neq \emptyset$. So we can apply Lemma 2.1 to write

$$\text{cl}(\text{Epi } \phi^* + \{0\} \times Y' \times \mathbb{R}_+) = \text{cl}(\text{Epi } \phi^* + \text{Epi } \delta_M^*) = \text{Epi } (\phi + \delta_M)^*.$$

Fix now an element $(x^*, y^*, r) \in \text{cl}(\text{Epi } \phi^* + \{0\} \times Y' \times \mathbb{R}_+)$. By the above equality we have that $(\phi + \delta_M)^*(x^*, y^*) \leq r$. So,

$$r \geq (\phi + \delta_M)^*(x^*, y^*) \geq -\inf_{x \in X} \{\phi(x, 0) - x^*(x)\}.$$

By assumption (i) we get

$$\inf_{x \in X} \{\phi(x, 0) - x^*(x)\} = \max_{v \in Y'} \{-\phi^*(x^*, v)\}.$$

Hence,

$$r \geq (\phi + \delta_M)^*(x^*, y^*) \geq \phi^*(x^*, v),$$

for some $v \in Y'$. This implies that $(x^*, v, r) \in \text{Epi } \phi^*$. Therefore, $(x^*, y^*, r) = (x^*, v, r) + (0, y^* - v, 0) \in \text{Epi } \phi^* + \{0\} \times Y' \times \mathbb{R}_+$ and (ii) holds. [(ii) $\implies$ (i)]. Note that the assumption $\alpha = \inf_{x \in X} \phi(x, 0) < +\infty$ yields $\alpha(x^*) := \inf_{x \in X} \phi(x, 0) + x^*(x) < +\infty$ for all $x^* \in X'$. If $x^* \in X'$ is such that $\alpha(x^*) = -\infty$, then Fenchel-Young inequality yields the equality in (i) for this choice of x^* . Now assume that $x^* \in X'$ is such that $\alpha(x^*) > -\infty$, so our basic assumption yields $\alpha(x^*) \in \mathbb{R}$. We can write

$$\alpha(x^*) = \inf_{x \in X} \phi(x, 0) + x^*(x) = -(\phi + \delta_M)^*(-x^*, 0) \in \mathbb{R}. \quad (3.4)$$

Note also that $\alpha(0) = \inf_{x \in X} \phi(x, 0) < +\infty$ and hence $M \cap \text{dom } \phi \neq \emptyset$, which gives

$$\begin{aligned} (-x^*, 0, -\alpha(x^*)) &\in \text{Epi } (\phi + \delta_M)^* = \text{cl}(\text{Epi } \phi^* + \{0\} \times Y' \times \mathbb{R}_+) \\ &= \text{Epi } \phi^* + \{0\} \times Y' \times \mathbb{R}_+, \end{aligned}$$

where we used (3.4) in the inclusion, Lemma 2.1 in the first equality and assumption (ii) in the last one. The above inclusion implies that $(-x^*, 0, -\alpha(x^*)) = (u, v, \gamma) + (0, w, r)$, with $\phi^*(u, v) \leq \gamma$ and $r \geq 0$. Therefore,

$$\alpha(x^*) = -\gamma - r \leq -\gamma \leq -\phi^*(-x^*, v) \leq \max_{z \in Y'} -\phi^*(-x^*, z) \leq \alpha(x^*),$$

where we used Fenchel-Young inequality in the last inequality. The above expression yields the equality in condition (i) for this choice of x^* . $\square$

It is worth noting that Theorem 3.1 gives a *necessary and sufficient condition* for the equality

$$\phi(\cdot, 0)^*(-x^*) = \min_{v \in Y'} \phi^*(x^*, v), \quad \forall x^* \in X',$$

which is equivalent to the equality (i) of Theorem 3.1. For numerous sufficient conditions for this equality, see Theorem 2.7.1 and Corollary 2.7.3 of [20].

