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A NEW PROOF OF BALINSKI'S THEOREM ON THE CONNECTIVITY OF POLYTOPES

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ABSTRACT. Balinski (1961) proved that the graph of a d -dimensional convex polytope is d -connected. We provide a new proof of this result. Our proof provides details on the nature of a separating set with exactly d vertices; some of which appear to be new.

1. INTRODUCTION

A (convex) polytope is the convex hull of a finite set X of points in $\mathbb{R}^d$; the *convex hull* of X is the smallest convex set containing X . The *dimension* of a polytope in $\mathbb{R}^d$ is one less than the maximum number of affinely independent points in the polytope; a set of points $\vec{p}_1, \dots, \vec{p}_k$ in $\mathbb{R}^d$ is *affinely independent* if the $k - 1$ vectors $\vec{p}_1 - \vec{p}_k, \dots, \vec{p}_{k-1} - \vec{p}_k$ are linearly independent. A polytope of dimension d is referred to as a d -polytope.

A polytope is structured around other polytopes, its faces. A *face* of a polytope P in $\mathbb{R}^d$ is P itself, or the intersection of P with a hyperplane in $\mathbb{R}^d$ that contains P in one of its closed halfspaces. A face of dimension 0, 1, and $d - 1$ in a d -polytope is a *vertex*, an *edge*, and a *facet*, respectively. The set of vertices and edges of a polytope or a graph are denoted by V and E , respectively. The *graph* $G(P)$ of a polytope P is the abstract graph with vertex set $V(P)$ and edge set $E(P)$.

A graph with at least $d + 1$ vertices is d -connected if removing any $d - 1$ vertices leaves a connected subgraph. Balinski (1961) showed that the graph of a d -polytope is d -connected. His proof considers a hyperplane in $\mathbb{R}^d$ passing through a set of $d - 1$ vertices of a d -polytope, and so do the proofs of Grünbaum (2003, Thm. 11.3.2), Ziegler (1995, Thm. 3.14), and Brøndsted (1983, Thm. 15.6). Such proofs yield a geometric structure of separators in the graph of the polytope (Lemma 1). A set X of vertices in a graph G *separates* two vertices x, y if every path in G between x and y contains an element of X , and $x, y \notin X$. And X *separates* G if it separates

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FIGURE 1. The link of a vertex in the four-dimensional cube, the convex hull of the 2^4 vectors $(\pm 1, \pm 1, \pm 1, \pm 1)$ in $\mathbb{R}^4$. (a) The four-dimensional cube with a vertex x highlighted. (b) The link of the vertex x in the cube. (c) The link of the vertex x as the boundary complex of the rhombic dodecahedron (Proposition 2).

two vertices of G . A separating set of vertices is a *separator* and a separator of cardinality r is an *r -separator*.

Lemma 1. *Let P be a d -polytope in $\mathbb{R}^d$ and let H be a hyperplane in $\mathbb{R}^d$. If X is a proper subset of $H \cap V(P)$, then removing X does not disconnect $G(P)$. In particular, a separator of $G(P)$ with exactly d vertices must form an affinely independent set in $\mathbb{R}^d$.*

Other proofs with a geometric flavour were given by Brøndsted & Maxwell (1989) and Barnette (1995). Our proof has a more combinatorial nature, relying on certain polytopal complexes in a polytope. Another combinatorial proof, based on a different idea, can be found in Barnette (1973).

The *boundary complex* of a polytope P is the set of faces of P other than P itself. And the *link* of a vertex x in P , denoted $\text{lk}(x)$, is the set of faces of P that do not contain x but lie in a facet of P that contains x (Fig. 1(b)). We require a result from Ziegler (1995).

Proposition 2 (Ziegler 1995, Ex. 8.6). *Let P be a d -polytope. Then the link of a vertex in P is combinatorially isomorphic to the boundary complex of a $(d-1)$ -polytope. In particular, for each $d \geq 3$, the graph of the link of a vertex is isomorphic to the graph of a $(d-1)$ -polytope.*

We proved Proposition 2 in Bui et al. (2018, Prop. 12) and exemplified it in Fig. 1. In this paper, we prove the following. The part about links appears to be new.

Theorem 3. *For $d \geq 1$, the graph of d -polytope P is d -connected. Besides, for each $d \geq 3$, each vertex x in a d -separator X of $G(P)$ lies in the link of every other vertex of X , and the set $X \setminus \{x\}$ is a separator of the link of x .*

As a corollary, we get a known result on d -separators in simplicial d -polytopes (Goodman et al., 2017, p. 509); see Corollary 4. A polytope is *simplicial* if all its facets are simplices, and a d -simplex is a d -polytope whose $d + 1$ vertices form an affinely independent set in $\mathbb{R}^d$. An *empty* $(d - 1)$ -simplex in a d -polytope P is a set of d vertices of P that does not form a face of P but every proper subset does. An empty $(d - 1)$ -simplex is also called a *missing* $(d - 1)$ -simplex.

