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# Transversality Properties: Primal Sufficient Conditions

Nguyen Duy Cuong · Alexander Y. Kruger

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**Abstract** The paper studies ‘good arrangements’ (transversality properties) of collections of sets in a normed vector space near a given point in their intersection. We target primal (metric and slope) characterizations of transversality properties in the nonlinear setting. The Hölder case is given a special attention. Our main objective is not formally extending our earlier results from the Hölder to a more general nonlinear setting, but rather to develop a general framework for quantitative analysis of transversality properties. The nonlinearity is just a simple setting, which allows us to unify the existing results on the topic. Unlike the well-studied subtransversality property, not many characterizations of the other two important properties: semitransversality and transversality have been known even in the linear case. Quantitative relations between nonlinear transversality properties and the corresponding regularity properties of set-valued mappings as well as nonlinear extensions of the new transversality properties of a set-valued mapping to a set in the range space due to Ioffe are also discussed.

**Keywords** Transversality · Subtransversality · Semitransversality · Regularity · Subregularity · Semiregularity · Slope · Chain Rule

**Mathematics Subject Classification (2010)** Primary 49J52 · 49J53 · Secondary 49K40 · 90C30 · 90C46

## 1 Introduction

This paper continues a series of publications by the authors [10, 14–16, 30–32, 36–44, 57] dedicated to studying ‘good arrangements’ of collections of sets in normed spaces near a point in their intersection. Following Ioffe [28], such arrangements are now commonly referred to as *transversality* properties. Here we refer to transversality broadly as a group

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of ‘good arrangement’ properties, which includes *semitransversality*, *subtransversality*, *transversality* (a specific property) and some others. The term *regularity* was extensively used for the same purpose in the earlier publications by the second author, and is still preferred by many authors.

Transversality (regularity) properties of collections of sets play an important role in optimization and variational analysis, e.g., as constraint qualifications, qualification conditions in subdifferential, normal cone and coderivative calculus, and convergence analysis of computational algorithms. Significant efforts have been invested into studying this class of properties and establishing their primal and dual necessary and/or sufficient characterizations in various settings (convex and nonconvex, finite and infinite dimensional, finite and infinite collections of sets). In addition to the references provided above, we refer the readers to [3–5, 7, 8, 19, 20, 24, 25, 46, 50, 51, 53, 55, 59, 60] for results and historical comments.

Our aim is to develop a general framework for quantitative analysis of transversality properties of collections of sets. In this paper, we focus on primal space conditions, and establish metric characterizations and slope-type sufficient conditions for three closely related general nonlinear transversality properties:  $\varphi$ -*semitransversality*,  $\varphi$ -*subtransversality* and  $\varphi$ -*transversality*.

The slope sufficient conditions stem from applying the Ekeland variational principle to the definitions of the respective properties; the proofs are rather straightforward. This type of conditions are often considered as just a first step on the way to producing more involved dual (subdifferential and normal cone) conditions, and the primal sufficient conditions remain hidden in the proofs. We believe that primal conditions (being in a sense analogues of very popular slope conditions for error bounds) can be of importance for applications. Moreover, subdividing the conventional regularity/transversality theory into the primal and dual parts clarifies the roles of the main tools employed within the theory: the Ekeland variational principle in the primal part and subdifferential sum rules in the dual part. As a result, the proofs in both the primal and the ‘more involved’ dual parts become straightforward. This observation goes beyond the transversality of collections of sets and applies also to the regularity of set-valued mappings and the error bound theory.

Unlike the earlier publications, here, besides estimates for the transversality moduli, we provide also quantitative estimates for the parameters  $\delta$ ’s involved in the definitions; cf. Definitions 1.1 and 2.1. This can be of importance from the computational point of view. We also examine quantitative relations between the nonlinear transversality properties of collections of sets and the corresponding nonlinear regularity properties of set-valued mappings as well as nonlinear extensions of the new *transversality properties of a set-valued mapping to a set in the range space* due to Ioffe.

We would like to emphasize that our main objective is not formally extending our earlier results from the Hölder to a more general nonlinear setting, but rather to develop a comprehensive theory of transversality. The nonlinearity is just a simple setting, which allows us to unify the existing (and hopefully also future) results on the topic. In fact, unlike the subtransversality property which has been well studied in the linear and Hölder settings (see, for instance, [3, 5, 19, 24, 25, 39, 42, 50, 59]), for the other two properties: semitransversality and transversality not many characterizations have been known even in the linear case; we fill this gap in the current paper.

Besides the conventional Hölder case, which is given a special attention in the paper, our general model covers also so called *Hölder-type* settings [6, 47] that have recently come into play in the closely related error bound theory due to their importance for applications. Such nonlinear settings of transversality properties have not been studied before. Some characterizations are new even in the linear setting.

Apart from being of interest on their own, the slope sufficient conditions for nonlinear transversality properties established in this paper lay the foundation for the dual sufficient

conditions for the respective properties in Banach and Asplund spaces in [14]. Primal and dual necessary conditions for the nonlinear transversality properties are studied in [15, 16].

There exist strong connections between transversality properties of collections of sets and the corresponding regularity properties of set-valued mappings. In this paper, we establish quantitative relations between the two models in the general nonlinear setting. Nonlinear regularity properties of set-valued mappings and closely related error bound properties of (extended-)real-valued functions have been intensively studied since 1980s; cf. [1, 2, 9, 13, 21–23, 26, 34, 35, 42, 45, 48, 54, 58, 61]. The slope sufficient conditions for  $\varphi$ -subtransversality in Section 4 can be interpreted in terms of the corresponding conditions for nonlinear error bounds. The semitransversality and transversality properties do not have exact counterparts within the conventional error bound theory.

As in most of our previous publications on the topic, our working model in this paper is a collection of  $n \geq 2$  arbitrary subsets  $\Omega_1, \dots, \Omega_n$  of a normed vector space  $X$ , having a common point  $\bar{x} \in \cap_{i=1}^n \Omega_i$ . The next definition introduces three most common Hölder transversality properties. It is a modification of [42, Definition 1].

**Definition 1.1** Let  $\alpha > 0$  and  $q > 0$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is

(i)  $\alpha$ -semitransversal of order  $q$  at  $\bar{x}$  if there exists a  $\delta > 0$  such that

$$\bigcap_{i=1}^n (\Omega_i - x_i) \cap B_\rho(\bar{x}) \neq \emptyset \quad (1)$$

for all  $\rho \in ]0, \delta[$  and  $x_i \in X$  ( $i = 1, \dots, n$ ) with  $\max_{1 \leq i \leq n} \|x_i\|^q < \alpha\rho$ ;

(ii)  $\alpha$ -subtransversal of order  $q$  at  $\bar{x}$  if there exist  $\delta_1 > 0$  and  $\delta_2 > 0$  such that

$$\bigcap_{i=1}^n \Omega_i \cap B_\rho(x) \neq \emptyset \quad (2)$$

for all  $\rho \in ]0, \delta_1[$  and  $x \in B_{\delta_2}(\bar{x})$  with  $\max_{1 \leq i \leq n} d^q(x, \Omega_i) < \alpha\rho$ ;

(iii)  $\alpha$ -transversal of order  $q$  at  $\bar{x}$  if there exist  $\delta_1 > 0$  and  $\delta_2 > 0$  such that

$$\bigcap_{i=1}^n (\Omega_i - \omega_i - x_i) \cap (\rho\mathbb{B}) \neq \emptyset \quad (3)$$

for all  $\rho \in ]0, \delta_1[$ ,  $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  and  $x_i \in X$  ( $i = 1, \dots, n$ ) with  $\max_{1 \leq i \leq n} \|x_i\|^q < \alpha\rho$ .

The three properties in the above definition were referred to in [42] as  $[q]$ -semi-regularity,  $[q]$ -subregularity and  $[q]$ -regularity, respectively. Property (ii) was defined in [42] in a slightly different but equivalent way, under an additional assumption that  $q \leq 1$ . When  $\cap_{i=1}^n \Omega_i$  is closed and  $\bar{x} \in \text{bd} \cap_{i=1}^n \Omega_i$ , the condition  $q \leq 1$  is indeed necessary for the  $\alpha$ -subtransversality and  $\alpha$ -transversality properties; see Remark 2.3. At the same time, as observed in [42], the property of  $\alpha$ -semitransversality can be meaningful with any positive  $q$  (and any positive  $\alpha$ ); see Example 2.1.

With  $q = 1$  (linear case), properties (i) and (iii) in Definition 1.1 were discussed in [31] (see also [32, Properties (R)<sub>S</sub> and (UR)<sub>S</sub>]), while property (ii) first appeared in [43]. If  $\cap_{i=1}^n \Omega_i$  is closed and  $\bar{x} \in \text{bd} \cap_{i=1}^n \Omega_i$ , then one can observe that properties (ii) and (iii) can only hold with  $\alpha \leq 1$ ; see Remark 2.3. If  $q = 1$ , when referring to the three properties in the above definition, we talk simply about  $\alpha$ -(semi-/sub-) transversality.

If a collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\alpha$ -semitransversal (respectively,  $\alpha$ -subtransversal or  $\alpha$ -transversal) of order  $q$  at  $\bar{x}$  with some  $\alpha > 0$  and  $\delta > 0$  (or  $\delta_1 > 0$  and  $\delta_2 > 0$ ), we often simply say that  $\{\Omega_1, \dots, \Omega_n\}$  is *semitransversal* (respectively, *subtransversal* or *transversal*) of order  $q$  at  $\bar{x}$ . The number  $\alpha$  characterizes the corresponding property quantitatively. The exact upper bound of all  $\alpha > 0$  such that the property holds with some

$\delta > 0$  (or  $\delta_1 > 0$  and  $\delta_2 > 0$ ) is called the *modulus* of this property. We use the notations  $\text{se}\text{tr}_q[\Omega_1, \dots, \Omega_n](\bar{x})$ ,  $\text{str}_q[\Omega_1, \dots, \Omega_n](\bar{x})$  and  $\text{tr}_q[\Omega_1, \dots, \Omega_n](\bar{x})$  for the moduli of the respective properties. If the property does not hold, then by convention the respective modulus equals 0.

If  $q < 1$ , the Hölder transversality properties in Definition 1.1 are obviously weaker than the corresponding conventional linear properties and can be satisfied for collections of sets when the conventional ones fail. This can happen in many natural situations (see examples in [42, Section 2.3]), which explains the growing interest of researchers to studying the more subtle nonlinear transversality properties.

Our basic notation is standard, see, e.g., [18, 49, 56]. Throughout the paper,  $X$  and  $Y$  are either metric or, more often, normed vector spaces. The open unit ball in any space is denoted by  $\mathbb{B}$ , and  $B_\delta(x)$  stands for the open ball with center  $x$  and radius  $\delta > 0$ . If not explicitly stated otherwise, products of normed vector spaces are assumed to be equipped with the maximum norm  $\|(x, y)\| := \max\{\|x\|, \|y\|\}$ ,  $(x, y) \in X \times Y$ . The symbols  $\mathbb{R}$  and  $\mathbb{R}_+$  denote the real line (with the usual norm) and the set of all nonnegative real numbers, respectively.

Given a set  $\Omega$ , its interior and boundary are denoted by  $\text{int}\Omega$  and  $\text{bd}\Omega$ , respectively. The distance from a point  $x$  to  $\Omega$  is defined by  $d(x, \Omega) := \inf_{u \in \Omega} \|u - x\|$ , and we use the convention  $d(x, \emptyset) = +\infty$ . The indicator function of  $\Omega$  is defined as follows:  $i_\Omega(x) = 0$  if  $x \in \Omega$  and  $i_\Omega(x) = +\infty$  if  $x \notin \Omega$ .

For an extended-real-valued function  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ , its domain and epigraph are defined, respectively, by  $\text{dom}f := \{x \in X \mid f(x) < +\infty\}$  and  $\text{epi}f := \{(x, \alpha) \in X \times \mathbb{R} \mid f(x) \leq \alpha\}$ . The inverse of  $f$  (if it exists) is denoted by  $f^{-1}$ . A set-valued mapping  $F : X \rightrightarrows Y$  between two sets  $X$  and  $Y$  is a mapping, which assigns to every  $x \in X$  a subset (possibly empty)  $F(x)$  of  $Y$ . We use the notations  $\text{gph}F := \{(x, y) \in X \times Y \mid y \in F(x)\}$  and  $\text{dom}F := \{x \in X \mid F(x) \neq \emptyset\}$  for the graph and the domain of  $F$ , respectively, and  $F^{-1} : Y \rightrightarrows X$  for the inverse of  $F$ . This inverse (which always exists with possibly empty values at some  $y$ ) is defined by  $F^{-1}(y) := \{x \in X \mid y \in F(x)\}$ ,  $y \in Y$ . Obviously  $\text{dom}F^{-1} = F(X)$ .

The closed and open intervals between points  $x_1$  and  $x_2$  in a normed space are defined, respectively, by

$$[x_1, x_2] := \{tx_1 + (1-t)x_2 \mid t \in [0, 1]\}, \quad ]x_1, x_2[ := \{tx_1 + (1-t)x_2 \mid t \in ]0, 1[ \}.$$

The semi-open intervals  $]x_1, x_2]$  and  $[x_1, x_2[$  are defined in a similar way.

The key tool in the proofs of the main results is the celebrated Ekeland variational principle; cf. [18, 28, 49, 55].

**Lemma 1.1** *Suppose  $X$  is a complete metric space,  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$  is lower semicontinuous,  $x \in X$ ,  $\varepsilon > 0$  and  $\lambda > 0$ . If*

$$f(x) < \inf_X f + \varepsilon,$$

*then there exists an  $\hat{x} \in X$  such that*

- (i)  $d(\hat{x}, x) < \lambda$ ;
- (ii)  $f(\hat{x}) \leq f(x)$ ;
- (iii)  $f(u) + (\varepsilon/\lambda)d(u, \hat{x}) \geq f(\hat{x})$  for all  $u \in X$ .

The *slope* [17] and *nonlocal slope* [33, 52] of a function  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$  on a metric space at  $x \in \text{dom}f$  are defined, respectively, by

$$|\nabla f|(x) := \limsup_{u \rightarrow x, u \neq x} \frac{[f(x) - f(u)]_+}{d(x, u)}, \quad |\nabla f|^\circ(x) := \sup_{u \neq x} \frac{[f(x) - f_+(u)]_+}{d(x, u)},$$

where  $\alpha_+ := \max\{0, \alpha\}$  for any  $\alpha \in \mathbb{R}$ . The limit  $|\nabla f|(x)$  provides the rate of steepest descent of  $f$  at  $x$ . If  $X$  is a normed space, and  $f$  is Fréchet differentiable at  $x$ , then  $|\nabla f|(x) = \|f'(x)\|$ . When  $x \notin \text{dom } f$ , we set  $|\nabla f|(x) = |\nabla f|^\circ(x) := +\infty$ . The next proposition is straightforward.

**Proposition 1.1** *Suppose  $X$  is a metric space,  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ , and  $x \in X$ .*

- (i) *If  $f$  is not lower semicontinuous at  $x$ , then  $|\nabla f|(x) = +\infty$ .*
- (ii) *If  $f(x) > 0$ , then  $|\nabla f|(x) \leq |\nabla f|^\circ(x)$ .*

When proving primal and dual characterizations of transversality properties in the nonlinear setting we use chain rules for slopes and subdifferentials, respectively. The next lemma provides a chain rule for slopes, which is used in Section 3. For its subdifferential counterparts we refer the reader to [14, Proposition 2.1].

**Lemma 1.2** *Let  $X$  be a metric space,  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ ,  $\varphi : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ ,  $x \in \text{dom } f$  and  $f(x) \in \text{dom } \varphi$ . Suppose  $\varphi$  is nondecreasing on  $\mathbb{R}$  and differentiable at  $f(x)$  with  $\varphi'(f(x)) > 0$ . Then  $|\nabla(\varphi \circ f)|(x) = \varphi'(f(x))|\nabla f|(x)$ .*

*Proof* If  $x$  is a local minimum of  $f$ , then, thanks to the monotonicity of  $\varphi$ , it is also a local minimum of  $\varphi \circ f$ , and consequently,  $|\nabla(\varphi \circ f)|(x) = |\nabla f|(x) = 0$ . Suppose  $x$  is not a local minimum of  $f$ . If  $f$  is not lower semicontinuous at  $x$ , i.e.  $\alpha := \lim_{k \rightarrow +\infty} f(x_k) < f(x)$  for some sequence  $x_k \rightarrow x$ , then, in view of the assumptions,  $\varphi$  is strictly increasing near  $f(x)$ , and consequently,  $\liminf_{k \rightarrow +\infty} \varphi(f(x_k)) \leq \varphi(\alpha) < \varphi(f(x))$  (with the convention that  $\varphi(-\infty) = -\infty$ , i.e.  $\varphi \circ f$  is not lower semicontinuous at  $x$ ; hence, in view of Proposition 1.1(i),  $|\nabla(\varphi \circ f)|(x) = |\nabla f|(x) = +\infty$ . Suppose  $f$  is lower semicontinuous at  $x$ , i.e.  $\liminf_{u \rightarrow x, u \neq x} f(u) = f(x)$ . Then, taking into account that  $x$  is not a local minimum of  $f$ ,

$$\begin{aligned} |\nabla(\varphi \circ f)|(x) &= \limsup_{u \rightarrow x, u \neq x} \frac{\varphi(f(x)) - \varphi(f(u))}{d(u, x)} \\ &= \limsup_{\substack{u \rightarrow x, u \neq x \\ f(u) < f(x)}} \frac{\varphi(f(x)) - \varphi(f(u))}{d(u, x)} = \limsup_{\substack{u \rightarrow x, u \neq x \\ f(u) \uparrow f(x)}} \frac{\varphi(f(x)) - \varphi(f(u))}{d(u, x)} \\ &= \limsup_{\substack{u \rightarrow x, u \neq x \\ f(u) \uparrow f(x)}} \left( \frac{\varphi(f(x)) - \varphi(f(u))}{f(x) - f(u)} \cdot \frac{f(x) - f(u)}{d(u, x)} \right) \\ &= \varphi'(f(x)) \limsup_{u \rightarrow x, u \neq x} \frac{f(x) - f(u)}{d(u, x)} = \varphi'(f(x))|\nabla f|(x). \end{aligned}$$

The proof is complete.  $\square$

- Remark 1.1* (i) The slope chain rule in Lemma 1.2 is a local result. Instead of assuming that  $\varphi$  is defined on the whole real line, one can assume that  $\varphi$  is defined and finite on a closed interval  $[\alpha, \beta]$  around the point  $f(x)$ :  $\alpha < f(x) < \beta$ . It is sufficient to define the composition  $\varphi \circ f$  for  $x$  with  $f(x) \notin [\alpha, \beta]$  as follows:  $(\varphi \circ f)(x) := \varphi(\alpha)$  if  $f(x) < \alpha$ , and  $(\varphi \circ f)(x) := \varphi(\beta)$  if  $f(x) > \beta$ . This does not affect the conclusion of the lemma.
- (ii) Lemma 1.2 slightly improves [2, Lemma 4.1], where  $f$  and  $\varphi$  are assumed lower semicontinuous and continuously differentiable, respectively.