Corollary 3.1 (*Generalized Conjugate Duality*) *Let $\phi : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper and lower semi-continuous convex function such that $\alpha := \inf_{x \in X} \phi(x, 0) < +\infty$. If $\text{Epi } \phi^* + (\{0\} \times Y' \times \mathbb{R}_+)$ is weak* closed then*

$$\inf_{x \in X} \phi(x, 0) = \max_{v \in Y'} -\phi^*(0, v).$$

Proof. The conclusion follows from Theorem 3.1 by taking $x^* = 0$. $\square$

Let us see how the epigraph closure condition in the conjugate duality results can be expressed in the particular case where ϕ is described in terms of the sum of two convex functions. Define the continuous linear projection map, $P_{X' \times \mathbb{R}} : X' \times Y' \times \mathbb{R} \rightarrow X' \times \mathbb{R}$, by $P_{X' \times \mathbb{R}}(u, v, r) = (u, r)$. The inverse of the projection map is denoted by $P_{X' \times \mathbb{R}}^{-1}$. Recall that for a continuous linear map $A : X \rightarrow Y$ the *adjoint operator* of A , denoted by A^* , is the unique linear application from Y' to X' with

the property $(A^*v)(x) = v(Ax)$ for all $v \in Y', x \in X$. Associated to the linear mapping A , we consider the mapping $A^* \times I : Y' \times \mathbb{R} \rightarrow X' \times \mathbb{R}$ defined as $(A^* \times I)(w, \alpha) = (A^*w, \alpha)$. Note that the image of $\text{Epi } g^*$ through this application is $(A^* \times I)(\text{Epi } g^*) = \{(A^*w, \alpha) | (w, \alpha) \in \text{Epi } g^*\} = \{(A^*w, \alpha) | g^*(w) \leq \alpha\}$. Let $(\text{Epi } g^*)_A := (A^* \times I)(\text{Epi } g^*)$.

Lemma 3.1 *Let X and Y be Banach spaces and take $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ two proper functions. Let $A : X \rightarrow Y$ be a continuous linear mapping. If $\phi : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ is given by $\phi(x, y) := f(x) + g(Ax + y)$, then*

$$\text{Epi } \phi^* + (\{0\} \times Y' \times \mathbb{R}_+) = P_{X' \times \mathbb{R}}^{-1}(\text{Epi } f^* + (\text{Epi } g^*)_A)$$

Proof. Call $E := P_{X' \times \mathbb{R}}^{-1}(\text{Epi } f^* + (\text{Epi } g^*)_A)$ and $F := \text{Epi } \phi^* + (\{0\} \times Y' \times \mathbb{R}_+)$. Let $(u, v, r) \in E$. Then, $(u, r) \in \text{Epi } f^* + (\text{Epi } g^*)_A$. So, there exist $(u_1, r_1) \in \text{Epi } f^*$ and $(u_2, r_2) \in \text{Epi } g^*$ such that

$$(u, r) = (u_1 + A^*u_2, r_1 + r_2).$$

Since $\phi^*(u_1, u_2) = f^*(u_1 - A^*u_2) + g^*(u_2)$, $\phi^*(u_1 + A^*u_2, u_2) = f^*(u_1) + g^*(u_2) \leq r_1 + r_2$, and so, $(u_1 + A^*u_2, u_2, r_1 + r_2) \in \text{Epi } \phi^*$. Hence,

$$(u, v, r) = (u_1 + A^*u_2, u_2, r_1 + r_2) + (0, v - u_2, 0) \in F;$$

thus, $E \subset F$.