Corollary 4. *Let P be a simplicial d -polytope with $d \geq 2$. A d -separator of $G(P)$ forms an empty $(d - 1)$ -simplex of P .*

We remark that the paragraph after Balinski's theorem in Goodman et al. (2017, p. 509) is meant to concern only simplicial d -polytopes, and not d -polytopes in general. While it is true that a d -separator of the graph of a d -polytope must form an affinely independent set in $\mathbb{R}^d$, it is not true that it must form an empty simplex. Take, for instance, the neighbours of a vertex in a d -dimensional cube (Fig. 1(a)).

We follow Diestel (2017) for the graph theoretical terminology that we have not defined.

2. PROOFS OF THEOREM 3 AND COROLLARY 4

A path between vertices x and y in a graph is an $x - y$ path, and two $x - y$ paths are *independent* if they share no inner vertex. For a path $L := x_0 \dots x_n$ and for $0 \leq i \leq j \leq n$, we write $x_i L x_j$ to denote the subpath $x_i \dots x_j$. We require a theorem of Whitney (1932) and one of Menger (1927).

Theorem 5 (Whitney 1932). *Let G be a graph with at least one pair of nonadjacent vertices. Then there is a minimum separator of G disconnecting two nonadjacent vertices.*

Theorem 6 (Menger 1927). *Let G be a graph, and let x and y be two nonadjacent vertices. Then the minimum number of vertices separating x from y in G equals the maximum number of independent $x - y$ paths in G .*

Proof of Theorem 3. Let P be a d -polytope and let G be its graph. Then G has at least $d + 1$ vertices. If G is a complete graph, there is nothing to prove, and suppose otherwise. In this case, G has at least one pair of nonadjacent vertices. For $d = 2$, G is d -connected. And so induct on d , assuming that $d \geq 3$ and that the theorem is true for $d - 1$. Let X be a separator in G of minimum cardinality, and let y and z be vertices separated by X . Then $y, z \notin X$. According to Whitney's theorem (Theorem 5), there is a minimum separator of G disconnecting two nonadjacent vertices. Hence we may assume that y and z are nonadjacent, and by Menger's theorem (Theorem 6), that there are $|X|$ independent $y - z$ paths in G , each containing precisely one vertex from X . Let L be one such $y - z$ paths and let x be the vertex in $X \cap V(L)$; say that $L = u_1 \dots u_m$ such that $y = u_1$, $u_j = x$, and $u_m = z$.

The graph G_x of the link of x in P is isomorphic to the graph of a $(d-1)$ -polytope (Proposition 2), and by the induction hypothesis it is $(d-1)$ -connected. The neighbours of x are all part of $\text{lk}(x)$, and so $u_{j-1}, u_{j+1} \in G_x$. Again, from Menger's theorem follows the existence of at least $d-1$ independent $u_{j-1} - u_{j+1}$ paths in G_x . We must have that $X \setminus \{x\}$ separates u_{j-1} from u_{j+1} in G_x , since X separates y from z . Hence $|X \setminus \{x\}| \geq d-1$, which establishes that G is d -connected.

Finally, let $d \geq 3$ and suppose X is a d -separator of G . As stated above, the set $X \setminus \{x\}$, of cardinality $d-1$, separates G_x , implying that $X \setminus \{x\} \subseteq V(G_x)$. The aforementioned path L was arbitrary among the $y - z$ paths separated by X , and each such path contains a unique vertex of X . It follows that every vertex in X is in the link of every other vertex of X , which concludes the proof of the theorem. $\square$

Proof of Corollary 4. Let P be a simplicial d -polytope and let G be its graph. Suppose that X is a d -separator of G , that x is a vertex of X , and that G_x is the graph of the link of x in P .

A simplicial 2-polytope is a polygon and a 2-separator in it satisfies the corollary. So assume that $d \geq 3$. From Theorem 3, it follows that every vertex in X is in the link of every other vertex of X , and that $X \setminus \{x\}$ is a $(d-1)$ -separator of G_x . Consequently, the subgraph $G[X]$ of G induced by X is a complete graph, as the set of neighbours of each vertex in P coincides with the vertex set of the link of the vertex.

If $d = 3$, then, from $G[X]$ being a complete graph, it follows that it is an empty 2-simplex. And so an inductive argument on d can start. Assume that $d \geq 4$. From the definition of a link and Proposition 2, we obtain that $\text{lk}(x)$ is combinatorially isomorphic to the boundary complex of a simplicial $(d-1)$ -polytope.

By the induction hypothesis on $\text{lk}(x)$, every proper subset of $X \setminus \{x\}$ forms a face F of $\text{lk}(x)$. And from the definition of $\text{lk}(x)$, that face F lies in a facet of P containing x , a $(d-1)$ -simplex containing x . As a consequence, if F is a face of dimension k , then the set $\text{conv}(F \cup \{x\})$ is a face of P of dimension $k+1$. Since the vertex x of X was taken arbitrarily, the corollary ensues. $\square$

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