The rest of the paper is organized as follows. In Section 2, we discuss transversality properties of finite collections of sets in the nonlinear setting. In Section 3, we establish metric characterizations of these properties. Section 4 is devoted to slope sufficient conditions for the nonlinear transversality properties. In Section 5, we discuss quantitative

relations between nonlinear transversality of collections of sets and the corresponding nonlinear regularity properties of set-valued mappings, and show that the two popular models are in a sense equivalent in the general nonlinear setting. As a consequence, we improve some results established in [42] in the Hölder setting. We also briefly discuss nonlinear extensions of the new transversality properties of a set-valued mapping to a set in the range space due to Ioffe [28].

## 2 Definitions and Basic Relations

The nonlinearity in the definitions of the transversality properties is determined by a continuous strictly increasing function  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying  $\varphi(0) = 0$  and  $\lim_{t \rightarrow +\infty} \varphi(t) = +\infty$ . The family of all such functions is denoted by  $\mathcal{C}$ . We denote by  $\mathcal{C}^1$  the subfamily of functions from  $\mathcal{C}$  which are differentiable on  $]0, +\infty[$  with  $\varphi'(t) > 0$  for all  $t > 0$ . Obviously, if  $\varphi \in \mathcal{C}$  ( $\varphi \in \mathcal{C}^1$ ), then  $\varphi^{-1} \in \mathcal{C}$  ( $\varphi^{-1} \in \mathcal{C}^1$ ). Observe that, for any  $\alpha > 0$  and  $q > 0$ , the function  $t \mapsto \alpha t^q$  on  $\mathbb{R}_+$  belongs to  $\mathcal{C}^1$ .

*Remark 2.1* For the purposes of the paper, it is sufficient to assume that functions  $\varphi \in \mathcal{C}$  are defined and invertible near 0.

In addition to our standing assumption that  $\Omega_1, \dots, \Omega_n$  are subsets of a normed space  $X$  and  $\bar{x} \in \bigcap_{i=1}^n \Omega_i$ , if not explicitly stated otherwise, we assume from now on that  $\varphi \in \mathcal{C}$ .

**Definition 2.1** The collection  $\{\Omega_1, \dots, \Omega_n\}$  is

- (i)  $\varphi$ -semitransversal at  $\bar{x}$  if there exists a  $\delta > 0$  such that condition (1) is satisfied for all  $\rho \in ]0, \delta[$  and  $x_i \in X$  ( $i = 1, \dots, n$ ) with  $\varphi(\max_{1 \leq i \leq n} \|x_i\|) < \rho$ ;
- (ii)  $\varphi$ -subtransversal at  $\bar{x}$  if there exist  $\delta_1 > 0$  and  $\delta_2 > 0$  such that condition (2) is satisfied for all  $\rho \in ]0, \delta_1[$  and  $x \in B_{\delta_2}(\bar{x})$  with  $\varphi(\max_{1 \leq i \leq n} d(x, \Omega_i)) < \rho$ ;
- (iii)  $\varphi$ -transversal at  $\bar{x}$  if there exist  $\delta_1 > 0$  and  $\delta_2 > 0$  such that condition (3) is satisfied for all  $\rho \in ]0, \delta_1[$ ,  $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  and  $x_i \in X$  ( $i = 1, \dots, n$ ) with  $\varphi(\max_{1 \leq i \leq n} \|x_i\|) < \rho$ .

Observe that conditions (1) and (3) are trivially satisfied when  $x_i = 0$  ( $i = 1, \dots, n$ ). Hence, in parts (i) and (iii) of Definition 2.1 (as well as Definition 1.1) one can additionally assume that  $\max_{1 \leq i \leq n} \|x_i\| > 0$ . Similarly, in part (ii) of Definition 2.1 (as well as Definition 1.1) one can assume that  $x \notin \bigcap_{i=1}^n \Omega_i$ .

Each of the properties in Definition 2.1 is determined by a function  $\varphi \in \mathcal{C}$ , and a number  $\delta > 0$  in item (i) or numbers  $\delta_1 > 0$  and  $\delta_2 > 0$  in items (ii) and (iii). The function plays the role of a kind of rate or modulus of the respective property, while the role of the  $\delta$ 's is more technical: they control the size of the interval for the values of  $\rho$  and, in the case of  $\varphi$ -subtransversality and  $\varphi$ -transversality in parts (ii) and (iii), the size of the neighbourhoods of  $\bar{x}$  involved in the respective definitions. Of course, if a property is satisfied with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , it is satisfied also with the single  $\delta := \min\{\delta_1, \delta_2\}$  in place of both  $\delta_1$  and  $\delta_2$ . Unlike our previous publications on (linear and Hölder) transversality properties, we use in the current paper two different parameters to emphasise their different roles in the definitions and the corresponding characterizations. Moreover, we are going to provide quantitative estimates for the values of these parameters.

Given a  $\delta > 0$  in item (i) ( $\delta_1 > 0$  and  $\delta_2 > 0$  in items (ii) and (iii)), if a property is satisfied for some function  $\varphi \in \mathcal{C}$ , it is obviously satisfied for any function  $\hat{\varphi} \in \mathcal{C}$  such that  $\hat{\varphi}^{-1}(t) \leq \varphi^{-1}(t)$  for all  $t \in ]0, \delta[$  ( $t \in ]0, \delta_1[$ ), or equivalently,  $\hat{\varphi}(t) \geq \varphi(t)$  for all  $t \in ]0, \varphi^{-1}(\delta)[$  ( $t \in ]0, \varphi^{-1}(\delta_1)[$ ). Thus, it makes sense looking for the smallest function in  $\mathcal{C}$  (if it exists) ensuring the corresponding property for the given sets. Observe also that taking a smaller  $\delta > 0$  (smaller  $\delta_1 > 0$  and  $\delta_2 > 0$ ) may allow each of the properties to be satisfied with a smaller  $\varphi$ . When the exact value of  $\delta$  ( $\delta_1$  and  $\delta_2$ ) in the definition of



the respective property is not important, it makes sense to look for the smallest function ensuring the corresponding property for some  $\delta > 0$  ( $\delta_1$  and  $\delta_2$ ).

The most important realization of the three properties in Definition 2.1 corresponds to the Hölder setting, i.e.  $\varphi$  being a power function, given for all  $t \geq 0$  by  $\varphi(t) := \alpha^{-1}t^q$  with some  $\alpha > 0$  and  $q > 0$ . In this case, Definition 2.1 reduces to Definition 1.1.

Another important for applications class of functions is given by the so called *Hölder-type* [6, 47] ones, i.e. functions of the form  $t \mapsto \alpha^{-1}(t^q + t)$ , frequently used in the error bound theory, or more generally, functions  $t \mapsto \alpha^{-1}(t^q + \beta t)$  with some  $\alpha > 0$ ,  $\beta > 0$  and  $q > 0$ . Depending on the value of  $q$ , transversality properties determined by such functions can be approximated by Hölder (if  $q < 1$ ) or even linear (if  $q \geq 1$ ) ones.

**Proposition 2.1** *Let  $\varphi(t) := \alpha^{-1}(t^q + \beta t)$  with some  $\alpha > 0$ ,  $\beta > 0$  and  $q > 0$ . If the collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -(semi-/sub-)transversal at  $\bar{x}$ , then it is  $\alpha'$ -(semi-/sub-)transversal of order  $q'$  at  $\bar{x}$ , where:*

- (i) if  $q < 1$ , then  $q' = q$  and  $\alpha'$  is any number in  $]0, \alpha[$ ;
- (ii) if  $q = 1$ , then  $q' = 1$  and  $\alpha' := \alpha(1 + \beta)^{-1}$ ;
- (iii) if  $q > 1$ , then  $q' = 1$  and  $\alpha'$  is any number in  $]0, \alpha\beta^{-1}[$ .

*Proof* The assertions follow from Definition 1.1 in view of the following observations:

- (i) if  $q < 1$  and  $\alpha' \in ]0, \alpha[$ , then, for all sufficiently small  $t > 0$ , it holds  $\alpha'(1 + \beta t^{1-q}) < \alpha$ , and consequently,  $\varphi(t) = \alpha^{-1}(1 + \beta t^{1-q})t^q < (\alpha')^{-1}t^q$ ;
- (ii) if  $q = 1$  and  $\alpha' = \alpha(1 + \beta)^{-1}$ , then  $\varphi(t) = \alpha^{-1}(1 + \beta)t = (\alpha')^{-1}t$ ;
- (iii) if  $q > 1$  and  $\alpha' \in ]0, \alpha\beta^{-1}[$ , then, for all sufficiently small  $t > 0$ , it holds  $\alpha'(\beta^{-1}t^{q-1} + 1) < \alpha\beta^{-1}$ , and consequently,  $\varphi(t) = \alpha^{-1}\beta(\beta^{-1}t^{q-1} + 1)t < (\alpha')^{-1}t$ .  $\square$

The next two propositions collect some simple facts about the properties in Definition 2.1 and clarify relationships between them.

- Proposition 2.2** (i) *If  $\Omega_1 = \dots = \Omega_n$ , and there exists a  $\delta_1 > 0$  such that  $\varphi(t) \geq t$  for all  $t \in ]0, \delta_1[$ , then  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -subtransversal at  $\bar{x}$  with  $\delta_1$  and any  $\delta_2 > 0$ .*
- (ii) *If  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then it is  $\varphi$ -semitransversal at  $\bar{x}$  with  $\delta_1$  and  $\varphi$ -subtransversal at  $\bar{x}$  with any  $\delta'_1 \in ]0, \delta_1[$  and  $\delta'_2 > 0$  such that  $\varphi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2$ .*
- (iii) *If  $\bar{x} \in \text{int} \cap_{i=1}^n \Omega_i$ , then  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ .*

*Proof* (i) Let  $\Omega := \Omega_1 = \dots = \Omega_n$ . Then condition (2) becomes  $\Omega \cap B_\rho(x) \neq \emptyset$ . This inclusion is trivially satisfied if  $\varphi(d(x, \Omega)) < \rho$  and  $\varphi(\rho) \geq \rho$ .

- (ii) Let  $\{\Omega_1, \dots, \Omega_n\}$  be  $\varphi$ -transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ . Since condition (1) is a particular case of condition (3) with  $\omega_i = \bar{x}$  ( $i = 1, \dots, n$ ), we can conclude that  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -semitransversal at  $\bar{x}$  with  $\delta_1$ . Let  $\delta'_1 \in ]0, \delta_1[$  and  $\delta'_2 > 0$  be such that  $\varphi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2$ , and let  $\rho \in ]0, \delta'_1[$  and  $x \in B_{\delta'_2}(\bar{x})$  with  $\varphi(\max_{1 \leq i \leq n} d(x, \Omega_i)) < \rho$ . Choose  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) such that  $\varphi(\max_{1 \leq i \leq n} \|x - \omega_i\|) < \rho$ . Then, for any  $i = 1, \dots, n$ ,

$$\|\omega_i - \bar{x}\| \leq \|x - \omega_i\| + \|x\| < \varphi^{-1}(\rho) + \delta'_2 < \delta_2.$$

Set  $x_i := x - \omega_i$  ( $i = 1, \dots, n$ ). We have  $\rho \in ]0, \delta_1[$ ,  $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and  $\varphi(\max_{1 \leq i \leq n} \|x_i\|) < \rho$ . By Definition 2.1(iii), condition (3) is satisfied. This is equivalent to condition (2). In view of Definition 2.1(ii),  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -subtransversal at  $\bar{x}$  with  $\delta'_1$  and  $\delta'_2$ .



- (iii) Let  $\bar{x} \in \text{int} \cap_{i=1}^n \Omega_i$ . Choose numbers  $\delta_1 > 0$  and  $\delta_2 > 0$  such that, with  $\delta := \varphi^{-1}(\delta_1) + \delta_2$ , it holds  $B_\delta(\bar{x}) \subset \cap_{i=1}^n \Omega_i$ . Then, for all  $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  and  $x_i \in X$  ( $i = 1, \dots, n$ ) with  $\varphi(\max_{1 \leq i \leq n} \|x_i\|) < \delta_1$ , it holds  $0 \in \cap_{i=1}^n (\Omega_i - \omega_i - x_i)$ , and consequently, condition (3) is satisfied with any  $\rho > 0$ . Hence,  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  with  $\delta_1$  and  $\delta_2$ .  $\square$

- Remark 2.2* (i) The inequality  $\varphi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2$  in Proposition 2.2(ii) and some statements below can obviously be replaced by the equality  $\varphi^{-1}(\delta'_1) + \delta'_2 = \delta_2$  providing in a sense the best estimate for the values of the parameters  $\delta'_1$  and  $\delta'_2$ .  
(ii) In the Hölder setting, parts (i) and (iii) of Proposition 2.2 recapture [42, Remarks 4 and 3], respectively, while part (ii) improves [42, Remark 1].  
(iii) The nonlinear semitransversality and subtransversality properties are in general independent; see examples in [42, Section 2.3] and [43, Section 3.2].

**Proposition 2.3** *Let  $\cap_{i=1}^n \Omega_i$  be closed and  $\bar{x} \in \text{bd} \cap_{i=1}^n \Omega_i$ . If  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -subtransversal (in particular, if it is  $\varphi$ -transversal) at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then there exists a  $\bar{t} \in ]0, \min\{\delta_2, \varphi^{-1}(\delta_1)\}]$  such that  $\varphi(t) \geq t$  for all  $t \in ]0, \bar{t}]$ .*

*Proof* Let  $\{\Omega_1, \dots, \Omega_n\}$  be  $\varphi$ -subtransversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ . Choose a point  $\hat{x} \notin \cap_{i=1}^n \Omega_i$  such that  $\|\hat{x} - \bar{x}\| < \min\{\varphi^{-1}(\delta_1), \delta_2\}$  and set  $\bar{t} := d(\hat{x}, \cap_{i=1}^n \Omega_i)$ . Then  $\bar{t} < \min\{\varphi^{-1}(\delta_1), \delta_2\}$ . Besides,  $\bar{t} > 0$  since  $\cap_{i=1}^n \Omega_i$  is closed. Thanks to the continuity of the function  $d(\cdot, \cap_{i=1}^n \Omega_i)$ , for any  $t \in ]0, \bar{t}]$  there is an  $x \in ]\bar{x}, \hat{x}]$  such that  $d(x, \cap_{i=1}^n \Omega_i) = t$ . We have  $\|x - \bar{x}\| \leq \|\hat{x} - \bar{x}\| < \delta_2$  and  $\varphi(t) \leq \varphi(\bar{t}) < \delta_1$ . Take a  $\rho \in ]\varphi(t), \delta_1[$ . Then  $\varphi(\max_{1 \leq i \leq n} d(x, \Omega_i)) \leq \varphi(t) < \rho$ . By Definition 2.1(ii),  $t = d(x, \cap_{i=1}^n \Omega_i) < \rho$ , and letting  $\rho \downarrow \varphi(t)$ , we arrive at  $t \leq \varphi(t)$ . If  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$ , the conclusion follows in view of Proposition 2.2(ii).  $\square$

*Remark 2.3* The conditions on  $\varphi$  in Proposition 2.3 in the Hölder setting can only be satisfied if either  $q < 1$ , or  $q = 1$  and  $\alpha \leq 1$ . This reflects the well known fact that the Hölder subtransversality and transversality properties are only meaningful when  $q \leq 1$  and, moreover, the linear case ( $q = 1$ ) is only meaningful when  $\alpha \leq 1$ ; cf. [39, p. 705], [36, p. 118]. The extreme case  $q = \alpha = 1$  is in a sense singular for subtransversality as in this case Definition 1.1(ii) yields  $d(x, \cap_{i=1}^n \Omega_i) = \max_{1 \leq i \leq n} d(x, \Omega_i)$  for all  $x$  near  $\bar{x}$ .

In accordance with Proposition 2.3, the  $\varphi$ -subtransversality and  $\varphi$ -transversality properties impose serious restrictions on the function  $\varphi$ . This is not the case with the  $\varphi$ -semitransversality property:  $\varphi$  can be, e.g., any power function.

*Example 2.1* Let  $\mathbb{R}^2$  be equipped with the maximum norm, and let  $q > 0$ ,  $\gamma > 0$ ,  $\Omega_1 := \{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid \gamma^{\frac{1}{q}} \xi_2 + |\xi_1|^{\frac{1}{q}} \geq 0\}$ ,  $\Omega_2 := \{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid \gamma^{\frac{1}{q}} \xi_2 - |\xi_1|^{\frac{1}{q}} \leq 0\}$  and  $\bar{x} := (0, 0)$ . Note that, when  $q > 1$ , the sets  $\Omega_1$  and  $\Omega_2$  are nonconvex. We claim that the pair  $\{\Omega_1, \Omega_2\}$  is  $\varphi$ -semitransversal at  $\bar{x}$  with  $\varphi(t) := \gamma t^q$  ( $t \geq 0$ ).

*Proof* Given an  $r > 0$ , set  $x_1 := (0, -r)$  and  $x_2 := (0, r)$ . Then  $\|x_1\| = \|x_2\| = r$  and  $(\pm \gamma r^q, 0) \in (\Omega_1 - x_1) \cap (\Omega_2 - x_2)$ . Moreover, it is easy to notice that either  $(\gamma r^q, 0)$  or  $(-\gamma r^q, 0)$  belongs to  $(\Omega_1 - x_1) \cap (\Omega_2 - x_2)$  for any choice of vectors  $x_1, x_2 \in \mathbb{R}^2$  with  $\max\{\|x_1\|, \|x_2\|\} \leq r$ . Hence,  $(\Omega_1 - x_1) \cap (\Omega_2 - x_2) \cap B_\rho(\bar{x}) \neq \emptyset$  for all such vectors  $x_1, x_2 \in \mathbb{R}^2$  as long as  $\rho > \gamma r^q$ , and consequently,  $\{\Omega_1, \Omega_2\}$  is  $\varphi$ -semitransversal at  $\bar{x}$ .  $\square$

### 3 Metric Characterizations

The three transversality properties are defined in Definition 2.1 geometrically. We now show that they can be characterized in metric terms. These metric characterizations can be used as equivalent definitions of the respective properties.

**Theorem 3.1** *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is*

(i)  $\varphi$ -semitransversal at  $\bar{x}$  with some  $\delta > 0$  if and only if

$$d\left(\bar{x}, \bigcap_{i=1}^n (\Omega_i - x_i)\right) \leq \varphi\left(\max_{1 \leq i \leq n} \|x_i\|\right) \quad (4)$$

for all  $x_i \in X$  ( $i = 1, \dots, n$ ) with  $\varphi(\max_{1 \leq i \leq n} \|x_i\|) < \delta$ ;

(ii)  $\varphi$ -subtransversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if and only if the following equivalent conditions hold:

(a) for all  $x \in B_{\delta_2}(\bar{x})$  with  $\varphi(\max_{1 \leq i \leq n} d(x, \Omega_i)) < \delta_1$ , it holds

$$d\left(x, \bigcap_{i=1}^n \Omega_i\right) \leq \varphi\left(\max_{1 \leq i \leq n} d(x, \Omega_i)\right); \quad (5)$$

(b) for all  $x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) with  $\varphi(\max_{1 \leq i \leq n} \|x_i\|) < \delta_1$  and  $\omega_1 + x_1 = \dots = \omega_n + x_n \in B_{\delta_2}(\bar{x})$ , it holds

$$d\left(0, \bigcap_{i=1}^n (\Omega_i - \omega_i - x_i)\right) \leq \varphi\left(\max_{1 \leq i \leq n} \|x_i\|\right); \quad (6)$$

(iii)  $\varphi$ -transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if and only if inequality (6) holds for all  $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  and  $x_i \in X$  ( $i = 1, \dots, n$ ) with  $\varphi(\max_{1 \leq i \leq n} \|x_i\|) < \delta_1$ .