Conversely, let $(u, v, r) \in F$. Then, $(u, v, r) = (u_1, u_2, r_1) + (0, v, r_2)$, where $\phi^*(u_1, u_2) = f^*(u_1 - A^*u_2) + g^*(u_2) \leq r_1$, $v \in Y'$ and $r_2 \in \mathbb{R}_+$. So, $(u_1 - A^*u_2, r_1 - g^*(u_2)) \in \text{Epi } f^*$ and $(u_2, g^*(u_2) + r_2) \in \text{Epi } g^*$. Hence,

$$(u, r) = (u_1 - A^*u_2 + A^*u_2, r_1 - g^*(u_2) + g^*(u_2) + r_2) \in (\text{Epi } f^* + (\text{Epi } g^*)_A);$$

thus, $F \subset E$. □

Note that if $\phi(x, y) := f(x) + g(x + y)$, then

$$\text{Epi } \phi^* + (\{0\} \times Y' \times \mathbb{R}_+) = P_{X' \times \mathbb{R}}^{-1}(\text{Epi } f^* + \text{Epi } g^*).$$

Remark 3.1 *Let $C \subset X' \times \mathbb{R}$ be an arbitrary set. Then it is easy to check that the set $P_{X' \times \mathbb{R}}^{-1}(C)$ can be identified with $C \times Y'$. Therefore, C is weak* closed if and only if $P_{X' \times \mathbb{R}}^{-1}(C)$ is weak* closed.*

Theorem 3.2 (Stable Fenchel-Rockafellar Duality) *Let $A : X \rightarrow Y$ be a continuous linear mapping. Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper and lower semi-continuous convex functions such that $A(\text{dom } f) \cap \text{dom } g \neq \emptyset$. Then the following statements are equivalent:*

$$(i) \inf_{x \in X} \{f(x) + g(Ax) + x^*(x)\} = \max_{v \in X'} \{-f^*(A^*v - x^*) - g^*(-v)\}, \forall x^* \in X'.$$

(ii) $\text{Epi } f^* + (\text{Epi } g^*)_A$ is weak* closed.

Proof. Define $\phi : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ by $\phi(x, y) := f(x) + g(Ax + y)$ for each $(x, y) \in X \times Y$. Then, $\phi^*(u, v) = f^*(u - A^*v) + g^*(v)$. So that condition (i) becomes

$$\inf_{x \in X} \phi(x, 0) + x^*(x) = \max_{v \in Y'} \{-\phi^*(-x^*, -v)\} \quad \forall x^* \in X'. \quad (3.5)$$

Since $A(\text{dom } f) \cap \text{dom } g \neq \emptyset$, there exists $y_0 \in \text{dom } g$ such that $y_0 = Ax_0$ with $x_0 \in \text{dom } f$. Hence, $\inf_{x \in X} \phi(x, 0) \leq f(x_0) + g(Ax_0) < +\infty$. So we can apply Theorem 3.1, and conclude that (3.5) is equivalent to the weak* closedness of $\text{Epi } \phi^* + \{0\} \times Y' \times \mathbb{R}_+$. Now the result follows from Lemma 3.1 and Remark 3.1. $\square$

Observe that, in the case where $X = Y$ and $A = I$, Theorem 3.2 shows that $\text{Epi } f^* + \text{Epi } g^*$ is weak* closed if and only if

$$(f + g)^*(x^*) = \min\{f^*(u) + g^*(v) \mid u + v = x^*\}, \quad \forall x^* \in X',$$

whenever $\text{dom } f \cap \text{dom } g \neq \emptyset$. For related results, see [5, 12, 19].

Corollary 3.2 (Stable Duality for Convex Programs) *Let $A : X \rightarrow Y$ be a continuous linear mapping, $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper and lower semi-continuous convex function. Let $C \subset \text{dom } f$ be a closed convex set, $b \in Y$ and let $K \subset Y$ be a closed convex cone. Assume that $A(C) \cap (K + b) \neq \emptyset$. Then the following statements are equivalent.*

$$(i) \quad \inf_{\substack{x \in C \\ Ax - b \in K}} \{f(x) + x^*(x)\} = \max_{v \in K^0} -(f + \delta_C)^*(A^*v - x^*) + v(b), \quad \forall x^* \in X'$$

$$(ii) \quad \text{Epi } (f + \delta_C)^* + (\text{Epi } \delta_{b+K}^*)_A \text{ is weak* closed,}$$

where $K^0 = \{v \in Y' \mid v(y) \leq 0, \forall y \in K\}$ is the polar cone of K .