*Proof* (i) Let  $\{\Omega_1, \dots, \Omega_n\}$  be  $\varphi$ -semitransversal at  $\bar{x}$  with some  $\delta > 0$ , and let  $x_i \in X$  ( $i = 1, \dots, n$ ) with  $\rho_0 := \varphi(\max_{1 \leq i \leq n} \|x_i\|) < \delta$ . Choose a  $\rho \in ]\rho_0, \delta[$ . By (1),  $d(\bar{x}, \bigcap_{i=1}^n (\Omega_i - x_i)) < \rho$ . Letting  $\rho \downarrow \rho_0$ , we arrive at inequality (4).

Conversely, let  $\delta > 0$  and inequality (4) hold for all  $x_i \in X$  ( $i = 1, \dots, n$ ) with  $\varphi(\max_{1 \leq i \leq n} \|x_i\|) < \delta$ . For all  $\rho \in ]0, \delta[$  and  $x_i \in X$  ( $i = 1, \dots, n$ ) with  $\varphi(\max_{1 \leq i \leq n} \|x_i\|) < \rho$ , we have  $d(\bar{x}, \bigcap_{i=1}^n (\Omega_i - x_i)) < \rho$ , which implies condition (1). By Definition 2.1(i),  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -semitransversal at  $\bar{x}$  with  $\delta$ .

(ii) We first prove the equivalence between (a) and (b). Suppose condition (a) is satisfied. Let  $\omega_i \in \Omega_i$  and  $x_i \in X$  ( $i = 1, \dots, n$ ) with  $\varphi(\max_{1 \leq i \leq n} \|x_i\|) < \delta_1$  and  $x := \omega_1 + x_1 = \dots = \omega_n + x_n \in B_{\delta_2}(\bar{x})$ . Then

$$\varphi(d(x, \Omega_i)) = \varphi(d(\omega_i + x_i, \Omega_i)) \leq \varphi(\|x_i\|) < \delta_1 \quad (i = 1, \dots, n),$$

and consequently, inequality (5) is satisfied. Hence,

$$\begin{aligned} d\left(0, \bigcap_{i=1}^n (\Omega_i - \omega_i - x_i)\right) &= d\left(x, \bigcap_{i=1}^n \Omega_i\right) \\ &\leq \varphi\left(\max_{1 \leq i \leq n} d(x, \Omega_i)\right) \leq \varphi\left(\max_{1 \leq i \leq n} \|x_i\|\right). \end{aligned}$$

Suppose condition (b) is satisfied. Let  $x \in B_{\delta_2}(\bar{x})$  with  $\varphi(\max_{1 \leq i \leq n} d(x, \Omega_i)) < \delta_1$ . Choose  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) such that  $\varphi(\max_{1 \leq i \leq n} \|x - \omega_i\|) < \delta_1$  and set  $x'_i := x - \omega_i$  ( $i = 1, \dots, n$ ). Then  $x = x'_i + \omega_i \in B_{\delta_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and  $\varphi(\max_{1 \leq i \leq n} \|x'_i\|) < \delta_1$ . In view of inequality (6) with  $x'_i$  in place of  $x_i$  ( $i = 1, \dots, n$ ), we obtain

$$d\left(x, \bigcap_{i=1}^n \Omega_i\right) \leq \varphi\left(\max_{1 \leq i \leq n} \|x - \omega_i\|\right).$$

Taking infimum in the right-hand side over  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ), we arrive at inequality (5).

Next we show that  $\varphi$ -subtransversality is equivalent to condition (a). Let  $\{\Omega_1, \dots, \Omega_n\}$  be  $\varphi$ -subtransversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , and let  $x \in B_{\delta_2}(\bar{x})$  with  $\rho_0 := \varphi(\max_{1 \leq i \leq n} d(x, \Omega_i)) < \delta_1$ . Choose a  $\rho \in ]\rho_0, \delta_1[$ . By Definition 2.1(ii),  $\bigcap_{i=1}^n \Omega_i \cap B_\rho(x) \neq \emptyset$ , and consequently,  $d(x, \bigcap_{i=1}^n \Omega_i) < \rho$ . Letting  $\rho \downarrow \rho_0$ , we arrive at inequality (5). Conversely, let  $\delta_1 > 0$  and  $\delta_2 > 0$ , and inequality (5) hold for all  $x \in B_{\delta_2}(\bar{x})$  with  $\varphi(\max_{1 \leq i \leq n} d(x, \Omega_i)) < \delta_1$ . For any  $\rho \in ]0, \delta_1[$  and  $x \in B_{\delta_2}(\bar{x})$  with  $\varphi(\max_{1 \leq i \leq n} d(x, \Omega_i)) < \rho$ , we have  $d(x, \bigcap_{i=1}^n \Omega_i) < \rho$ , which implies condition (2). By Definition 2.1(ii),  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -subtransversal at  $\bar{x}$  with  $\delta_1$  and  $\delta_2$ .

- (iii) Let  $\{\Omega_1, \dots, \Omega_n\}$  be  $\varphi$ -transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , and let  $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  and  $x_i \in X$  ( $i = 1, \dots, n$ ) with  $\rho_0 := \varphi(\max_{1 \leq i \leq n} \|\omega_i - x_i\|) < \delta_1$ . Choose a  $\rho \in ]\rho_0, \delta_1[$ . By (3),  $d(0, \bigcap_{i=1}^n (\Omega_i - \omega_i - x_i)) < \rho$ . Letting  $\rho \downarrow \rho_0$ , we arrive at inequality (6).

Conversely, let  $\delta_1 > 0$  and  $\delta_2 > 0$ , and inequality (6) hold for all  $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  and  $x_i \in X$  ( $i = 1, \dots, n$ ) with  $\varphi(\max_{1 \leq i \leq n} \|\omega_i - x_i\|) < \delta_1$ . For any  $\rho \in ]0, \delta_1[$ ,  $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  and  $x_i \in X$  ( $i = 1, \dots, n$ ) with  $\varphi(\max_{1 \leq i \leq n} \|\omega_i - x_i\|) < \rho$ , we have  $d(0, \bigcap_{i=1}^n (\Omega_i - \omega_i - x_i)) < \rho$ , which is equivalent to condition (3). By Definition 2.1(iii),  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  with  $\delta_1$  and  $\delta_2$ .  $\square$

*Example 3.1* Let  $\mathbb{R}^2$  be equipped with the maximum norm, and let  $\Omega_1 := \{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid \xi_2 \geq 0\}$ ,  $\Omega_2 := \{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid \xi_2 \leq \xi_1^2\}$  and  $\bar{x} := (0, 0)$ . Thus,  $\Omega_1 \cap \Omega_2 = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid 0 \leq \xi_2 \leq \xi_1^2\}$ , and no shift of the sets can make their intersection empty. We claim that the pair  $\{\Omega_1, \Omega_2\}$  is  $\varphi$ -semitransversal at  $\bar{x}$  with  $\varphi(t) := \sqrt{2t}$  ( $t \geq 0$ ) and  $\delta := 2$ .

*Proof* Observe that, given any  $\varepsilon \geq 0$ , the vertical shifts of the sets determined by  $x_{1\varepsilon} := (0, -\varepsilon)$  and  $x_{2\varepsilon} = (0, \varepsilon)$  produce the largest ‘gap’ between them compared to all possible shifts  $x_1$  and  $x_2$  with  $\max\{\|x_1\|, \|x_2\|\} \leq \varepsilon$ . Indeed,

$$\begin{aligned} (\Omega_1 - x_{1\varepsilon}) \cap (\Omega_2 - x_{2\varepsilon}) &= \{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid \varepsilon \leq \xi_2 \leq \xi_1^2 - \varepsilon\} \\ &\subset (\Omega_1 - x_1) \cap (\Omega_2 - x_2), \end{aligned}$$

as long as  $\max\{\|x_1\|, \|x_2\|\} \leq \varepsilon$ . Observe also that  $(\sqrt{2\varepsilon}, \varepsilon) \in (\Omega_1 - x_{1\varepsilon}) \cap (\Omega_2 - x_{2\varepsilon})$ . Hence, for any  $x_1, x_2 \in \mathbb{R}^2$  with  $\varepsilon := \max\{\|x_1\|, \|x_2\|\} < \varphi^{-1}(\delta) = 2$ , we have

$$d(\bar{x}, (\Omega_1 - x_1) \cap (\Omega_2 - x_2)) \leq \|(\sqrt{2\varepsilon}, \varepsilon)\| = \sqrt{2\varepsilon} = \varphi(\max\{\|x_1\|, \|x_2\|\}).$$

In view of Theorem 3.1(i),  $\{\Omega_1, \Omega_2\}$   $\varphi$ -semitransversal at  $\bar{x}$  with  $\delta$ .  $\square$

*Example 3.2* Let  $\mathbb{R}^2$  be equipped with the maximum norm, and let  $\Omega_1 := \{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid \xi_2 = \xi_1^2\}$ ,  $\Omega_2 := \{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid \xi_2 = -\xi_1^2\}$  and  $\bar{x} := (0, 0)$ . Thus,  $\Omega_1 \cap \Omega_2 = \{\bar{x}\}$ . We claim that, for any  $\gamma > 1$ , the pair  $\{\Omega_1, \Omega_2\}$  is  $\varphi$ -subtransversal at  $\bar{x}$  with  $\varphi(t) := \gamma\sqrt{t}$  ( $t \geq 0$ ) and any  $\delta_1 > 0$  and  $\delta_2 > 0$  satisfying  $\delta_2 + \frac{1}{2} + \sqrt{\delta_2 + \frac{1}{4}} < \gamma^2$ .

*Proof* Observe that,  $d(x, \Omega_1 \cap \Omega_2) = \|x\|$  for all  $x \in \mathbb{R}^2$  and, given any  $\varepsilon \geq 0$  and the corresponding point  $x_\varepsilon = (0, \varepsilon)$ , one has

$$\min_{\|x\|=\varepsilon} \max\{d(x, \Omega_1), d(x, \Omega_2)\} = d(x_\varepsilon, \Omega_1) = d(x_\varepsilon, \Omega_2) = \min_{t \geq 0} \max\{\varepsilon - t, t^2\}.$$

It is easy to see that the minimum in the rightmost minimization problem is attained at  $t := \sqrt{\varepsilon + \frac{1}{4}} - \frac{1}{2}$  satisfying  $\varepsilon - t = t^2$ . Thus,

$$\min_{\|x\|=\varepsilon} \max\{d(x, \Omega_1), d(x, \Omega_2)\} = \varepsilon + \frac{1}{2} - \sqrt{\varepsilon + \frac{1}{4}} = \frac{\varepsilon^2}{\varepsilon + \frac{1}{2} + \sqrt{\varepsilon + \frac{1}{4}}}.$$

Hence, for any  $x \in \mathbb{R}^2$  with  $\|x\| < \delta_2$ , we have

$$d(x, \Omega_1 \cap \Omega_2) = \|x\| \leq \frac{\gamma \|x\|}{\sqrt{\|x\| + \frac{1}{2}} + \sqrt{\|x\| + \frac{1}{4}}} \leq \varphi(\max\{d(x, \Omega_1), d(x, \Omega_2)\}).$$

In view of Theorem 3.1(ii),  $\{\Omega_1, \Omega_2\}$  is  $\varphi$ -subtransversal at  $\bar{x}$  with  $\delta_1$  and  $\delta_2$ .

The next statement provides alternative metric characterizations of  $\varphi$ -transversality. These characterizations differ from the one in Theorem 3.1(iii) by values of the parameters  $\delta_1$  and  $\delta_2$  and have certain advantages, e.g., when establishing connections with metric regularity of set-valued mappings. The relations between the values of the parameters in the two groups of metric characterizations can be estimated.

**Theorem 3.2** *Let  $\delta_1 > 0$  and  $\delta_2 > 0$ . The following conditions are equivalent:*

- (i) *inequality (6) is satisfied for all  $x_i \in X$  and  $\omega_i \in \Omega_i$  with  $\omega_i + x_i \in B_{\delta_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and  $\varphi(\max_{1 \leq i \leq n} \|x_i\|) < \delta_1$ ;*
- (ii) *for all  $x_i \in \delta_2 \mathbb{B}$  ( $i = 1, \dots, n$ ) with  $\varphi(\max_{1 \leq i \leq n} d(\bar{x}, \Omega_i - x_i)) < \delta_1$ , it holds*

$$d\left(\bar{x}, \bigcap_{i=1}^n (\Omega_i - x_i)\right) \leq \varphi\left(\max_{1 \leq i \leq n} d(\bar{x}, \Omega_i - x_i)\right); \quad (7)$$

- (iii) *for all  $x, x_i \in X$  with  $x + x_i \in B_{\delta_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and  $\varphi(\max_{1 \leq i \leq n} d(x, \Omega_i - x_i)) < \delta_1$ , it holds*

$$d\left(x, \bigcap_{i=1}^n (\Omega_i - x_i)\right) \leq \varphi\left(\max_{1 \leq i \leq n} d(x, \Omega_i - x_i)\right). \quad (8)$$

Moreover, if  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then conditions (i)–(iii) hold with any  $\delta'_1 \in ]0, \delta_1]$  and  $\delta'_2 > 0$  satisfying  $\varphi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2$  in place of  $\delta_1$  and  $\delta_2$ .

Conversely, if conditions (i)–(iii) hold with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  with any  $\delta'_1 \in ]0, \delta_1]$  and  $\delta'_2 > 0$  satisfying  $\varphi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2$ .

*Proof* We first prove the equivalence of conditions (i)–(iii).

(i)  $\Rightarrow$  (ii). Let  $x_i \in \delta_2 \mathbb{B}$  ( $i = 1, \dots, n$ ) with  $\varphi(\max_{1 \leq i \leq n} d(\bar{x}, \Omega_i - x_i)) < \delta_1$ . Choose  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) such that  $\varphi(\max_{1 \leq i \leq n} \|\bar{x} + x_i - \omega_i\|) < \delta_1$ . Set  $x'_i := \bar{x} + x_i - \omega_i$  ( $i = 1, \dots, n$ ). Then  $\omega_i + x'_i \in B_{\delta_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and  $\varphi(\max_{1 \leq i \leq n} \|x'_i\|) < \delta_1$ . By (i), inequality (6) is satisfied with  $x'_i$  in place of  $x_i$  ( $i = 1, \dots, n$ ), i.e.

$$d\left(\bar{x}, \bigcap_{i=1}^n (\Omega_i - x_i)\right) \leq \varphi\left(\max_{1 \leq i \leq n} \|\bar{x} + x_i - \omega_i\|\right).$$

Taking the infimum in the right-hand side over  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ), we arrive at inequality (7).

(ii)  $\Rightarrow$  (iii). Let  $x, x_i \in X$  with  $x + x_i \in B_{\delta_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and  $\varphi(\max_{1 \leq i \leq n} d(x, \Omega_i - x_i)) < \delta_1$ . Set  $x'_i := x + x_i - \bar{x}$  ( $i = 1, \dots, n$ ). Then  $x'_i \in \delta_2 \mathbb{B}$  ( $i = 1, \dots, n$ ) and  $\varphi(\max_{1 \leq i \leq n} d(\bar{x}, \Omega_i - x'_i)) < \delta_1$ . By (ii), inequality (7) is satisfied with  $x'_i$  in place of  $x_i$  ( $i = 1, \dots, n$ ). This is equivalent to inequality (8).

(iii)  $\Rightarrow$  (i). Let  $x_i \in X$  and  $\omega_i \in \Omega_i$  with  $\omega_i + x_i \in B_{\delta_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and  $\varphi(\max_{1 \leq i \leq n} \|x_i\|) < \delta_1$ . Set  $x'_i := \omega_i + x_i - \bar{x}$  ( $i = 1, \dots, n$ ). Then,  $\bar{x} + x'_i \in B_{\delta_2}(\bar{x})$  and

$\varphi(\max_{1 \leq i \leq n} d(\bar{x}, \Omega_i - x'_i)) \leq \varphi(\|x_i\|) < \delta_1$ . By (iii), inequality (8) is satisfied with  $\bar{x}$  and  $x'_i$  in place of  $x$  and  $x_i$  ( $i = 1, \dots, n$ ), respectively, i.e.

$$d\left(0, \bigcap_{i=1}^n (\Omega_i - \omega_i - x_i)\right) \leq \varphi\left(\max_{1 \leq i \leq n} d(0, \Omega_i - \omega_i - x_i)\right).$$

Since  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ), inequality (6) is satisfied.

Suppose  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , and let  $\delta'_1 \in ]0, \delta_1]$  and  $\delta'_2 > 0$  be such that  $\varphi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2$ . Then, for all  $x_i \in X$  and  $\omega_i \in \Omega_i$  with  $\omega_i + x_i \in B_{\delta'_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and  $\varphi(\max_{1 \leq i \leq n} \|x_i\|) < \delta'_1$ , we have  $\|\omega_i - \bar{x}\| \leq \|x_i\| + \|\omega_i + x_i - \bar{x}\| < \varphi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2$  ( $i = 1, \dots, n$ ). By Theorem 3.1(iii), inequality (6) is satisfied, and consequently, condition (i) (as well as conditions (ii) and (iii)) holds with  $\delta'_1$  and  $\delta'_2$ .

Conversely, suppose conditions (i)–(iii) hold with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , and let  $\delta'_1 \in ]0, \delta_1]$  and  $\delta'_2 > 0$  be such that  $\varphi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2$ . Then, for all  $\omega_i \in \Omega_i \cap B_{\delta'_2}(\bar{x})$  and  $x_i \in X$  ( $i = 1, \dots, n$ ) with  $\varphi(\max_{1 \leq i \leq n} \|x_i\|) < \delta'_1$ , we have  $\|\omega_i + x_i - \bar{x}\| \leq \|x_i\| + \|\omega_i - \bar{x}\| < \varphi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2$  ( $i = 1, \dots, n$ ). By (i), inequality (6) is satisfied, and consequently,  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  with  $\delta'_1$  and  $\delta'_2$  according to Theorem 3.1(iii).  $\square$

*Remark 3.1* (i) In the Hölder case, i.e. when  $\varphi(t) := \alpha^{-1}t^q$  ( $t \geq 0$ ) for some  $\alpha > 0$  and  $q \in ]0, 1]$ , condition (8) served as the main metric characterization of transversality; cf. [42, 43]. In the linear case, condition (7) has been picked up recently in [10, 11]. This condition seems an important advancement as it replaces an arbitrary point  $x$  in (8) with the given reference point  $\bar{x}$ . Condition (6) in part (i) seems new. In view of Theorem 3.1(iii), it is the most straightforward metric counterpart of the original geometric property (3).  
(ii) The metric characterizations of the three  $\varphi$ -transversality properties in the above theorems look similar: each of them provides an upper error bound type estimate for the distance from a point to the intersection of sets, which can be useful from the computational point of view. For the account of nonlinear error bounds theory, we refer the reader to [1, 2, 13, 58].

The next corollary provides qualitative metric characterizations of the three nonlinear transversality properties. They are direct consequences of Theorems 3.1 and 3.2.

**Corollary 3.1** *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is*

- (i)  $\varphi$ -semitransversal at  $\bar{x}$  if and only if there exists a  $\delta > 0$  such that inequality (4) holds for all  $x_i \in \delta\mathbb{B}$  ( $i = 1, \dots, n$ );
- (ii)  $\varphi$ -subtransversal at  $\bar{x}$  if and only if the following equivalent conditions hold:
  - (a) there exists a  $\delta > 0$  such that inequality (5) holds for all  $x \in B_\delta(\bar{x})$ ;
  - (b) there exists a  $\delta > 0$  such that inequality (6) holds for all  $\omega_i \in \Omega_i \cap B_\delta(\bar{x})$  and  $x_i \in \delta\mathbb{B}$  ( $i = 1, \dots, n$ ) with  $\omega_1 + x_1 = \dots = \omega_n + x_n$ ;
- (iii)  $\varphi$ -transversal at  $\bar{x}$  if and only if the following equivalent conditions hold:
  - (a) there exists a  $\delta > 0$  such that inequality (6) holds for all  $\omega_i \in \Omega_i \cap B_\delta(\bar{x})$  and  $x_i \in \delta\mathbb{B}$  ( $i = 1, \dots, n$ );
  - (b) there exists a  $\delta > 0$  such that inequality (7) holds for all  $x_i \in \delta\mathbb{B}$  ( $i = 1, \dots, n$ );
  - (c) there exists a  $\delta > 0$  such that inequality (8) holds for all  $x \in B_\delta(\bar{x})$  and  $x_i \in \delta\mathbb{B}$  ( $i = 1, \dots, n$ ).

*Remark 3.2* In the Hölder setting, i.e. when  $\varphi(t) := \alpha^{-1}t^q$  ( $t \geq 0$ ) with some  $\alpha > 0$  and  $q > 0$ , the above corollary improves [42, Theorem 1]. In the linear case, the equivalence of the three characterizations of transversality in Corollary 3.1(iii) has been established in [10]. We refer the readers to [36, 39, 40] for more discussions and historical comments.

The next two propositions identify important situations when ‘restricted’ versions of the metric characterizations of nonlinear transversality properties in Theorem 3.1 can be used: with all but one sets being translated in the cases of  $\varphi$ -semitransversality and  $\varphi$ -transversality, and with the point  $x$  restricted to one of the sets in the case of  $\varphi$ -subtransversality. The latter restricted version is of importance, for instance, when dealing with alternating (or cyclic) projections. The first proposition formulates simplified necessary conditions for the transversality properties which are direct consequences of the respective statements, while the second one gives conditions under which these conditions become sufficient in the case of two sets.

**Proposition 3.1** (i) *If  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -semitransversal at  $\bar{x}$  with some  $\delta > 0$ , then*

$$d\left(\bar{x}, \bigcap_{i=1}^{n-1} (\Omega_i - x_i) \cap \Omega_n\right) \leq \varphi\left(\max_{1 \leq i \leq n-1} \|x_i\|\right)$$

*for all  $x_i \in X$  ( $i = 1, \dots, n-1$ ) with  $\varphi(\max_{1 \leq i \leq n-1} \|x_i\|) < \delta$ .*

(ii) *If  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -subtransversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then*

$$d\left(x, \bigcap_{i=1}^n \Omega_i\right) \leq \varphi\left(\max_{1 \leq i \leq n-1} d(x, \Omega_i)\right)$$

*for all  $x \in \Omega_n \cap B_{\delta_2}(\bar{x})$  with  $\varphi(\max_{1 \leq i \leq n-1} d(x, \Omega_i)) < \delta_1$ .*

(iii) *If  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then*

$$d\left(0, \bigcap_{i=1}^{n-1} (\Omega_i - \omega_i - x_i) \cap (\Omega_n - \omega_n)\right) \leq \varphi\left(\max_{1 \leq i \leq n-1} \|x_i\|\right)$$

*for all  $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and  $x_i \in X$  ( $i = 1, \dots, n-1$ ) with  $\varphi(\max_{1 \leq i \leq n-1} \|x_i\|) < \delta_1$ .*

**Proposition 3.2** *Let  $\Omega_1, \Omega_2$  be subsets of a normed space  $X$ , and  $\bar{x} \in \Omega_1 \cap \Omega_2$ . Let  $\alpha > 0$ ,  $\bar{t} > 0$ ,  $\varphi(t) \leq \alpha t$  for all  $t \in ]0, \bar{t}]$ , and  $\alpha' := (1 + 2\alpha)^{-1}$ .*

(i) *If for all  $x \in \bar{t}\mathbb{B}$ ,*

$$d(\bar{x}, (\Omega_1 - x) \cap \Omega_2) \leq \varphi(\|x\|), \quad (9)$$

*then  $\{\Omega_1, \Omega_2\}$  is  $\alpha'$ -semitransversal at  $\bar{x}$  with  $\delta := (\alpha + \frac{1}{2})\bar{t}$ .*

(ii) *If there exists a  $\delta_2 > 0$  such that, for all  $x \in \Omega_2 \cap B_{2\delta_2}(\bar{x})$  with  $d(x, \Omega_1) < \bar{t}$ ,*

$$d(x, \Omega_1 \cap \Omega_2) \leq \varphi(d(x, \Omega_1)), \quad (10)$$

*then  $\{\Omega_1, \Omega_2\}$  is  $\alpha'$ -subtransversal at  $\bar{x}$  with  $\delta_1 := (\alpha + \frac{1}{2})\bar{t}$  and  $\delta_2$ .*

(iii) *If there exists a  $\delta_2 > 0$  such that, for all  $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  ( $i = 1, 2$ ) and  $x \in \bar{t}\mathbb{B}$ ,*

$$d(0, (\Omega_1 - \omega_1 - x) \cap (\Omega_2 - \omega_2)) \leq \varphi(\|x\|),$$

*then  $\{\Omega_1, \Omega_2\}$  is  $\alpha'$ -transversal at  $\bar{x}$  with  $\delta_1 := (\alpha + \frac{1}{2})\bar{t}$  and  $\delta_2$ .*

*Proof* (i) Let  $\delta := (\alpha + \frac{1}{2})\bar{t}$ , and inequality (9) be satisfied for all  $x \in \bar{t}\mathbb{B}$ . Let  $\rho \in ]0, \delta[$  and  $x_1, x_2 \in X$  with  $\max\{\|x_1\|, \|x_2\|\} < \alpha'\rho$ . Set  $x' := x_1 - x_2$ . Thus,  $\|x'\| \leq 2\max\{\|x_1\|, \|x_2\|\} < 2\alpha'\delta = \bar{t}$ . Hence, by (9) with  $x'$  in place of  $x$ ,

$$\begin{aligned} d(\bar{x}, (\Omega_1 - x_1) \cap (\Omega_2 - x_2)) &\leq \|x_2\| + d(\bar{x} - x_2, (\Omega_1 - x_1) \cap (\Omega_2 - x_2)) \\ &= \|x_2\| + d(\bar{x}, (\Omega_1 - x') \cap \Omega_2) \\ &\leq \|x_2\| + \varphi(\|x'\|) \leq \|x_2\| + \alpha\|x'\| \\ &\leq (1 + 2\alpha)\max\{\|x_1\|, \|x_2\|\} < \rho. \end{aligned}$$

Hence,  $(\Omega_1 - x_1) \cap (\Omega_2 - x_2) \cap B_\rho(\bar{x}) \neq \emptyset$  and, by Definition 1.1(i),  $\{\Omega_1, \Omega_2\}$  is  $\alpha'$ -semitransversal at  $\bar{x}$  with  $\delta$ .

- (ii) Let  $\delta_1 := (\alpha + \frac{1}{2})\bar{r}$ ,  $\delta_2 > 0$ , and inequality (10) be satisfied for all  $x \in \Omega_2 \cap B_{2\delta_2}(\bar{x})$  with  $d(x, \Omega_1) < \bar{r}$ . Let  $\rho \in ]0, \delta_1[$  and  $x \in B_{\delta_2}(\bar{x})$  with  $\max\{d(x, \Omega_1), d(x, \Omega_2)\} < \alpha'\rho$ . Choose a number  $\gamma > 1$  such that

$$\|x - \bar{x}\| < \gamma^{-1}\delta_2 \quad \text{and} \quad \max\{d(x, \Omega_1), d(x, \Omega_2)\} < \gamma^{-1}\alpha'\rho,$$

and a point  $x' \in \Omega_2$  such that  $\|x - x'\| \leq \gamma d(x, \Omega_2)$ . Then

$$\begin{aligned} \|x' - \bar{x}\| &\leq \|x - x'\| + \|x - \bar{x}\| \leq \gamma d(x, \Omega_2) + \|x - \bar{x}\| \\ &\leq (\gamma + 1)\|x - \bar{x}\| < (1 + \gamma^{-1})\delta_2 < 2\delta_2, \\ d(x', \Omega_1) &\leq \|x - x'\| + d(x, \Omega_1) \leq (\gamma + 1)\max\{d(x, \Omega_1), d(x, \Omega_2)\} \\ &< (1 + \gamma^{-1})\alpha'\delta_1 < 2\alpha'\delta_1 = \bar{r}. \end{aligned}$$

Hence, by (10) with  $x'$  in place of  $x$ ,

$$\begin{aligned} d(x, \Omega_1 \cap \Omega_2) &\leq \|x - x'\| + d(x', \Omega_1 \cap \Omega_2) \leq \|x - x'\| + \varphi(d(x', \Omega_1)) \\ &\leq \|x - x'\| + \alpha d(x', \Omega_1) \leq (1 + \alpha)\|x - x'\| + \alpha d(x, \Omega_1) \\ &\leq (1 + \alpha)\gamma d(x, \Omega_2) + \alpha d(x, \Omega_1) \\ &\leq ((1 + \alpha)\gamma + \alpha)\max\{d(x, \Omega_1), d(x, \Omega_2)\}. \end{aligned}$$

Letting  $\gamma \downarrow 1$ , we arrive at

$$d(x, \Omega_1 \cap \Omega_2) \leq (1 + 2\alpha)\max\{d(x, \Omega_1), d(x, \Omega_2)\} < \rho.$$

Hence,  $\Omega_1 \cap \Omega_2 \cap B_\rho(x) \neq \emptyset$  and, by Definition 1.1(ii),  $\{\Omega_1, \Omega_2\}$  is  $\alpha'$ -subtransversal at  $\bar{x}$  with  $\delta_1$  and  $\delta_2$ .

- (iii) The proof follows that of assertion (i) with the sets  $\Omega_1 - \omega_1$  and  $\Omega_2 - \omega_2$  in place of  $\Omega_1$  and  $\Omega_2$ , respectively.  $\square$

*Remark 3.3* (i) In the linear case, Proposition 3.2(ii) recaptures [40, Theorem 1(iii)], while parts (i) and (iii) seem new.

- (ii) Restricted versions of the metric conditions in Theorem 3.2 can be produced in a similar way.

Checking the metric estimates of the  $\varphi$ -subtransversality and  $\varphi$ -transversality can be simplified as illustrated by the following proposition referring to condition (5) in Theorem 3.1(ii). Equivalent versions of conditions (7) and (8) in Theorem 3.2 look similar.

**Proposition 3.3** *The following conditions are equivalent:*

- (i) inequality (5) holds true;  
(ii) for all  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ), it holds

$$d\left(x, \bigcap_{i=1}^n \Omega_i\right) \leq \varphi\left(\max_{1 \leq i \leq n} \|x - \omega_i\|\right); \quad (11)$$

- (iii) inequality (11) holds true for all  $\omega_i \in \Omega_i$  with  $\|\omega_i - \bar{x}\| < \|x - \bar{x}\| + \varphi^{-1}(\|x - \bar{x}\|)$  ( $i = 1, \dots, n$ );  
(iv) inequality (11) holds true for all  $\omega_i \in \Omega_i$  with  $\varphi(\|\omega_i - x\|) < \|x - \bar{x}\|$  ( $i = 1, \dots, n$ ).

*Proof* The equivalence (i)  $\Leftrightarrow$  (ii) and implications (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv) are straightforward. We next show that (iv)  $\Rightarrow$  (ii). Let condition (iv) hold true,  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ), and  $\varphi(\|\omega_i - x\|) \geq \|x - \bar{x}\|$  for some  $i$ . Then

$$d\left(x, \bigcap_{i=1}^n \Omega_i\right) \leq \|x - \bar{x}\| \leq \varphi\left(\max_{1 \leq i \leq n} \|x - \omega_i\|\right),$$

i.e. inequality (11) is satisfied, and consequently condition (ii) holds true.  $\square$



#### 4 Slope Sufficient Conditions

In this section, we formulate slope sufficient conditions for the properties in Definition 2.1. The conditions are straightforward consequences of the Ekeland variational principle (Lemma 1.1) applied to appropriate lower semicontinuous functions. Throughout this section,  $X$  is a Banach space, and the sets  $\Omega_1, \dots, \Omega_n$  are closed. These are exactly the assumptions which ensure that the Ekeland variational principle is applicable. In view of Proposition 2.2(iii), it suffices to assume that  $\bar{x} \in \text{bd} \cap_{i=1}^n \Omega_i$ .

The sufficient conditions for the three properties follow the same pattern. We first establish nonlocal slope sufficient conditions arising from the Ekeland variational principle. These nonlocal conditions are largely of theoretical interest (unless the sets are convex): they encapsulate the application of the Ekeland variational principle and serve as a source of more practical local (infinitesimal) conditions. The corresponding local slope sufficient conditions, their Hölder as well as simplified  $\delta$ -free versions are formulated as corollaries. This way we expose the hierarchy of this type of conditions.

Along with the standard maximum norm on  $X^{n+1}$ , we are going to use also the following norm depending on a parameter  $\gamma > 0$ :

$$\|(x_1, \dots, x_n, x)\|_\gamma := \max \left\{ \|x\|, \gamma \max_{1 \leq i \leq n} \|x_i\| \right\}, \quad x_1, \dots, x_n, x \in X. \quad (12)$$

##### 4.1 Semitransversality

**Theorem 4.1** *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -semitransversal at  $\bar{x}$  with some  $\delta > 0$  if, for some  $\gamma > 0$  and any  $x_i \in X$  ( $i = 1, \dots, n$ ) satisfying*

$$0 < \max_{1 \leq i \leq n} \|x_i\| < \varphi^{-1}(\delta), \quad (13)$$

*there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$  such that*

$$\sup_{\substack{u_i \in \Omega_i \ (i=1, \dots, n), u \in X \\ (u_1, \dots, u_n, u) \neq (\omega_1, \dots, \omega_n, x)}} \frac{\varphi \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right) - \varphi \left( \max_{1 \leq i \leq n} \|u_i - x_i - u\| \right)}{\|(u_1, \dots, u_n, u) - (\omega_1, \dots, \omega_n, x)\|_\gamma} \geq 1 \quad (14)$$

*for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying*

$$\|x - \bar{x}\| < \lambda, \quad \max_{1 \leq i \leq n} \|\omega_i - \bar{x}\| < \frac{\lambda}{\gamma}, \quad (15)$$

$$0 < \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \leq \max_{1 \leq i \leq n} \|x_i\|. \quad (16)$$

The proof below employs two closely related nonnegative functions on  $X^{n+1}$  determined by the given function  $\varphi \in \mathcal{C}$  and vectors  $x_1, \dots, x_n \in X$ :

$$f(u_1, \dots, u_n, u) := \varphi \left( \max_{1 \leq i \leq n} \|u_i - x_i - u\| \right), \quad u_1, \dots, u_n, u \in X, \quad (17)$$

$$\hat{f} := f + i_{\Omega_1 \times \dots \times \Omega_n}. \quad (18)$$

*Proof* Suppose  $\{\Omega_1, \dots, \Omega_n\}$  is not  $\varphi$ -semitransversal at  $\bar{x}$  with some  $\delta > 0$ , and let  $\gamma > 0$  be given. By Definition 2.1(i), there exist a  $\rho \in ]0, \delta[$  and  $x_i \in X$  ( $i = 1, \dots, n$ ) with  $\varphi(\max_{1 \leq i \leq n} \|x_i\|) < \rho$  such that  $\cap_{i=1}^n (\Omega_i - x_i) \cap B_\rho(\bar{x}) = \emptyset$ . Thus,  $\max_{1 \leq i \leq n} \|x_i\| > 0$ . Let

$\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$  and  $\lambda' := \min\{\lambda, \rho\}$ . Then  $\lambda' > \varphi(\max_{1 \leq i \leq n} \|x_i\|)$ ,  $\cap_{i=1}^n (\Omega_i - x_i) \cap B_{\lambda'}(\bar{x}) = \emptyset$ , and consequently,

$$\max_{1 \leq i \leq n} \|u_i - x_i - u\| > 0 \quad \text{for all } u_i \in \Omega_i (i = 1, \dots, n), u \in B_{\lambda'}(\bar{x}). \quad (19)$$

Let  $f$  and  $\widehat{f}$  be defined by (17) and (18), respectively, while  $X^{n+1}$  be equipped with the metric induced by the norm (12). We have  $\widehat{f}(\bar{x}, \dots, \bar{x}, \bar{x}) = \varphi(\max_{1 \leq i \leq n} \|x_i\|) < \lambda'$ . Choose a number  $\varepsilon$  such that  $\widehat{f}(\bar{x}, \dots, \bar{x}, \bar{x}) < \varepsilon < \lambda'$ . Applying the Ekeland variational principle, we can find points  $\omega_i \in \Omega_i (i = 1, \dots, n)$  and  $x \in X$  such that

$$\|(\omega_1, \dots, \omega_n, x) - (\bar{x}, \dots, \bar{x}, \bar{x})\|_\gamma < \lambda' \leq \lambda, \quad f(\omega_1, \dots, \omega_n, x) \leq f(\bar{x}, \dots, \bar{x}, \bar{x}), \quad (20)$$

$$f(\omega_1, \dots, \omega_n, x) - f(u_1, \dots, u_n, u) \leq \frac{\varepsilon}{\lambda'} \|(u_1, \dots, u_n, u) - (\omega_1, \dots, \omega_n, x)\|_\gamma \quad (21)$$

for all  $(u_1, \dots, u_n, u) \in \Omega_1 \times \dots \times \Omega_n \times X$ . In view of (19) and the definitions of  $\lambda'$  and  $f$ , conditions (20) yield (15) and (16). Since  $\varepsilon/\lambda' < 1$ , condition (21) contradicts (14).  $\square$

*Remark 4.1* The expression in the left-hand side of (14) is the nonlocal  $\gamma$ -slope [33, p. 60] at  $(\omega_1, \dots, \omega_n, x)$  of the function (18).

The next statement is a localized version of Theorem 4.1.

**Corollary 4.1** (i) *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -semitransversal at  $\bar{x}$  with some  $\delta > 0$  if, for some  $\gamma > 0$  and any  $x_i \in X (i = 1, \dots, n)$  satisfying (13), there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$  such that*

$$\limsup_{\substack{u_i \xrightarrow{\Omega_i} \omega_i (i=1, \dots, n), u \rightarrow x \\ (u_1, \dots, u_n, u) \neq (\omega_1, \dots, \omega_n, x)}} \frac{\varphi\left(\max_{1 \leq i \leq n} \|\omega_i - x_i - x\|\right) - \varphi\left(\max_{1 \leq i \leq n} \|u_i - x_i - u\|\right)}{\|(u_1, \dots, u_n, u) - (\omega_1, \dots, \omega_n, x)\|_\gamma} \geq 1 \quad (22)$$

for all  $x \in X$  and  $\omega_i \in \Omega_i (i = 1, \dots, n)$  satisfying (15) and (16).