Proof. By Theorem 3.2, applied to the functions $f + \delta_C$ and $g := \delta_{b+K}$, we readily see the equivalence between (i) and (ii). $\square$

Corollary 3.3 (Generalized Fenchel-Rockafellar Duality) *Let $A : X \rightarrow Y$ be a continuous linear mapping. Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper and lower semi-continuous convex functions such that $A(\text{dom } f) \cap \text{dom } g \neq \emptyset$. If the set $\text{Epi } f^* + (\text{Epi } g^*)_A$ is weak* closed, then*

$$\inf_{x \in X} \{f(x) + g(Ax)\} = \max_{v \in Y'} \{-f^*(A^*v) - g^*(-v)\}. \quad (3.6)$$

Proof. The conclusion follows from Theorem 3.2 by taking $x^* = 0$. $\square$

When the functions f and g are sublinear, we have that stable Fenchel duality is equivalent to the subdifferential sum formula.

Corollary 3.4 (Subdifferential Sum Formula) *Let $f, g : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous sublinear functions with $\text{dom } f \cap \text{dom } g \neq \emptyset$. Then the following statements are equivalent:*

$$(i) \quad \inf_{x \in X} \{f(x) + g(x) + x^*(x)\} = \max_{v \in X'} \{-f^*(v - x^*) - g^*(-v)\}, \quad \forall x^* \in X'.$$

$$(ii) \quad \partial(f + g)(x) = \partial f(x) + \partial g(x), \quad \forall x \in \text{dom } f \cap \text{dom } g.$$

(iii) $\text{Epi } f^* + \text{Epi } g^*$ is weak* closed.

Proof. [(i) $\implies$ (ii)]. This implication is well known and holds for arbitrary proper convex and lower semicontinuous functions, see for instance Theorem 2.1 of [13]. [(ii) $\implies$ (iii)]. If (ii) holds then $\partial(f + g)(0) = \partial f(0) + \partial g(0)$. Since f and g are proper lower semi-continuous sublinear functions, $\partial(f + g)(0) = \text{cl}(\partial f(0) + \partial g(0))$. So, $\text{cl}(\partial f(0) + \partial g(0)) = \partial f(0) + \partial g(0)$; thus, $\partial f(0) + \partial g(0)$ is weak*closed. Hence,

$$\text{Epi } f^* + \text{Epi } g^* = \partial f(0) \times \mathbb{R}_+ + \partial g(0) \times \mathbb{R}_+ = (\partial f(0) + \partial g(0)) \times \mathbb{R}_+$$

is weak*closed. [(iii) $\implies$ (i)]. This follows from Theorem 3.2, where $X = Y$ and $A = I$. $\square$

Corollary 3.5 (Subdifferential Composition Formula) *Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper and lower semi-continuous convex functions such that $A(\text{dom } f) \cap \text{dom } g \neq \emptyset$. Let $A : X \rightarrow Y$ be a continuous linear mapping. If the set $\text{Epi } f^* + (\text{Epi } g^*)_A$ is weak* closed then for each $x_0 \in X$,*

$$\partial(f + (g \circ A))(x_0) = \partial f(x_0) + A^* \partial g(Ax_0).$$

Proof. Let $v \in \partial(f + (g \circ A))(x_0)$. Then, for each $x \in X$, $(f + g \circ A)(x) - (f + (g \circ A))(x_0) \geq v(x - x_0)$, and so, it follows from Corollary 3.3 that there exists $y_0 \in Y'$ such that

$$\begin{aligned} (f - v)(x_0) + (g \circ A)(x_0) &= \inf\{(f - v)(x) + g(Ax) \mid x \in X\} \\ &= -(f - v)^*(A^*y_0) - g^*(-y_0). \end{aligned}$$