(ii) *If  $\varphi \in \mathcal{C}^1$ , then inequality (22) in part (i) can be replaced by*

$$\varphi' \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right) \times \limsup_{\substack{u_i \xrightarrow{\Omega_i} \omega_i (i=1, \dots, n), u \rightarrow x \\ (u_1, \dots, u_n, u) \neq (\omega_1, \dots, \omega_n, x)}} \frac{\max_{1 \leq i \leq n} \|\omega_i - x_i - x\| - \max_{1 \leq i \leq n} \|u_i - x_i - u\|}{\|(u_1, \dots, u_n, u) - (\omega_1, \dots, \omega_n, x)\|_\gamma} \geq 1. \quad (23)$$

*Proof* The expression in the left-hand side of (22) is the  $\gamma$ -slope [33, p. 61] of the function (18) at  $(\omega_1, \dots, \omega_n, x)$ . The first assertion follows from Theorem 4.1 in view of Proposition 1.1(ii), while the second one is a consequence of Lemma 1.2 in view of Remark 1.1(i).  $\square$

In the Hölder setting, Theorem 4.1 and Corollary 4.1 yield the following statement.

**Corollary 4.2** *Let  $\alpha > 0$  and  $q > 0$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\alpha$ -semitransversal of order  $q$  at  $\bar{x}$  with some  $\delta > 0$  if, for some  $\gamma > 0$  and any  $x_i \in X (i = 1, \dots, n)$  with  $0 < \max_{1 \leq i \leq n} \|x_i\| < (\alpha\delta)^{\frac{1}{q}}$ , there exists a  $\lambda \in ]\alpha^{-1}(\max_{1 \leq i \leq n} \|x_i\|)^q, \delta[$  such that*

$$\sup_{\substack{u_i \in \Omega_i (i=1, \dots, n), u \in X \\ (u_1, \dots, u_n, u) \neq (\omega_1, \dots, \omega_n, x)}} \frac{\left(\max_{1 \leq i \leq n} \|\omega_i - x_i - x\|\right)^q - \left(\max_{1 \leq i \leq n} \|u_i - x_i - u\|\right)^q}{\|(u_1, \dots, u_n, u) - (\omega_1, \dots, \omega_n, x)\|_\gamma} \geq \alpha \quad (24)$$

for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (15) and (16), or all the more, such that

$$q \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right)^{q-1} \times \limsup_{\substack{\Omega_i \\ u_i \rightarrow \omega_i (i=1, \dots, n), u \rightarrow x \\ (u_1, \dots, u_n, u) \neq (\omega_1, \dots, \omega_n, x)}} \frac{\max_{1 \leq i \leq n} \|\omega_i - x_i - x\| - \max_{1 \leq i \leq n} \|u_i - x_i - u\|}{\|(u_1, \dots, u_n, u) - (\omega_1, \dots, \omega_n, x)\|_\gamma} \geq \alpha. \quad (25)$$

*Proof* The statement is a direct consequence of Theorem 4.1 and Corollary 4.1 with  $\varphi(t) := \alpha^{-1}t^q$  for all  $t \geq 0$ . Observe that  $\varphi^{-1}(t) = (\alpha t)^{\frac{1}{q}}$ .  $\square$

*Remark 4.2* (i) On top of the explicitly given restriction  $\|\omega_i - \bar{x}\| < \lambda/\gamma$  in Theorem 4.1 (and similar conditions in its corollaries) on the choice of the points  $\omega_i \in \Omega_i$ , which involves  $\gamma$ , the other conditions implicitly impose another one:

$$\|\omega_i - \bar{x}\| \leq \|x - \bar{x}\| + \|\omega_i - x_i - x\| + \|x_i\| \leq \|x - \bar{x}\| + 2 \max_{1 \leq i \leq n} \|x_i\|,$$

and consequently,  $\|\omega_i - \bar{x}\| < \lambda + 2\varphi^{-1}(\delta)$ . This alternative restriction can be of importance when  $\gamma$  is small.

- (ii) The statements of Theorem 4.1 and its corollaries can be simplified (and weakened!) by dropping condition (16).
- (iii) Inequalities (14), (22)–(25), which are crucial for checking nonlinear semitransversality, involve two groups of parameters: on one hand, sufficiently small vectors  $x_i \in X$ , not all zero, and on the other hand, points  $x \in X$  and  $\omega_i \in \Omega_i$  near  $\bar{x}$ . Note an important difference between these two groups. The magnitudes of  $x_i$  are directly controlled by the value of  $\delta$  in the definition of  $\varphi$ -semitransversality:  $\varphi(\max_{1 \leq i \leq n} \|x_i\|) < \delta$ . At the same time, taking into account that  $\lambda$  can be made arbitrarily close to  $\varphi(\max_{1 \leq i \leq n} \|x_i\|)$ , the magnitudes of  $x - \bar{x}$  and  $\omega_i - \bar{x}$  (as well as  $\omega_i - x_i - x$ ) are determined by  $\delta$  indirectly; they are controlled by  $\max_{1 \leq i \leq n} \|x_i\|$ : cf. conditions (15) and (16).
- (iv) In view of the definition of the parametric norm (12), if any of the inequalities (14), (22)–(25) holds true for some  $\gamma > 0$ , then it also holds for any  $\gamma' \in ]0, \gamma[$ .
- (v) Even in the linear setting, the characterizations in Corollary 4.2 are new.

The next corollary provides a simplified (and weaker!) version of Theorem 4.1. The simplification comes at the expense of eliminating the difference between the two groups of parameters highlighted in Remark 4.2(iii).

**Corollary 4.3** *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -semitransversal at  $\bar{x}$  with some  $\delta > 0$  if, for some  $\gamma > 0$  and any  $x_i \in X$  ( $i = 1, \dots, n$ ) satisfying (13), inequality (14) holds for all  $x \in B_\delta(\bar{x})$  and  $\omega_i \in \Omega_i \cap B_{\delta/\gamma}(\bar{x})$  ( $i = 1, \dots, n$ ) satisfying (16).*

Sacrificing the estimates for  $\delta$  in Theorem 4.1, and Corollaries 4.1 and 4.3, we arrive at the following ‘ $\delta$ -free’ statement.

**Corollary 4.4** *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -semitransversal at  $\bar{x}$  if, for some  $\gamma > 0$  and all  $x_i \in X$  ( $i = 1, \dots, n$ ) near 0 with  $\max_{1 \leq i \leq n} \|x_i\| > 0$ ,  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) near  $\bar{x}$  satisfying (16), inequality (14) holds true. Moreover, inequality (14) can be replaced by its localized version (22), or by (23) if  $\varphi \in \mathcal{C}^1$ .*

## 4.2 Subtransversality

**Theorem 4.2** *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -subtransversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\gamma > 0$  and any  $x' \in X$  satisfying*

$$\|x' - \bar{x}\| < \delta_2, \quad 0 < \max_{1 \leq i \leq n} d(x', \Omega_i) < \varphi^{-1}(\delta_1), \quad (26)$$

*there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)), \delta_1[$  such that*

$$\sup_{\substack{u_i \in \Omega_i \ (i=1, \dots, n), u \in X \\ (u_1, \dots, u_n, u) \neq (\omega_1, \dots, \omega_n, x)}} \frac{\varphi\left(\max_{1 \leq i \leq n} \|\omega_i - x\|\right) - \varphi\left(\max_{1 \leq i \leq n} \|u_i - u\|\right)}{\|(u_1, \dots, u_n, u) - (\omega_1, \dots, \omega_n, x)\|_\gamma} \geq 1 \quad (27)$$

*for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying*

$$\|x - x'\| < \lambda, \quad \max_{1 \leq i \leq n} \|\omega_i - \omega'_i\| < \frac{\lambda}{\gamma}, \quad (28)$$

$$0 < \max_{1 \leq i \leq n} \|\omega_i - x\| \leq \max_{1 \leq i \leq n} \|\omega'_i - x'\| < \varphi^{-1}(\lambda). \quad (29)$$

The proof below follows the pattern of that of Theorem 4.1. It employs a continuous real-valued function  $f : X^{n+1} \rightarrow \mathbb{R}_+$  determined by the given function  $\varphi \in \mathcal{C}$ :

$$f(u_1, \dots, u_n, u) := \varphi\left(\max_{1 \leq i \leq n} \|u_i - u\|\right), \quad u_1, \dots, u_n, u \in X, \quad (30)$$

and its restriction to  $\Omega_1 \times \dots \times \Omega_n \times X$  given by (18). Note that the function (30) is a particular case of (17) corresponding to setting  $x_i := 0$  ( $i = 1, \dots, n$ ). We provide here the proof of Theorem 4.2 for completeness and to expose the differences in handling the two transversality properties, but we skip the proofs of most of its corollaries.

*Proof* Suppose  $\{\Omega_1, \dots, \Omega_n\}$  is not  $\varphi$ -subtransversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , and let  $\gamma > 0$  be given. By Definition 2.1(ii), there exist a number  $\rho \in ]0, \delta_1[$  and a point  $x' \in B_{\delta_2}(\bar{x})$  such that  $\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)) < \rho$  and  $\cap_{i=1}^n \Omega_i \cap B_\rho(x') = \emptyset$ . Hence,  $x' \notin \cap_{i=1}^n \Omega_i$  and

$$0 < \varphi\left(\max_{1 \leq i \leq n} d(x', \Omega_i)\right) < \rho \leq d\left(x', \bigcap_{i=1}^n \Omega_i\right).$$

Let  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)), \delta_1[$ . Choose numbers  $\varepsilon$  and  $\lambda'$  such that

$$\varphi\left(\max_{1 \leq i \leq n} d(x', \Omega_i)\right) < \varepsilon < \lambda' < \min\{\lambda, \rho\},$$

and points  $\omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) such that  $\varphi(\max_{1 \leq i \leq n} \|\omega'_i - x'\|) < \varepsilon$ . Let  $f$  and  $\widehat{f}$  be defined by (30) and (18), respectively, while  $X^{n+1}$  be equipped with the metric induced by the norm (12). We have  $\widehat{f}(\omega'_1, \dots, \omega'_n, x') < \varepsilon$ . Applying the Ekeland variational principle, we can find points  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) and  $x \in X$  such that

$$\|(\omega_1, \dots, \omega_n, x) - (\omega'_1, \dots, \omega'_n, x')\|_\gamma < \lambda', \quad f(\omega_1, \dots, \omega_n, x) \leq f(\omega'_1, \dots, \omega'_n, x'), \quad (31)$$

$$f(\omega_1, \dots, \omega_n, x) - f(u_1, \dots, u_n, u) \leq \frac{\varepsilon}{\lambda'} \|(u_1, \dots, u_n, u) - (\omega_1, \dots, \omega_n, x)\|_\gamma \quad (32)$$

for all  $(u_1, \dots, u_n, u) \in \Omega_1 \times \dots \times \Omega_n \times X$ . Thanks to (31), we have  $\|x - x'\| < \lambda'$ , and consequently,

$$d\left(x, \bigcap_{i=1}^n \Omega_i\right) \geq d\left(x', \bigcap_{i=1}^n \Omega_i\right) - \|x - x'\| > d\left(x', \bigcap_{i=1}^n \Omega_i\right) - \lambda' > 0.$$

Hence,  $x \notin \bigcap_{i=1}^n \Omega_i$ , and  $\max_{1 \leq i \leq n} \|\omega_i - x\| > 0$ . In view of the definitions of  $\lambda'$  and  $f$ , conditions (31) together with the last inequality yield (28) and (29). Since  $\varepsilon/\lambda' < 1$ , condition (32) contradicts (27).  $\square$

The next statement is a localized version of Theorem 4.2.

**Corollary 4.5** (i) *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -subtransversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\gamma > 0$  and any  $x' \in X$  satisfying (26), there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)), \delta_1[$  such that*

$$\limsup_{\substack{\Omega_i \\ u_i \rightarrow \omega_i \ (i=1, \dots, n), \ u \rightarrow x \\ (u_1, \dots, u_n, u) \neq (\omega_1, \dots, \omega_n, x)}} \frac{\varphi\left(\max_{1 \leq i \leq n} \|\omega_i - x\|\right) - \varphi\left(\max_{1 \leq i \leq n} \|u_i - u\|\right)}{\|(u_1, \dots, u_n, u) - (\omega_1, \dots, \omega_n, x)\|_\gamma} \geq 1 \quad (33)$$

for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (28) and (29).

(ii) *If  $\varphi \in \mathcal{C}^1$ , then inequality (33) in part (i) can be replaced by*

$$\begin{aligned} & \varphi'\left(\max_{1 \leq i \leq n} \|\omega_i - x\|\right) \\ & \times \limsup_{\substack{\Omega_i \\ u_i \rightarrow \omega_i \ (i=1, \dots, n), \ u \rightarrow x \\ (u_1, \dots, u_n, u) \neq (\omega_1, \dots, \omega_n, x)}} \frac{\max_{1 \leq i \leq n} \|\omega_i - x\| - \max_{1 \leq i \leq n} \|u_i - u\|}{\|(u_1, \dots, u_n, u) - (\omega_1, \dots, \omega_n, x)\|_\gamma} \geq 1. \end{aligned} \quad (34)$$

In the Hölder setting, Theorem 4.2 and Corollary 4.5 yield the following statement. In view of Remark 2.3, we assume that  $q \leq 1$ .

**Corollary 4.6** *Let  $\alpha > 0$  and  $q \in ]0, 1]$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\alpha$ -subtransversal of order  $q$  at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\gamma > 0$  and any  $x' \in B_{\delta_2}(\bar{x})$  with  $0 < \max_{1 \leq i \leq n} d(x', \Omega_i) < (\alpha \delta_1)^{\frac{1}{q}}$ , there exists a  $\lambda \in ]\alpha^{-1}(\max_{1 \leq i \leq n} d(x', \Omega_i))^q, \delta_1[$  such that*

$$\sup_{\substack{u_i \in \Omega_i \ (i=1, \dots, n), \ u \in X \\ (u_1, \dots, u_n, u) \neq (\omega_1, \dots, \omega_n, x)}} \frac{\left(\max_{1 \leq i \leq n} \|\omega_i - x\|\right)^q - \left(\max_{1 \leq i \leq n} \|u_i - u\|\right)^q}{\|(u_1, \dots, u_n, u) - (\omega_1, \dots, \omega_n, x)\|_\gamma} \geq \alpha, \quad (35)$$

for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (28) and

$$0 < \max_{1 \leq i \leq n} \|\omega_i - x\| \leq \max_{1 \leq i \leq n} \|\omega'_i - x'\| < (\alpha \lambda)^{\frac{1}{q}},$$

or all the more, such that

$$\begin{aligned} & q \left(\max_{1 \leq i \leq n} \|\omega_i - x\|\right)^{q-1} \\ & \times \limsup_{\substack{\Omega_i \\ u_i \rightarrow \omega_i \ (i=1, \dots, n), \ u \rightarrow x \\ (u_1, \dots, u_n, u) \neq (\omega_1, \dots, \omega_n, x)}} \frac{\max_{1 \leq i \leq n} \|\omega_i - x\| - \max_{1 \leq i \leq n} \|u_i - u\|}{\|(u_1, \dots, u_n, u) - (\omega_1, \dots, \omega_n, x)\|_\gamma} \geq \alpha. \end{aligned} \quad (36)$$

**Remark 4.3** (i) The expressions in the left-hand sides of (27) and (33) are, respectively, the nonlocal  $\gamma$ -slope and the  $\gamma$ -slope at  $(\omega_1, \dots, \omega_n, x)$  of the function (18).

(ii) Under the conditions of Theorem 4.2, there are two ways for estimating  $\|\omega_i - \bar{x}\|$ :

$$\begin{aligned} \|\omega_i - \bar{x}\| &\leq \|x' - \bar{x}\| + \|\omega_i - \omega'_i\| + \|\omega'_i - x'\| < \delta_2 + \lambda/\gamma + \varphi^{-1}(\lambda), \\ \|\omega_i - \bar{x}\| &\leq \|x - \bar{x}\| + \|\omega_i - x\| \\ &\leq \|x' - \bar{x}\| + \|x - x'\| + \max_{1 \leq i \leq n} \|\omega'_i - x'\| < \delta_2 + \lambda + \varphi^{-1}(\lambda). \end{aligned}$$

The second estimate does not involve  $\gamma$  and is better than the first one when  $\gamma < 1$ . A similar observation can be made about Corollary 4.7.

- (iii) It can be observed from the proof of Theorem 4.2 that the sufficient conditions for  $\varphi$ -subtransversality can be strengthened by adding another restriction on the choice of  $x'$ :  $\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)) < d(x', \cap_{i=1}^n \Omega_i)$ .
- (iv) The statement of Theorem 4.2 and its corollaries can be simplified by dropping condition (29).
- (v) Inequalities (27), (33)–(36), which are crucial for checking nonlinear subtransversality, involve points  $x \in X$  and  $\omega_i \in \Omega_i$  near  $\bar{x}$ . Their distance from  $\bar{x}$  is determined in Theorem 4.2 via other points:  $x' \notin \cap_{i=1}^n \Omega_i$  and  $\omega'_i \in \Omega_i$ ; cf. conditions (28) and (29). Only the distance from  $x'$  to  $\bar{x}$  and to the sets  $\Omega_i$  is directly controlled by the values of  $\delta_1$  and  $\delta_2$  in the definition of  $\varphi$ -subtransversality:  $x' \in B_{\delta_2}(\bar{x})$  and  $\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)) < \delta_1$ . All the other distances are controlled by  $\lambda$ , which can be made arbitrarily close to  $\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i))$ .
- (vi) In view of the definition of the parametric norm (12), if any of the inequalities (27), (33)–(36) holds true for some  $\gamma > 0$ , then it also holds for any  $\gamma' \in ]0, \gamma[$ .
- (vii) Corollary 4.6 strengthens [42, Proposition 6]. In the linear case, it improves [39, Proposition 10].

The next corollary provides a simplified (and weaker!) version of Theorem 4.2; cf. Remark 4.3(v).

**Corollary 4.7** *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -subtransversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\gamma > 0$ , inequality (27) holds for all  $x \in B_{\delta_1 + \delta_2}(\bar{x})$  and  $\omega_i \in \Omega_i \cap B_{\delta_2 + \delta_1/\gamma + \varphi^{-1}(\delta_1)}(\bar{x})$  ( $i = 1, \dots, n$ ) satisfying  $0 < \max_{1 \leq i \leq n} \|\omega_i - x\| < \varphi^{-1}(\delta_1)$ .*

*Proof* Let  $\delta_1 > 0$  and  $\delta_2 > 0$ ,  $x' \in B_{\delta_2}(\bar{x}) \setminus \cap_{i=1}^n \Omega_i$ ,  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)), \delta_1[$ , and points  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfy conditions (28) and (29). Then

$$\begin{aligned} \|x - \bar{x}\| &\leq \|x - x'\| + \|x' - \bar{x}\| < \lambda + \delta_2 < \delta_1 + \delta_2, \\ \|\omega_i - \bar{x}\| &\leq \|x' - \bar{x}\| + \|\omega_i - \omega'_i\| + \|\omega'_i - x'\| \\ &< \delta_2 + \lambda/\gamma + \varphi^{-1}(\lambda) < \delta_2 + \delta_1/\gamma + \varphi^{-1}(\delta_1), \\ \|\omega_i - x\| &< \varphi^{-1}(\lambda) < \varphi^{-1}(\delta_1), \end{aligned}$$

i.e. points  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfy all the conditions in the corollary. Hence, inequality (27) holds. It follows from Theorem 4.2 that  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -subtransversal at  $\bar{x}$  with  $\delta_1$  and  $\delta_2$ .  $\square$

Sacrificing the estimates for  $\delta_1$  and  $\delta_2$  in Theorem 4.2, and Corollaries 4.5 and 4.7, we can formulate the following ‘ $\delta$ -free’ statement.

**Corollary 4.8** *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -subtransversal at  $\bar{x}$  if inequality (27) holds true for some  $\gamma > 0$  and all  $x \in X$  near  $\bar{x}$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) near  $\bar{x}$  satisfying  $\max_{1 \leq i \leq n} \|\omega_i - x\| > 0$ . Moreover, inequality (27) can be replaced by its localized version (33), or by (34) if  $\varphi \in \mathcal{C}^1$ .*

### 4.3 Transversality

Since  $\widehat{\varphi}$ -transversality is in a sense an overarching property covering both  $\varphi$ -semitransversality and  $\varphi$ -subtransversality (see Proposition 2.2(iii)), the next theorem contains some elements of both Theorems 4.1 and 4.2, and its proof goes along the same lines. Similar to the proof of Theorem 4.1, it employs functions (17) and (18).

**Theorem 4.3** *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\gamma > 0$  and any  $\omega'_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and  $\xi \in ]0, \varphi^{-1}(\delta_1)[$ , there exists a  $\lambda \in ]\varphi(\xi), \delta_1[$  such that inequality (14) holds for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying*

$$\|x - \bar{x}\| < \lambda, \quad \max_{1 \leq i \leq n} \|\omega_i - \omega'_i\| < \frac{\lambda}{\gamma}, \quad (37)$$

$$0 < \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \leq \max_{1 \leq i \leq n} \|\omega'_i - x_i - \bar{x}\| = \xi. \quad (38)$$

*Proof* Suppose  $\{\Omega_1, \dots, \Omega_n\}$  is not  $\varphi$ -transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , and let  $\gamma > 0$  be given. By Definition 2.1(iii), there exist a number  $\rho \in ]0, \delta_1[$  and points  $\omega'_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  and  $x'_i \in X$  ( $i = 1, \dots, n$ ) with  $\varphi(\max_{1 \leq i \leq n} \|x'_i\|) < \rho$  such that  $\cap_{i=1}^n (\Omega_i - \omega'_i - x'_i) \cap (\rho\mathbb{B}) = \emptyset$ . Thus,  $\xi := \max_{1 \leq i \leq n} \|x'_i\| > 0$  and  $\xi < \varphi^{-1}(\rho) < \varphi^{-1}(\delta_1)$ . Set  $x_i := \omega'_i + x'_i - \bar{x}$  ( $i = 1, \dots, n$ ). Then

$$\max_{1 \leq i \leq n} \|\omega'_i - x_i - \bar{x}\| = \max_{1 \leq i \leq n} \|x'_i\| = \xi.$$

Let  $\lambda \in ]\varphi(\xi), \delta_1[$  and  $\lambda' := \min\{\lambda, \rho\}$ . Then  $\cap_{i=1}^n (\Omega_i - x_i) \cap B_{\lambda'}(\bar{x}) = \emptyset$ , and consequently, condition (19) holds true. Let  $f$  and  $\widehat{f}$  be defined by (17) and (18), respectively, while  $X^{n+1}$  be equipped with the metric induced by the norm (12). We have  $\widehat{f}(\omega'_1, \dots, \omega'_n, \bar{x}) = \varphi(\max_{1 \leq i \leq n} \|x'_i\|) = \varphi(\xi) < \lambda'$ . Choose a number  $\varepsilon$  such that  $\widehat{f}(\omega'_1, \dots, \omega'_n, \bar{x}) < \varepsilon < \lambda'$ . Applying the Ekeland variational principle, we can find points  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) and  $x \in X$  such that

$$\|(\omega_1, \dots, \omega_n, x) - (\omega'_1, \dots, \omega'_n, \bar{x})\|_\gamma < \lambda', \quad f(\omega_1, \dots, \omega_n, x) \leq f(\omega'_1, \dots, \omega'_n, \bar{x}), \quad (39)$$

and condition (21) holds for all  $u \in X$  and  $u_i \in \Omega_i$  ( $i = 1, \dots, n$ ). In view of (19) and the definitions of  $\lambda'$  and  $f$ , conditions (39) yield (37) and (38). Since  $\varepsilon/\lambda' < 1$ , condition (21) contradicts (14).  $\square$

The next statement is a localized version of Theorem 4.3.

**Corollary 4.9** (i) *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\gamma > 0$  and any  $\omega'_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and  $\xi \in ]0, \varphi^{-1}(\delta_1)[$ , there exists a  $\lambda \in ]\varphi(\xi), \delta_1[$  such that inequality (22) holds for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (37) and (38).*

(ii) *If  $\varphi \in \mathcal{C}^1$ , then inequality (22) in part (i) can be replaced by (23).*

In the Hölder setting, Theorem 4.3 and Corollary 4.9 yield the following statement. In view of Remark 2.3, we assume that  $q \leq 1$ .

**Corollary 4.10** *Let  $\alpha > 0$  and  $q \in ]0, 1]$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\alpha$ -transversal of order  $q$  at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\gamma > 0$  and any  $\omega'_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and  $\xi \in ]0, (\alpha\delta_1)^{\frac{1}{q}}[$ , there exists a  $\lambda \in ]\alpha^{-1}\xi^q, \delta_1[$  such that inequality (24) holds true for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (37) and (38), or all the more, such that inequality (25) holds true.*



*Remark 4.4* (i) On top of the explicitly given restriction  $\|\omega_i - \omega'_i\| < \lambda/\gamma$  in Theorem 4.3 (and similar conditions in its corollaries), which involves  $\gamma$ , the other conditions implicitly impose another one:

$$\begin{aligned} \|\omega_i - \omega'_i\| &\leq \|x - \bar{x}\| + \|\omega_i - x_i - x\| + \|\omega'_i - x_i - \bar{x}\| \\ &\leq \|x - \bar{x}\| + 2\xi < \lambda + 2\varphi^{-1}(\delta_1). \end{aligned}$$

This alternative restriction can be of importance when  $\gamma$  is small.

- (ii) It can be observed from the proof of Theorem 4.3 that the sufficient conditions for  $\varphi$ -transversality can be strengthened by adding another restriction on the choice of  $\xi$  and  $x_i$ :  $\varphi(\xi) < d(\bar{x}, \cap_{i=1}^n (\Omega_i - x_i))$ .
- (iii) The sufficient conditions for  $\varphi$ -semitransversality and  $\varphi$ -subtransversality in Theorems 4.1 and 4.2 are particular cases of those in Theorem 4.3, corresponding to setting  $\omega'_i := \bar{x}$  and  $x_1 = \dots = x_n$ , respectively.
- (iv) The statement of Theorem 4.3 and its corollaries can be simplified by dropping condition (38).
- (v) Inequalities (14), (22)–(25), which are crucial for checking nonlinear transversality, involve a collection of parameters:  $x, x_i \in X$  and  $\omega_i \in \Omega_i$ , which are related to another collection: a small number  $\xi > 0$  and points  $\omega'_i \in \Omega_i$  near  $\bar{x}$ . The value of  $\xi$  and magnitudes of  $\omega'_i - \bar{x}$  are directly controlled by the values of  $\delta_1$  and  $\delta_2$  in the definition of  $\varphi$ -transversality:  $\varphi(\xi) < \delta_1$  and  $\omega'_i \in B_{\delta_2}(\bar{x})$ . At the same time, taking into account that  $\lambda$  can be made arbitrarily close to  $\varphi(\xi)$ , the magnitudes of  $x - \bar{x}$ ,  $\omega_i - \omega'_i$  and  $x_i$  are determined by  $\delta_1$  and  $\delta_2$  indirectly; they are controlled by  $\xi$ : cf. conditions (37) and (38). Thus, the derived parameters  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  involved in (14) possess the natural properties: when  $\delta_1$  and  $\delta_2$  are small, the points  $x$  and  $\omega_i$  are near  $\bar{x}$  and the vectors  $x_i$  are small.
- (vi) In view of the definition of the parametric norm (12), if any of the inequalities (14), (22)–(25) holds true for some  $\gamma > 0$ , then it also holds for any  $\gamma' \in ]0, \gamma[$ .
- (vii) Even in the linear setting, the characterizations in Corollary 4.10 are new.

The next corollary provides a simplified (and weaker!) version of Theorem 4.3; cf. Remark 4.4(v).

**Corollary 4.11** *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\gamma > 0$ , inequality (14) holds for all  $x \in B_{\delta_1}(\bar{x})$ ,  $x_i \in X$  and  $\omega_i \in \Omega_i \cap B_{\delta_2 + \delta_1/\gamma}(\bar{x})$  ( $i = 1, \dots, n$ ) satisfying  $\varphi(\max_{1 \leq i \leq n} d(x_i + \bar{x}, \Omega_i)) < \delta_1$  and  $0 < \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| < \varphi^{-1}(\delta_1)$ .*

*Proof* Let  $\delta_1 > 0$ ,  $\delta_2 > 0$ ,  $\omega'_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ ,  $\xi \in ]0, \varphi^{-1}(\delta_1)[$ ,  $\lambda \in ]\varphi(\xi), \delta_1[$ , and points  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfy conditions (37) and (38). Then

$$\begin{aligned} \|x - \bar{x}\| &< \lambda < \delta_1, \quad \|\omega_i - \bar{x}\| \leq \|\omega'_i - \bar{x}\| + \|\omega_i - \omega'_i\| < \delta_2 + \lambda/\gamma < \delta_2 + \delta_1/\gamma, \\ d(x_i + \bar{x}, \Omega_i) &\leq \|x_i + \bar{x} - \omega'_i\| \leq \xi < \varphi^{-1}(\delta_1), \\ 0 < \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| &\leq \xi < \varphi^{-1}(\delta_1), \end{aligned}$$

i.e. points  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfy all the conditions in the corollary. Hence, inequality (14) holds. It follows from Theorem 4.3 that  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  with  $\delta_1$  and  $\delta_2$ .  $\square$

Sacrificing the estimates for  $\delta_1$  and  $\delta_2$  in Theorem 4.3, and Corollaries 4.9 and 4.11, we can formulate the following ‘ $\delta$ -free’ statement.

**Corollary 4.12** *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  if, for some  $\gamma > 0$  and all  $x \in X$  near  $\bar{x}$ ,  $x_i \in X$  ( $i = 1, \dots, n$ ) near 0 and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) near  $\bar{x}$  satisfying  $\max_{1 \leq i \leq n} \|\omega_i - x_i - x\| > 0$ , inequality (14) holds true. Moreover, inequality (14) can be replaced by its localized version (22), or by (23) if  $\varphi \in \mathcal{C}^1$ .*

*Remark 4.5* The sufficient conditions for  $\varphi$ -semitransversality and  $\varphi$ -transversality in Theorems 4.1 and 4.3 and their corollaries use the same (slope) inequalities (14), (22) and (23). Nevertheless, the sufficient conditions in Theorem 4.3 and Corollary 4.12 are stronger than the corresponding ones in Theorem 4.1 and Corollary 4.4, respectively, as they require the inequalities to be satisfied on a larger set of points. This is natural as  $\varphi$ -transversality is a stronger property than  $\varphi$ -semitransversality. At the same time, the ‘ $\delta$ -free’ versions in Corollaries 4.4 and 4.12 are almost identical: the only difference is the additional condition

$$\max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \leq \max_{1 \leq i \leq n} \|x_i\|$$

in Corollary 4.4. The sufficient condition in Corollary 4.12 is still acceptable for characterizing  $\varphi$ -transversality, but the one in Corollary 4.4 seems a little too strong for  $\varphi$ -semitransversality. That is why we prefer not to oversimplify these sufficient conditions.

## 5 Transversality and Regularity

In this section, we provide quantitative relations between the nonlinear transversality of collections of sets and the corresponding nonlinear regularity properties of set-valued mappings. Besides, nonlinear extensions of the new *transversality properties of a set-valued mapping to a set in the range space* due to Ioffe [28] are discussed.

### 5.1 Regularity of Set-Valued Mappings

Our model here is a set-valued mapping  $F : X \rightrightarrows Y$  between metric spaces. We consider its local regularity properties near a given point  $(\bar{x}, \bar{y}) \in \text{gph} F$ . The nonlinearity in the definitions of the properties is determined by a function  $\varphi \in \mathcal{C}$ .

Regularity of set-valued mappings have been intensively studied for decades due to their numerous important applications; see monographs [18, 28, 29, 49]. Nonlinear regularity properties have also been considered by many authors; cf. [9, 21–23, 26, 34, 35, 45, 48, 54, 61]. The relations between transversality and regularity properties are well known in the linear case [25, 27, 30–32, 36, 39, 40, 43] as well as in the Hölder setting [42]. Below we briefly discuss more general nonlinear models.

**Definition 5.1** The mapping  $F$  is

- (i)  $\varphi$ -semiregular at  $(\bar{x}, \bar{y})$  if there exists a  $\delta > 0$  such that

$$d(\bar{x}, F^{-1}(y)) \leq \varphi(d(y, \bar{y}))$$

for all  $y \in Y$  with  $\varphi(d(y, \bar{y})) < \delta$ ;

- (ii)  $\varphi$ -subregular at  $(\bar{x}, \bar{y})$  if there exist  $\delta_1 > 0$  and  $\delta_2 > 0$  such that

$$d(x, F^{-1}(\bar{y})) \leq \varphi(d(\bar{y}, F(x)))$$

for all  $x \in B_{\delta_2}(\bar{x})$  with  $\varphi(d(\bar{y}, F(x))) < \delta_1$ ;

(iii)  $\varphi$ -regular at  $(\bar{x}, \bar{y})$  if there exist  $\delta_1 > 0$  and  $\delta_2 > 0$  such that

$$d(x, F^{-1}(y)) \leq \varphi(d(y, F(x))) \quad (40)$$

for all  $x \in X$  and  $y \in Y$  with  $d(x, \bar{x}) + d(y, \bar{y}) < \delta_2$  and  $\varphi(d(y, F(x))) < \delta_1$ .

The function  $\varphi \in \mathcal{C}$  in the above definition plays the role of a kind of rate or modulus of the respective property. In the Hölder setting, i.e. when  $\varphi(t) := \alpha^{-1}t^q$  with  $\alpha > 0$  and  $q > 0$ , we refer to the respective properties in Definition 5.1 as  $\alpha$ -semiregularity,  $\alpha$ -subregularity and  $\alpha$ -regularity of order  $q$ . These regularity properties have been studied in [22, 23, 34, 42, 45, 48]. It is usually assumed that  $q \leq 1$ . The exact upper bound of all  $\alpha > 0$  such that a property holds with some  $\delta > 0$ , or  $\delta_1 > 0$  and  $\delta_2 > 0$ , is called the *modulus* of this property. We use notations  $\text{srg}_q[F](\bar{x}, \bar{y})$ ,  $\text{srg}_q[F](\bar{x}, \bar{y})$  and  $\text{rg}_q[F](\bar{x}, \bar{y})$  for the moduli of the respective properties. If a property does not hold, then by convention the respective modulus equals 0. With  $q = 1$  (linear case), the properties are called *metric semiregularity*, *subregularity* and *regularity*, respectively; cf. [12, 18, 28, 32, 49, 56].

The following assertion is a direct consequence of Definition 5.1.

**Proposition 5.1** *If  $F$  is  $\varphi$ -regular at  $(\bar{x}, \bar{y})$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then it is  $\varphi$ -semiregular at  $(\bar{x}, \bar{y})$  with  $\delta := \min\{\delta_1, \varphi(\delta_2)\}$  and  $\varphi$ -subregular at  $(\bar{x}, \bar{y})$  with  $\delta_1$  and  $\delta_2$ .*

Note the combined inequality  $d(x, \bar{x}) + d(y, \bar{y}) < \delta_2$  employed in part (iii) of Definition 5.1 instead of the more traditional separate conditions  $x \in B_{\delta_2}(\bar{x})$  and  $y \in B_{\delta_2}(\bar{y})$ . This replacement does not affect the property of  $\varphi$ -regularity itself, but can have an effect on the value of  $\delta_2$ . Employing this inequality makes the property a direct analogue of the metric characterization of  $\varphi$ -transversality in Theorem 3.2 and is convenient for establishing relations between the regularity and transversality properties. The next proposition provides also an important special case when the point  $x$  in (40) can be fixed:  $x = \bar{x}$ .

**Proposition 5.2** *Let  $\delta_1 > 0$  and  $\delta_2 > 0$ . Consider the following conditions:*

- (a) *inequality (40) holds for all  $x \in B_{\delta_2}(\bar{x})$  and  $y \in B_{\delta_2}(\bar{y})$  with  $\varphi(d(y, F(x))) < \delta_1$ ;*
- (b) *inequality (40) holds for all  $x \in X$  and  $y \in Y$  with  $d(x, \bar{x}) + d(y, \bar{y}) < \delta_2$  and  $\varphi(d(y, F(x))) < \delta_1$ ;*
- (c)  *$d(\bar{x}, F^{-1}(y)) \leq \varphi(d(y, F(\bar{x})))$  for all  $y \in B_{\delta_2}(\bar{y})$  with  $\varphi(d(y, F(\bar{x}))) < \delta_1$ .*

*Then*

- (i) *(a)  $\Rightarrow$  (b)  $\Rightarrow$  (c). Moreover, condition (b) implies (a) with  $\delta'_2 := \delta_2/2$  in place of  $\delta_2$ .*
- (ii) *If  $X$  is a normed space,  $Y = X^n$  for some  $n \in \mathbb{N}$ ,  $\bar{y} = (\bar{x}_1, \dots, \bar{x}_n)$  and  $F : X \rightrightarrows X^n$  is given by*

$$F(x) := (\Omega_1 - x) \times \dots \times (\Omega_n - x), \quad x \in X, \quad (41)$$

*where  $\Omega_1, \dots, \Omega_n \subset X$ , then (b)  $\Leftrightarrow$  (c).*

*Proof* (i) All the implications are straightforward.

- (ii) In view of (i), we only need to prove (c)  $\Rightarrow$  (b). Suppose condition (c) is satisfied. Let  $x \in X$ ,  $y = (x_1, \dots, x_n) \in X^n$ ,  $\|x - \bar{x}\| + \|y - \bar{y}\| < \delta_2$  and  $\varphi(d(y, F(x))) < \delta_1$ . Set  $x'_i := x_i + x - \bar{x}$  ( $i = 1, \dots, n$ ) and  $y' := (x'_1, \dots, x'_n)$ . Then

$$\begin{aligned} \|y' - \bar{y}\| &\leq \|y' - y\| + \|y - \bar{y}\| = \|x - \bar{x}\| + \|y - \bar{y}\| < \delta_2, \\ d(x, F^{-1}(y)) &= d(x, \cap_{i=1}^n (\Omega_i - x_i)) = d(\bar{x}, \cap_{i=1}^n (\Omega_i - x'_i)) = d(\bar{x}, F^{-1}(y')), \\ d(y, F(x)) &= \max_{1 \leq i \leq n} d(x_i, \Omega_i - x) = \max_{1 \leq i \leq n} d(x'_i, \Omega_i - \bar{x}) = d(y', F(\bar{x})). \end{aligned}$$

and, thanks to (c),  $d(x, F^{-1}(y)) \leq \varphi(d(y, F(x)))$ .  $\square$

The set-valued mapping (41) plays the key role in establishing relations between the regularity and transversality properties. It was most likely first used by Ioffe in [25]. Observe that  $F^{-1}(x_1, \dots, x_n) = (\Omega_1 - x_1) \cap \dots \cap (\Omega_n - x_n)$  for all  $x_1, \dots, x_n \in X$  and, if  $\bar{x} \in \bigcap_{i=1}^n \Omega_i$ , then  $(0, \dots, 0) \in F(\bar{x})$ .