Now, it is easy to show that $A^*y_0 \in \partial(f - v)(x_0) = \partial f(x_0) - \{v\}$, and $-y_0 \in \partial g(Ax_0)$. Indeed, the above equality yields

$$\begin{aligned} -(f - v)(x_0) - (g \circ A)(x_0) &= (f - v)^*(A^*y_0) + g^*(-y_0) \\ &\geq [(A^*y_0)(x) - (f - v)(x)] + [(-y_0)(y) - g(y)], \end{aligned}$$

For all $x \in X$ and $y \in Y$. Taking $y = Ax_0$, we see that $A^*y_0 \in \partial(f - v)(x_0) = \partial f(x_0) - \{v\}$, and choosing $x = x_0$, we obtain $-y_0 \in \partial g(Ax_0)$. Thus, $v \in \partial f(x_0) + A^* \partial g(Ax_0)$. Hence, $\partial(f + (g \circ A))(x_0) \subset \partial f(x_0) + A^* \partial g(Ax_0)$. The required equality now follows as the reverse inclusion easily holds. $\square$

We now show how a stable minimax theorem can be derived from the stable Fenchel-Rockafellar conjugate duality theorem. For related general stable minimax theorems see [10, 11] and other references therein.

Corollary 3.6 (Stable Minimax Theorem) *Let $A : X \rightarrow Y$ be a continuous linear mapping. Let C and D be closed convex subsets of X and Y' respectively. Suppose that $A(C) \cap \text{dom } \sigma_D \neq \emptyset$. Then the set $\text{Epi } \sigma_C + A^*(D) \times \mathbb{R}_+$ is weak* closed if and only if*

$$\inf_{x \in C} \sup_{v \in D} v(Ax) + x^*(x) = \max_{v \in D} \inf_{x \in C} v(Ax) + x^*(x), \quad \forall x^* \in X'.$$

Proof. Let $f = \delta_C$ and $g = \delta_D^* = \sigma_D$ in Theorem 3.2. Then $A(\text{dom } f) \cap \text{dom } g = A(C) \cap \text{dom } \sigma_D \neq \emptyset$, and it is easy to check that $(\text{Epi } \delta_D)_A = A^*(D) \times \mathbb{R}_+$. Now, it follows from Theorem 3.2 that $\text{Epi } \sigma_C + A^*(D) \times \mathbb{R}_+$ is weak* closed if and only if

$$\inf_{x \in X} \{\delta_C(x) + \sigma_D(Ax) + x^*(x)\} = \max_{u \in Y'} \{-\sigma_C(A^*u - x^*) - \delta_D(-u)\},$$

which is equivalent to the equalities

$$\begin{aligned} \inf_{x \in C} \sup_{v \in D} v(Ax) + x^*(x) &= \max_{-u \in D} \{-\sigma_C(A^*u - x^*)\} \\ &= \max_{-u \in D} \inf_{x \in C} [x^*(x) + (A^*(-u))(x)] \\ &= \max_{v \in D} \inf_{x \in C} (x^* + A^*v)(x) \\ &= \max_{v \in D} \inf_{x \in C} v(Ax) + x^*(x). \end{aligned}$$

□

4 Duality and Polyhedral Convex Functions

This section studies the case in which one of the convex functions is a polyhedral convex function, i.e., a function which has as epigraph a polyhedral convex set. Recall that, for $f, g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ convex and lower-semicontinuous functions, the classical Fenchel's Duality Theorem [17, Theorem 31.1] states that one has

$$\inf_{x \in X} \{f(x) + g(x)\} = \max_{v \in X'} \{-f^*(v) - g^*(-v)\}, \quad (4.7)$$

when $\text{ri}(\text{dom } f) \cap \text{ri}(\text{dom } g) \neq \emptyset$. However, if one of these functions, say g , is polyhedral, the latter condition can be weakened to $\text{ri}(\text{dom } f) \cap \text{dom } g \neq \emptyset$. Moreover, when both functions are polyhedral, then the relative interiors can be replaced by the domains of the functions. This fundamental result has been recently extended to arbitrary Banach spaces in [16, Theorem 3.3], where the concept of relative interior has been replaced by the *strong quasi relative interior* (see [14, 15, 20]), which is denoted by $\text{sqri}(C)$, and defined as

$$\text{sqri}(C) = \{x \in C \mid \text{cone}(C - x) \text{ is a closed subspace}\}.$$

Note that if the set C is contained in a finite dimensional space, then $\text{ri}(C) = \text{sqri}(C)$.