**Theorem 5.1** *Let  $\Omega_1, \dots, \Omega_n$  be subsets of a normed space  $X$ ,  $\bar{x} \in \bigcap_{i=1}^n \Omega_i$ ,  $\varphi \in \mathcal{C}$ , and  $F$  be defined by (41).*

- (i) *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -semitransversal at  $\bar{x}$  with some  $\delta > 0$  if and only if  $F$  is  $\varphi$ -semiregular at  $(\bar{x}, (0, \dots, 0))$  with  $\delta$ .*
- (ii) *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -subtransversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if and only if  $F$  is  $\varphi$ -subregular at  $(\bar{x}, (0, \dots, 0))$  with  $\delta_1$  and  $\delta_2$ .*
- (iii) *If  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then  $F$  is  $\varphi$ -regular at  $(\bar{x}, (0, \dots, 0))$  with any  $\delta'_1 \in ]0, \delta_1]$  and  $\delta'_2 > 0$  satisfying  $\varphi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2$ .*  
*Conversely, if  $F$  is  $\varphi$ -regular at  $(\bar{x}, (0, \dots, 0))$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  with any  $\delta'_1 \in ]0, \delta_1]$  and  $\delta'_2 > 0$  satisfying  $\varphi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2$ .*

*Proof* (i) and (ii) follow from Theorem 3.1(i) and (ii), respectively, while (iii) is a consequence of Theorem 3.2.  $\square$

The next corollary provides  $\delta$ -free versions of the assertions in Theorem 5.1.

**Corollary 5.1** *Let  $\Omega_1, \dots, \Omega_n$  be subsets of a normed space  $X$ ,  $\bar{x} \in \bigcap_{i=1}^n \Omega_i$ ,  $\varphi \in \mathcal{C}$ , and  $F$  be defined by (41). The collection  $\{\Omega_1, \dots, \Omega_n\}$  is*

- (i)  *$\varphi$ -semitransversal at  $\bar{x}$  if and only if  $F$  is  $\varphi$ -semiregular at  $(\bar{x}, (0, \dots, 0))$ ;*
- (ii)  *$\varphi$ -subtransversal at  $\bar{x}$  if and only if  $F$  is  $\varphi$ -subregular at  $(\bar{x}, (0, \dots, 0))$ ;*
- (iii)  *$\varphi$ -transversal at  $\bar{x}$  if and only if  $F$  is  $\varphi$ -regular at  $(\bar{x}, (0, \dots, 0))$ .*

*Remark 5.1* (i) In the Hölder setting Corollary 5.1 reduces to [42, Proposition 9].  
(ii) Apart from the mapping  $F$  defined by (41), in the case of two sets other set-valued mappings can be used to ensure similar equivalences between the transversality and regularity properties; see [28].

In view of Theorem 5.1, the nonlinear transversality properties of collections of sets can be viewed as particular cases of the corresponding nonlinear regularity properties of set-valued mappings. We are going to show that the two popular models are in a sense equivalent.

Given an arbitrary set-valued mapping  $F : X \rightrightarrows Y$  between metric spaces and a point  $(\bar{x}, \bar{y}) \in \text{gph}F$ , we can consider the two sets:

$$\Omega_1 := \text{gph}F, \quad \Omega_2 := X \times \{\bar{y}\} \quad (42)$$

in the product space  $X \times Y$ . Note that  $(\bar{x}, \bar{y}) \in \Omega_1 \cap \Omega_2 = F^{-1}(\bar{y}) \times \{\bar{y}\}$ . To establish the relationship between the two sets of properties, we have to assume in the next two theorems that  $X$  and  $Y$  are normed vector spaces.

**Theorem 5.2** *Let  $X$  and  $Y$  be normed spaces,  $F : X \rightrightarrows Y$ ,  $(\bar{x}, \bar{y}) \in \text{gph}F$ , and  $\varphi \in \mathcal{C}$ . Let  $\Omega_1$  and  $\Omega_2$  be defined by (42), and  $\psi(t) := \varphi(2t) + t$  for all  $t \geq 0$ .*

- (i) *If  $F$  is  $\varphi$ -semiregular at  $(\bar{x}, \bar{y})$  with some  $\delta > 0$ , then  $\{\Omega_1, \Omega_2\}$  is  $\psi$ -semitransversal at  $(\bar{x}, \bar{y})$  with  $\delta' := \delta + \varphi^{-1}(\delta)/2$ .*
- (ii) *If  $F$  is  $\varphi$ -subregular at  $(\bar{x}, \bar{y})$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then  $\{\Omega_1, \Omega_2\}$  is  $\psi$ -subtransversal at  $(\bar{x}, \bar{y})$  with any  $\delta'_1 > 0$  and  $\delta'_2 > 0$  such that  $\varphi(2\psi^{-1}(\delta'_1)) \leq \delta_1$  and  $\psi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2$ .*

- (iii) If  $F$  is  $\varphi$ -regular at  $(\bar{x}, \bar{y})$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then  $\{\Omega_1, \Omega_2\}$  is  $\psi$ -transversal at  $(\bar{x}, \bar{y})$  with any  $\delta'_1 > 0$  and  $\delta'_2 > 0$  such that  $\varphi(2\psi^{-1}(\delta'_1)) \leq \delta_1$  and  $\psi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2/2$ .

*Proof* Observe that  $\psi \in \mathcal{C}$ ,  $\varphi(2\psi^{-1}(t)) + \psi^{-1}(t) = t$  and  $\psi(\varphi^{-1}(t)/2) = t + \varphi^{-1}(t)/2$  for all  $t \geq 0$ .

- (i) Let  $F$  be  $\varphi$ -semiregular at  $(\bar{x}, \bar{y})$  with some  $\delta > 0$ . Set  $\delta' := \delta + \varphi^{-1}(\delta)/2 = \psi(\varphi^{-1}(\delta)/2)$ . Let  $\rho \in ]0, \delta'[$  and  $(u_1, v_1), (u_2, v_2) \in \psi^{-1}(\rho)\mathbb{B}$ . Set  $y' := \bar{y} + v_1 - v_2$ . Observe that

$$\begin{aligned} (\Omega_1 - (u_1, v_1)) \cap (\Omega_2 - (u_2, v_2)) &= (\text{gph } F - (u_1, v_1)) \cap (X \times \{\bar{y} - v_2\}) \\ &= (F^{-1}(y') - u_1) \times \{\bar{y} - v_2\}. \end{aligned}$$

We have  $\|y' - \bar{y}\| = \|v_1 - v_2\| \leq \|v_1\| + \|v_2\| < 2\psi^{-1}(\rho)$ , and consequently,  $\varphi(\|y' - \bar{y}\|) < \varphi(2\psi^{-1}(\rho)) < \varphi(2\psi^{-1}(\delta')) = \delta$ . By Definition 5.1(i),

$$\begin{aligned} d(\bar{x}, F^{-1}(y') - u_1) &\leq d(\bar{x}, F^{-1}(y')) + \|u_1\| \\ &\leq \varphi(\|y' - \bar{y}\|) + \|u_1\| < \varphi(2\psi^{-1}(\rho)) + \psi^{-1}(\rho) = \rho, \end{aligned}$$

and consequently,

$$\begin{aligned} d((\bar{x}, \bar{y}), (\Omega_1 - (u_1, v_1)) \cap (\Omega_2 - (u_2, v_2))) &\leq \max\{d(\bar{x}, F^{-1}(y') - u_1), \|v_2\|\} \\ &< \max\{\rho, \psi^{-1}(\rho)\} = \rho; \end{aligned}$$

hence,

$$(\Omega_1 - (u_1, v_1)) \cap (\Omega_2 - (u_2, v_2)) \cap B_\rho(\bar{x}, \bar{y}) \neq \emptyset. \quad (43)$$

By Definition 2.1(i),  $\{\Omega_1, \Omega_2\}$  is  $\varphi$ -semitransversal at  $(\bar{x}, \bar{y})$  with  $\delta'$ .

- (ii) Let  $F$  be  $\varphi$ -subregular at  $(\bar{x}, \bar{y})$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ . Choose numbers  $\delta'_1 > 0$  and  $\delta'_2 > 0$  such that  $\varphi(2\psi^{-1}(\delta'_1)) \leq \delta_1$  and  $\psi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2$ . Let  $\rho \in ]0, \delta'_1[$  and  $(x, y) \in B_{\delta'_2}(\bar{x}, \bar{y})$  with  $\psi(\max\{d((x, y), \Omega_1), d((x, y), \Omega_2)\}) < \rho$ , i.e.  $\|y - \bar{y}\| < \psi^{-1}(\rho)$  and there exists a point  $(x_1, y_1) \in \text{gph } F$  such that  $\|(x, y) - (x_1, y_1)\| < \psi^{-1}(\rho)$ . Then

$$\begin{aligned} \|x_1 - \bar{x}\| &\leq \|x_1 - x\| + \|x - \bar{x}\| < \psi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2, \\ d(\bar{y}, F(x_1)) &\leq \|y_1 - \bar{y}\| \leq \|y - \bar{y}\| + \|y_1 - y\| < 2\psi^{-1}(\rho), \end{aligned}$$

and consequently,  $\varphi(d(\bar{y}, F(x_1))) < \varphi(2\psi^{-1}(\rho)) < \varphi(2\psi^{-1}(\delta'_1)) \leq \delta_1$ . Choose a positive  $\varepsilon < 2\psi^{-1}(\rho) - d(\bar{y}, F(x_1))$ . By Definition 5.1(ii), there exists an  $x' \in F^{-1}(\bar{y})$  such that  $\|x' - x_1\| < \varphi(d(\bar{y}, F(x_1)) + \varepsilon) < \varphi(2\psi^{-1}(\rho))$ . Hence,  $(x', \bar{y}) \in \Omega_1 \cap \Omega_2$  and

$$\begin{aligned} \|x - x'\| &\leq \|x_1 - x'\| + \|x - x_1\| < \varphi(2\psi^{-1}(\rho)) + \psi^{-1}(\rho) = \rho, \\ \|y - \bar{y}\| &< \psi^{-1}(\rho) < \rho. \end{aligned}$$

Thus,  $\Omega_1 \cap \Omega_2 \cap B_\rho(x, y) \neq \emptyset$ . By Definition 2.1(ii),  $\{\Omega_1, \Omega_2\}$  is  $\psi$ -subtransversal at  $(\bar{x}, \bar{y})$  with  $\delta'_1$  and  $\delta'_2$ .

- (iii) Let  $F$  be  $\varphi$ -regular at  $(\bar{x}, \bar{y})$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ . Choose numbers  $\delta'_1 > 0$  and  $\delta'_2 > 0$  such that  $\varphi(2\psi^{-1}(\delta'_1)) \leq \delta_1$  and  $\psi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2/2$ . Let  $\rho \in ]0, \delta'_1[$ ,  $(x_1, y_1) \in \text{gph } F \cap B_{\delta'_2}(\bar{x}, \bar{y})$ ,  $x_2 \in B_{\delta'_2}(\bar{x})$  and  $(u_1, v_1), (u_2, v_2) \in \psi^{-1}(\rho)\mathbb{B}$ . Set  $y' := y_1 + v_1 - v_2$ . Then

$$\begin{aligned} \|x_1 - \bar{x}\| + \|y' - \bar{y}\| &\leq \|v_1\| + \|v_2\| + \|x_1 - \bar{x}\| + \|y_1 - \bar{y}\| < 2\psi^{-1}(\delta'_1) + 2\delta'_2 \leq \delta_2, \\ \varphi(d(y', F(x_1))) &\leq \varphi(\|y' - y_1\|) \leq \varphi(\|v_1\| + \|v_2\|) < \varphi(2\psi^{-1}(\delta'_1)) \leq \delta_1. \end{aligned}$$

Choose a positive  $\varepsilon < 2(\psi^{-1}(\rho) - \max\{\|v_1\|, \|v_2\|\})$ . By Definition 5.1(iii), there exists an  $x' \in F^{-1}(y')$  such that

$$\|x_1 - x'\| < \varphi(\|y' - y_1\| + \varepsilon) \leq \varphi(2\max\{\|v_1\|, \|v_2\|\} + \varepsilon) < \varphi(2\psi^{-1}(\rho)).$$

Denote  $\hat{x} := x' - x_1 - u_1$  and  $\hat{y} := y' - y_1 - v_1$ . Thus,  $(x', y') \in \Omega_1$  and  $(\hat{x}, \hat{y}) \in \Omega_1 - (x_1, y_1) - (u_1, v_1)$ . At the same time,  $\hat{y} = -v_2$  and  $(\hat{x}, \hat{y}) \in \Omega_2 - (x_2, \bar{y}) - (u_2, v_2)$ . Moreover,

$$\begin{aligned} \|\hat{x}\| &\leq \|x' - x_1\| + \|u_1\| < \varphi(2\psi^{-1}(\rho)) + \psi^{-1}(\rho) = \rho, \\ \|\hat{y}\| &= \|v_2\| < \psi^{-1}(\rho) < \rho; \end{aligned}$$

hence  $(x', y') \in \rho\mathbb{B}$ . By Definition 2.1(iii),  $\{\Omega_1, \Omega_2\}$  is  $\psi$ -transversal at  $(\bar{x}, \bar{y})$  with  $\delta'_1$  and  $\delta'_2$ .  $\square$

**Theorem 5.3** *Let  $X$  and  $Y$  be normed spaces,  $F : X \rightrightarrows Y$ ,  $(\bar{x}, \bar{y}) \in \text{gph}F$ , and  $\varphi \in \mathcal{C}$ . Let  $\Omega_1$  and  $\Omega_2$  be defined by (42), and  $\psi(t) := \varphi(t/2)$  for all  $t \geq 0$ .*

- (i) *If  $\{\Omega_1, \Omega_2\}$  is  $\varphi$ -semitransversal at  $(\bar{x}, \bar{y})$  with some  $\delta > 0$ , then  $F$  is  $\psi$ -semiregular at  $(\bar{x}, \bar{y})$  with  $\delta$ .*
- (ii) *If  $\{\Omega_1, \Omega_2\}$  is  $\varphi$ -subtransversal at  $(\bar{x}, \bar{y})$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then  $F$  is  $\psi$ -subregular at  $(\bar{x}, \bar{y})$  with  $\delta'_1 := \min\{\delta_1, \psi(2\delta_2)\}$  and  $\delta_2$ .*
- (iii) *If  $\{\Omega_1, \Omega_2\}$  is  $\varphi$ -transversal at  $(\bar{x}, \bar{y})$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then  $F$  is  $\psi$ -regular at  $(\bar{x}, \bar{y})$  with any  $\delta'_1 \in ]0, \delta_1[$  and  $\delta'_2 > 0$  such that  $\psi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2$ .*

*Proof* Observe that  $\psi \in \mathcal{C}$ .

- (i) Let  $\{\Omega_1, \Omega_2\}$  be  $\varphi$ -semitransversal at  $(\bar{x}, \bar{y})$  with some  $\delta > 0$ . By Definition 2.1(i), condition (43) is satisfied for all  $\rho \in ]0, \delta[$  and  $(u_1, v_1), (u_2, v_2) \in \varphi^{-1}(\rho)\mathbb{B}$ . Let  $y \in Y$  with  $\rho_0 := \psi(\|y - \bar{y}\|) < \delta$ . Choose a  $\rho \in ]\rho_0, \delta[$  and observe that

$$\varphi\left(\left\|\left(0, \frac{y - \bar{y}}{2}\right)\right\|\right) = \varphi\left(\frac{\|y - \bar{y}\|}{2}\right) = \psi(\|y - \bar{y}\|) < \rho.$$

In view of (43), we can find  $(x_1, y_1) \in \text{gph}F$  and  $x_2 \in X$  such that

$$(x_1, y_1) - \left(0, \frac{y - \bar{y}}{2}\right) = (x_2, \bar{y}) - \left(0, \frac{\bar{y} - y}{2}\right) \in B_\rho(\bar{x}, \bar{y}).$$

Hence,  $y_1 = \bar{y} + 2\frac{y - \bar{y}}{2} = y$ ,  $x_1 \in F^{-1}(y)$ ,  $\|x_1 - \bar{x}\| < \rho$ , and consequently,  $d(\bar{x}, F^{-1}(y)) < \rho$ . Letting  $\rho \downarrow \rho_0$ , we obtain  $d(\bar{x}, F^{-1}(y)) \leq \psi(\|y - \bar{y}\|)$ . By Definition 5.1(i),  $F$  is  $\psi$ -semiregular at  $(\bar{x}, \bar{y})$  with  $\delta$ .

- (ii) Let  $\{\Omega_1, \Omega_2\}$  be  $\varphi$ -subtransversal at  $(\bar{x}, \bar{y})$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ . By Definition 2.1(ii),  $\text{gph}F \cap (X \times \{\bar{y}\}) \cap B_\rho(x, y) \neq \emptyset$  for all  $\rho \in ]0, \delta_1[$  and  $(x, y) \in B_{\delta_2}(\bar{x}, \bar{y})$  with  $\varphi(d((x, y), \text{gph}F)) < \rho$  and  $\varphi(\|y - \bar{y}\|) < \rho$ . Set  $\delta'_1 := \min\{\delta_1, \psi(2\delta_2)\}$ . Let  $x \in B_{\delta_2}(\bar{x})$  and  $\psi(d(\bar{y}, F(x))) < \delta'_1$ . Choose a  $y \in F(x)$  such that  $\rho_0 := \psi(\|\bar{y} - y\|) < \delta'_1$ , and a  $\rho \in ]\rho_0, \delta'_1[$ . Set  $\hat{y} := \frac{y + \bar{y}}{2}$ . Observe that

$$\begin{aligned} \|\hat{y} - y\| &= \|\hat{y} - \bar{y}\| = \frac{\|\bar{y} - y\|}{2} = \frac{\psi^{-1}(\rho_0)}{2} < \frac{\psi^{-1}(\rho)}{2} = \varphi^{-1}(\rho), \\ \|\hat{y} - \bar{y}\| &< \frac{\psi^{-1}(\rho)}{2} < \frac{\psi^{-1}(\delta'_1)}{2} \leq \delta_2. \end{aligned}$$

Thus,  $\rho \in ]0, \delta_1[$ ,  $(x, \hat{y}) \in B_{\delta_2}(\bar{x}, \bar{y})$ ,  $\varphi(d((x, \hat{y}), \text{gph}F)) \leq \varphi(\|\hat{y} - y\|) < \rho$  and  $\varphi(\|\hat{y} - \bar{y}\|) < \rho$ . Hence,  $\text{gph}F \cap (X \times \{\bar{y}\}) \cap B_\rho(x, \hat{y}) \neq \emptyset$ , and consequently,  $d(x, F^{-1}(\bar{y})) < \rho$ . Letting  $\rho \downarrow \rho_0$ , we obtain  $d(x, F^{-1}(\bar{y})) \leq \psi(\|\bar{y} - y\|)$ . Taking the infimum in the right-hand side of this inequality over  $y \in F(x)$ , we conclude that  $F$  is  $\psi$ -subregular at  $(\bar{x}, \bar{y})$  with  $\delta'_1$  and  $\delta_2$  in view of Definition 5.1(ii).