In the case in which both functions f and g are lower semi-continuous polyhedral convex functions on $\mathbb{R}^n$, then it is well known that both sets $\text{Epi } f^*$ and $\text{Epi } g^*$ are closed polyhedrons, and their sum $\text{Epi } f^* + \text{Epi } g^*$, being also a polyhedral, is also closed.

We now show that, when g is polyhedral, our closure condition holds whenever $\text{sqri}(\text{dom } f) \cap \text{dom } g \neq \emptyset$. This condition is used in [16, Theorem 3.3], which is stated below.

Theorem 4.1 *Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be convex proper and lower semicontinuous functions, and let $A : X \rightarrow Y$ be a linear and continuous operator. Suppose that $\text{sqri}(A(\text{dom } f)) \cap \text{dom } g \neq \emptyset$ and that g is polyhedral. Then*

$$\inf_{x \in X} \{f(x) + g(Ax)\} = \max_{v \in X'} \{-f^*(A^*v) - g^*(-v)\}. \quad (4.8)$$

Theorem 4.2 *Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be convex proper and lower semicontinuous functions, and let $A : X \rightarrow Y$ be a linear and continuous operator. Suppose that g is polyhedral and that $\text{dom } g \cap \text{sqli}(A(\text{dom } f)) \neq \emptyset$. Then $\text{Epi } f^* + (\text{Epi } g^*)_A$ is weak* closed.*

Proof. Let $x^* \in X'$ and define $\hat{f}(x) := f(x) + x^*(x)$. Then, $\hat{f}$ is also a proper convex function and $\text{dom } \hat{f} = \text{dom } f$, so that we still have $\text{dom } g \cap \text{sqli}(A(\text{dom } \hat{f})) \neq \emptyset$. Moreover, for each $v \in X'$, $\hat{f}^*(v) = f^*(v - x^*)$. By Theorem 4.1, we know that if g is polyhedral and $\text{dom } g \cap \text{sqli}(A(\text{dom } \hat{f})) \neq \emptyset$, then

$$\inf_{x \in X} (\hat{f}(x) + g(Ax)) = \max_{v \in X^*} (-\hat{f}^*(A^*v) - g^*(-v)).$$

Re-writing the expression above we get

$$\inf_{x \in X} (f(x) + g(Ax) + x^*(x)) = \max_{v \in X^*} (-f^*(A^*v - x^*) - g^*(-v)).$$

Having shown the above equality for arbitrary $x^* \in X'$, Theorem 3.2 applies and we can conclude that $\text{Epi } f^* + (\text{Epi } g^*)_A$ is weak* closed. $\square$

A corollary of the above result is an application to convex programs with polyhedral constraints.

Corollary 4.1 (*Duality for Convex Programs with Polyhedral Constraints*)

Let $A : X \rightarrow Y$ be a continuous linear mapping, $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper and lower semi-continuous convex function. Let C be a closed convex subset of $\text{dom } f$, $b \in Y$ and let $K \subset Y$ be a polyhedral cone. Assume that $A(C) \cap (K + b) \neq \emptyset$. Under one of the following conditions:

- (i) $Y = \mathbb{R}^n$ and $\text{ri}(A(C)) \cap (b + K) \neq \emptyset$,
- (ii) $\text{sqli}(A(C)) \cap (b + K) \neq \emptyset$,
- (iii) $\text{Epi } (f + \delta_C)^* + (\text{Epi } \delta_{b+K}^*)_A$ is weak* closed,

one has

$$\inf \{f(x) \mid x \in C, Ax - b \in K\} = \max \{-(f + \delta_C)^*(A^*v) + v(b) \mid v \in K^0\}. \quad (4.9)$$

Proof. By Corollary 3.2, we have that condition (iii) above implies (4.9), which is condition (i) of Corollary 3.2 for the choice $x^* = 0$. Using now Theorem 4.2 applied to the functions $f + \delta_C$ and $g := \delta_{b+K}$, we have that both (ii) and its finite dimensional version (i) are stronger than (iii). $\square$

References

- [1] H. Attouch and H. Brézis, Duality for the sum of convex functions in general Banach spaces, in *Aspects of Mathematics and its applications*, J. A. Barroso (ed.), North Holland, Amsterdam, (1986), 125-133.

- [2] H. H. Bauschke, J. M. Borwein and W. Li, Strong conical hull intersection property, bounded linear regularity, Jameson's property (G), and error bounds in convex optimization, *Math. Progr.*, 86 (1999), 135-160.
- [3] J. M. Borwein and A. Lewis, Partially finite convex programming, Part I: Quasi-relative interiors and duality theory, *Math. Progr.*, 57 (1992), 15 - 48.
- [4] R. S. Burachik and V. Jeyakumar, A simple closure condition for the normal cone intersection formula, Applied Mathematics Research Report AMR04, University of New South Wales, Sydney, Australia. To appear in *Proc. Amer. Math. Soc.* (2005).
- [5] R. S. Burachik and V. Jeyakumar, A new geometric condition for Fenchel's duality in infinite dimensions, To appear in *Math. Progr., Series B*, 2005.
- [6] R. S. Burachik and V. Jeyakumar, A dual condition for the convex subdifferential sum formula with applications. To appear in *J. Convex Analysis*, 12(2) (2005).
- [7] F. Deutsch, *Best approximation in inner product spaces*, Springer-Verlag, New York, 2000.
- [8] F. Deutsch, W. Li and J. Swetits, Fenchel duality and the strong conical hull intersection property, *J. Optim. Theory Appl.*, 102 (1999), 681-695.
- [9] F. Deutsch, W. Li and J. D. Ward, Best approximation from the intersection of a closed convex set and a polyhedron in Hilbert space, weak Slater conditions, and the strong conical hull intersection property, *SIAM J. Optim.*, 10 (1999), 252-268.
- [10] J. Gwinner and V. Jeyakumar, Stable minimax on noncompact sets. Fixed point theory and applications (Marseille, 1989), Pitman Res. Notes Math. Ser., 252, Longman Sci. Tech., Harlow, 215-220, 1991.
- [11] J. Gwinner and J-C. Pomerol, On weak* closedness, coerciveness, and inf-sup theorems, *Arch. Math.*, 52(2) (1989), 159-167.
- [12] J.-B. Hiriart-Urruty, ϵ -subdifferential calculus, in *Convex Analysis and Optimization*, Pitman, Boston, 43-92, 1982.
- [13] J.-B. Hiriart-Urruty and R. R. Phelps, Subdifferential Calculus using ϵ -subdifferentials, *J. of Funct. Anal.*, **118** (1993), 154-166.
- [14] V. Jeyakumar, Duality and infinite dimensional optimization, *Nonlinear Anal.*, 15 (1990), 1111-1122.
- [15] V. Jeyakumar and H. Wolkowicz, Generalizations of Slater's constraint qualification for infinite convex programs, *Math. Progr.*, 57(1) (1992), 85-102.
- [16] K. F. Ng and W. Song, Fenchel duality in infinite-dimensional setting and its applications, *Nonlinear Analysis* 55(7-8) (2003), 845-858.

- [17] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.
- [18] T. Strömberg, The operation of infimal convolution, *Diss. Math.*, 352 (1996), 1-61.
- [19] C. Zalinescu, Stability for a class of nonlinear optimization problems and applications, in *Nonsmooth Optimization and Related Topics*, F. H. Clarke, V. F. Demyanov and F. Giannessi (eds.), Plenum Press, New York, 1989, 437-458.
- [20] C. Zalinescu, *Convex analysis in general vector spaces*, World Scientific, London, 2002.