- (iii) Let  $\{\Omega_1, \Omega_2\}$  be  $\varphi$ -transversal at  $(\bar{x}, \bar{y})$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , i.e. for all  $\rho \in ]0, \delta_1[$ ,  $(x', y') \in \text{gph}F \cap B_{\delta_2}(\bar{x}, \bar{y})$ ,  $u_1 \in X$  and  $v_1, v_2 \in Y$  with  $\varphi(\max\{\|u_1\|, \|v_1\|, \|v_2\|\}) < \rho$ , it holds

$$(\text{gph}F - (x', y') - (u_1, v_1)) \cap (X \times \{-v_2\}) \cap (\rho\mathbb{B}) \neq \emptyset,$$

or equivalently,  $d(x' + u_1, F^{-1}(y' + v_1 - v_2)) < \rho$ . In other words,  $d(x, F^{-1}(y)) < \rho$  for all  $\rho \in ]0, \delta_1[$ ,  $(x', y') \in \text{gph}F \cap B_{\delta_2}(\bar{x}, \bar{y})$ ,  $x \in X$  and  $y \in Y$  with  $\|x - x'\| < \varphi^{-1}(\rho)$  and  $\|y - y'\| < 2\varphi^{-1}(\rho)$ . Choose numbers  $\delta'_1 \in ]0, \delta_1[$  and  $\delta'_2 > 0$  such that  $\psi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2$ . Let  $x \in X$  and  $y \in Y$  with  $\|x - \bar{x}\| + \|y - \bar{y}\| < \delta'_2$  and  $\psi(d(y, F(x))) < \delta'_1$ . Choose a  $y' \in F(x)$  such that  $\rho_0 := \psi(\|y - y'\|) < \delta'_1$  and a  $\rho \in ]\rho_0, \delta'_1[$ . Then  $\rho \in ]0, \delta_1[$ ,  $(x, y') \in \text{gph}F$ ,  $\|x - \bar{x}\| < \delta'_2 < \delta_2$ ,  $\|y' - \bar{y}\| \leq \|y' - y\| + \|y - \bar{y}\| < \psi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2$  and  $\|y - y'\| < \psi^{-1}(\rho) = 2\varphi^{-1}(\rho)$ . Hence,  $d(x, F^{-1}(y)) < \rho$ . Letting  $\rho \downarrow \rho_0$ , we obtain  $d(x, F^{-1}(y)) \leq \psi(\|y - y'\|)$ . Taking the infimum in the right-hand side of this inequality over  $y' \in F(x)$ , we conclude that  $F$  is  $\psi$ -regular at  $(\bar{x}, \bar{y})$  with  $\delta'_1$  and  $\delta'_2$  in view of Definition 5.1(iii).  $\square$

The next corollary of Theorems 5.2 and 5.3 provides qualitative relations between the regularity and transversality properties.

**Corollary 5.2** *Let  $X$  and  $Y$  be normed spaces,  $F : X \rightrightarrows Y$ ,  $(\bar{x}, \bar{y}) \in \text{gph}F$ , and  $\varphi \in \mathcal{C}$ . Let  $\Omega_1$  and  $\Omega_2$  be defined by (42),  $\psi_1(t) := \varphi(2t) + t$  and  $\psi_2(t) := \varphi(t/2)$  for all  $t \geq 0$ .*

- (i) *If  $F$  is  $\varphi$ -(semi-/sub-)regular at  $(\bar{x}, \bar{y})$ , then  $\{\Omega_1, \Omega_2\}$  is  $\psi_1$ -(semi-/sub-)transversal at  $(\bar{x}, \bar{y})$ .*  
(ii) *If  $\{\Omega_1, \Omega_2\}$  is  $\varphi$ -(semi-/sub-)transversal at  $(\bar{x}, \bar{y})$ , then  $F$  is  $\psi_2$ -(semi-/sub-)regular at  $(\bar{x}, \bar{y})$ .*

The next statement addresses the Hölder setting. It is a consequence of Theorems 5.2 and 5.3 with  $\varphi(t) := \alpha^{-1}t^q$  for some  $\alpha > 0$ ,  $q > 0$  and all  $t \geq 0$ .

**Corollary 5.3** *Let  $X$  and  $Y$  be normed spaces,  $F : X \rightrightarrows Y$ ,  $(\bar{x}, \bar{y}) \in \text{gph}F$ ,  $\alpha > 0$  and  $q > 0$ . Let  $\Omega_1$  and  $\Omega_2$  be defined by (42),  $\alpha_1 := 2^{-q}\alpha$ ,  $\alpha_2 := 2^q\alpha$ , and  $\psi(t) := \alpha^{-1}t^q + t$  for all  $t \geq 0$ .*

- (i) *If  $F$  is  $\alpha$ -semiregular of order  $q$  at  $(\bar{x}, \bar{y})$  with some  $\delta > 0$ , then  $\{\Omega_1, \Omega_2\}$  is  $\psi$ -semitransversal at  $(\bar{x}, \bar{y})$  with  $\delta' := \delta + (\alpha\delta)^{\frac{1}{q}}/2$ .  
If  $\{\Omega_1, \Omega_2\}$  is  $\alpha$ -semitransversal of order  $q$  at  $(\bar{x}, \bar{y})$  with some  $\delta > 0$ , then  $F$  is  $\alpha_2$ -semiregular of order  $q$  at  $(\bar{x}, \bar{y})$  with  $\delta$ .*  
(ii) *Let  $q \leq 1$ . If  $F$  is  $\alpha$ -subregular of order  $q$  at  $(\bar{x}, \bar{y})$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then  $\{\Omega_1, \Omega_2\}$  is  $\psi$ -subtransversal at  $(\bar{x}, \bar{y})$  with any  $\delta'_1 > 0$  and  $\delta'_2 > 0$  such that  $(2\psi^{-1}(\delta'_1))^q \leq \alpha\delta_1$  and  $\psi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2$ .  
If  $\{\Omega_1, \Omega_2\}$  is  $\alpha$ -subtransversal of order  $q$  at  $(\bar{x}, \bar{y})$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then  $F$  is  $\alpha_2$ -subregular of order  $q$  at  $(\bar{x}, \bar{y})$  with  $\delta'_1 := \min\{\delta_1, \alpha^{-1}\delta_2^q\}$  and  $\delta_2$ .*  
(iii) *Let  $q \leq 1$ . If  $F$  is  $\alpha$ -regular of order  $q$  at  $(\bar{x}, \bar{y})$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then  $\{\Omega_1, \Omega_2\}$  is  $\psi$ -transversal at  $(\bar{x}, \bar{y})$  with any  $\delta'_1 > 0$  and  $\delta'_2 > 0$  such that  $(2\psi^{-1}(\delta'_1))^q \leq \alpha\delta_1$  and  $\psi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2/2$ .  
If  $\{\Omega_1, \Omega_2\}$  is  $\alpha$ -transversal of order  $q$  at  $(\bar{x}, \bar{y})$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then  $F$  is  $\alpha_2$ -regular of order  $q$  at  $(\bar{x}, \bar{y})$  with any  $\delta'_1 \in ]0, \delta_1[$  and  $\delta'_2 > 0$  such that  $2(\alpha\delta'_1)^{\frac{1}{q}} + \delta'_2 \leq \delta_2$ .*

In view of Corollary 5.3, Hölder transversality properties of  $\{\Omega_1, \Omega_2\}$  imply the corresponding Hölder regularity properties of  $F$ , while Hölder regularity properties of  $F$  imply certain ‘Hölder-type’ transversality properties of  $\{\Omega_1, \Omega_2\}$  determined by the function  $\psi$ . Utilizing Proposition 2.1, they can be approximated by proper Hölder (or even linear) transversality properties.



**Corollary 5.4** *Let  $X$  and  $Y$  be normed spaces,  $F : X \rightrightarrows Y$ ,  $(\bar{x}, \bar{y}) \in \text{gph} F$ ,  $\alpha > 0$  and  $q > 0$ . Let  $\Omega_1$  and  $\Omega_2$  be defined by (42) and  $\alpha_1 := 2^{-q}\alpha$ . If  $F$  is  $\alpha$ –(semi-/sub-) transversal at  $(\bar{x}, \bar{y})$ , then  $\{\Omega_1, \Omega_2\}$  is  $\alpha'$ –(semi-/sub-)transversal of order  $q'$  at  $\bar{x}$ , where:*

- (i) *if  $q < 1$ , then  $q' = q$  and  $\alpha'$  is any number in  $]0, \alpha_1[$ ;*
- (ii) *if  $q = 1$ , then  $q' = 1$  and  $\alpha' := (1 + \alpha_1^{-1})^{-1}$ ;*
- (iii) *if  $q > 1$ , then  $q' = 1$  and  $\alpha'$  is any number in  $]0, 1[$ .*

Thanks to Corollaries 5.3 and 5.4, in the case  $q \in ]0, 1[$  we have full equivalence between the two sets of properties. The following corollary recaptures [42, Proposition 10].

**Corollary 5.5** *Let  $X$  and  $Y$  be normed spaces,  $F : X \rightrightarrows Y$ ,  $(\bar{x}, \bar{y}) \in \text{gph} F$ , and  $q \in ]0, 1[$ . Let  $\Omega_1$  and  $\Omega_2$  be defined by (42).*

- (i)  *$\{\Omega_1, \Omega_2\}$  is  $\alpha$ –semitransversal of order  $q$  at  $(\bar{x}, \bar{y})$  if and only if  $F$  is semiregular of order  $q$  at  $(\bar{x}, \bar{y})$ . Moreover,*

$$\frac{\text{se}rg_q[F](\bar{x}, \bar{y})}{\text{se}rg_q[F](\bar{x}, \bar{y}) + 2^q} \leq \text{se}tr_q[\Omega_1, \Omega_2](\bar{x}) \leq \frac{\text{se}rg_q[F](\bar{x}, \bar{y})}{2^q}.$$

- (ii)  *$\{\Omega_1, \Omega_2\}$  is  $\alpha$ –subtransversal of order  $q$  at  $(\bar{x}, \bar{y})$  if and only if  $F$  is subregular of order  $q$  at  $(\bar{x}, \bar{y})$ . Moreover,*

$$\frac{\text{srg}_q[F](\bar{x}, \bar{y})}{\text{srg}_q[F](\bar{x}, \bar{y}) + 2^q} \leq \text{str}_q[\Omega_1, \Omega_2](\bar{x}) \leq \frac{\text{srg}_q[F](\bar{x}, \bar{y})}{2^q}.$$

- (iii)  *$\{\Omega_1, \Omega_2\}$  is  $\alpha$ –transversal of order  $q$  at  $(\bar{x}, \bar{y})$  if and only if  $F$  is regular of order  $q$  at  $(\bar{x}, \bar{y})$ . Moreover,*

$$\frac{\text{rg}_q[F](\bar{x}, \bar{y})}{\text{rg}_q[F](\bar{x}, \bar{y}) + 2^q} \leq \text{tr}_q[\Omega_1, \Omega_2](\bar{x}) \leq \frac{\text{rg}_q[F](\bar{x}, \bar{y})}{2^q}.$$

## 5.2 Transversality of a Mapping to a Set in the Range Space

Finally, we briefly discuss metric characterizations of nonlinear extensions of the new transversality properties of a set-valued mapping to a set in the range space due to Ioffe [27, 28]. For geometric and subdifferential/normal cone characterizations of the properties, we refer the reader to [14–16]. In the rest of this section,  $F : X \rightrightarrows Y$  is a set-valued mapping between normed spaces,  $(\bar{x}, \bar{y}) \in \text{gph} F$ ,  $S$  is a subset of  $Y$ ,  $\bar{y} \in S$ , and  $\varphi \in \mathcal{C}$ .

**Definition 5.2** The mapping  $F$  is

- (i)  $\varphi$ –semitransversal to  $S$  at  $(\bar{x}, \bar{y})$  if  $\{\text{gph} F, X \times S\}$  is  $\varphi$ –semitransversal at  $(\bar{x}, \bar{y})$ , i.e. there exists a  $\delta > 0$  such that

$$(\text{gph} F - (u_1, v_1)) \cap (X \times (S - v_2)) \cap B_\rho(\bar{x}, \bar{y}) \neq \emptyset$$

for all  $\rho \in ]0, \delta[$ ,  $u_1 \in X$ ,  $v_1, v_2 \in Y$  with  $\varphi(\max\{\|u_1\|, \|v_1\|, \|v_2\|\}) < \rho$ ;

- (ii)  $\varphi$ –subtransversal to  $S$  at  $(\bar{x}, \bar{y})$  if  $\{\text{gph} F, X \times S\}$  is  $\varphi$ –subtransversal at  $(\bar{x}, \bar{y})$ , i.e. there exist  $\delta_1 > 0$  and  $\delta_2 > 0$  such that

$$\text{gph} F \cap (X \times S) \cap B_\rho(x, y) \neq \emptyset$$

for all  $\rho \in ]0, \delta_1[$  and  $(x, y) \in B_{\delta_2}(\bar{x}, \bar{y})$  with  $\varphi(\max\{d((x, y), \text{gph} F), d(y, S)\}) < \rho$ ;

- (iii)  $\varphi$ -transversal to  $S$  at  $(\bar{x}, \bar{y})$  if  $\{\text{gph}F, X \times S\}$  is  $\varphi$ -transversal at  $(\bar{x}, \bar{y})$ , i.e. there exist  $\delta_1 > 0$  and  $\delta_2 > 0$  such that

$$(\text{gph}F - (x_1, y_1) - (u_1, v_1)) \cap (X \times (S - y_2 - v_2)) \cap (\rho \mathbb{B}) \neq \emptyset$$

for all  $\rho \in ]0, \delta_1[$ ,  $(x_1, y_1) \in \text{gph}F \cap B_{\delta_2}(\bar{x}, \bar{y})$ ,  $y_2 \in S \cap B_{\delta_2}(\bar{y})$ ,  $u_1 \in X$ ,  $v_1, v_2 \in Y$  with  $\varphi(\max\{\|u_1\|, \|v_1\|, \|v_2\|\}) < \rho$ .

The two-set model  $\{\text{gph}F, X \times S\}$  employed in Definition 5.2 is an extension of the model (42), which corresponds to the case when  $S$  is a singleton:  $S := \{\bar{y}\}$ .

The metric characterizations of the properties in the next two statements are consequences of Theorems 3.1 and 3.2, respectively. Each characterization can be used as an equivalent definition for the respective property.

**Corollary 5.6** *The mapping  $F$  is*

- (i)  $\varphi$ -semitransversal to  $S$  at  $(\bar{x}, \bar{y})$  with some  $\delta > 0$  if and only if

$$d((\bar{x}, \bar{y}), (\text{gph}F - (x_1, y_1)) \cap (X \times (S - y_2))) \leq \varphi(\max\{\|x_1\|, \|y_1\|, \|y_2\|\})$$

for all  $x_1 \in X$ ,  $y_1, y_2 \in Y$  with  $\varphi(\max\{\|x_1\|, \|y_1\|, \|y_2\|\}) < \delta$ ;

- (ii) is  $\varphi$ -subtransversal to  $S$  at  $(\bar{x}, \bar{y})$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if and only if the following equivalent conditions hold:

- (a) for all  $(x, y) \in B_{\delta_2}(\bar{x}, \bar{y})$  with  $\varphi(\max\{d((x, y), \text{gph}F), d(y, S)\}) < \delta_1$ , it holds

$$d((x, y), \text{gph}F \cap (X \times S)) \leq \varphi(\max\{d((x, y), \text{gph}F), d(y, S)\});$$

- (b) for all  $(x_1, y_1) \in \text{gph}F \cap B_{\delta_2}(\bar{x}, \bar{y})$ ,  $y_2 \in S \cap B_{\delta_2}(\bar{y})$  and  $u_1 \in X$ ,  $v_1, v_2 \in Y$  with  $\varphi(\max\{\|u_1\|, \|v_1\|, \|v_2\|\}) < \delta_1$  and  $x_1 + u_1 \in B_{\delta_2}(\bar{x})$ ,  $y_1 + v_1 = y_2 + v_2 \in B_{\delta_2}(\bar{y})$ , it holds

$$\begin{aligned} d((0, 0), (\text{gph}F - (x_1, y_1) - (u_1, v_1)) \cap (X \times (S - y_2 - v_2))) \\ \leq \varphi(\max\{\|u_1\|, \|v_1\|, \|v_2\|\}); \end{aligned} \quad (44)$$

- (iii)  $\varphi$ -transversal to  $S$  at  $(\bar{x}, \bar{y})$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if and only if inequality (44) holds for all  $(x_1, y_1) \in \text{gph}F \cap B_{\delta_2}(\bar{x}, \bar{y})$ ,  $y_2 \in S \cap B_{\delta_2}(\bar{y})$  and  $u_1 \in X$ ,  $v_1, v_2 \in Y$  with  $\varphi(\max\{\|u_1\|, \|v_1\|, \|v_2\|\}) < \delta_1$ .

**Corollary 5.7** *Let  $\delta_1 > 0$  and  $\delta_2 > 0$ . The following conditions are equivalent:*

- (i) for all  $(x_1, y_1) \in \text{gph}F \cap B_{\delta_2}(\bar{x}, \bar{y})$ ,  $y_2 \in S \cap B_{\delta_2}(\bar{y})$  and  $u_1 \in X$ ,  $v_1, v_2 \in Y$  with  $x_1 + u_1 \in B_{\delta_2}(\bar{x})$ ,  $y_1 + v_1, y_2 + v_2 \in B_{\delta_2}(\bar{y})$  and  $\varphi(\max\{\|u_1\|, \|v_1\|, \|v_2\|\}) < \delta_1$ , inequality (44) holds true;
- (ii) for all  $x_1, y_1, y_2 \in \delta_2 \mathbb{B}$  with  $\varphi(\max\{d((\bar{x}, \bar{y}), \text{gph}F - (x_1, y_1)), d(\bar{y}, S - y_2)\}) < \delta_1$ , it holds

$$\begin{aligned} d((\bar{x}, \bar{y}), (\text{gph}F - (x_1, y_1)) \cap (X \times (S - y_2))) \\ \leq \varphi(\max\{d((\bar{x}, \bar{y}), \text{gph}F - (x_1, y_1)), d(\bar{y}, S - y_2)\}); \end{aligned} \quad (45)$$

- (iii) for all  $x, x_1 \in X$ ,  $y, y_1, y_2 \in Y$  such that  $x + x_1 \in B_{\delta_2}(\bar{x})$ ,  $y + y_1, y + y_2 \in B_{\delta_2}(\bar{y})$  and  $\varphi(\max\{d((x, y), \text{gph}F - (x_1, y_1)), d(y, S - y_2)\}) < \delta_1$ , it holds

$$\begin{aligned} d((x, y), (\text{gph}F - (x_1, y_1)) \cap (X \times (S - y_2))) \\ \leq \varphi(\max\{d((x, y), \text{gph}F - (x_1, y_1)), d(y, S - y_2)\}). \end{aligned}$$

Moreover, if  $F$  is  $\varphi$ -transversal to  $S$  at  $(\bar{x}, \bar{y})$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then conditions (i)–(iii) hold with any  $\delta'_1 \in ]0, \delta_1]$  and  $\delta'_2 > 0$  satisfying  $\varphi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2$  in place of  $\delta_1$  and  $\delta_2$ .

Conversely, if conditions (i)–(iii) hold with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then  $F$  is  $\varphi$ -transversal to  $S$  at  $(\bar{x}, \bar{y})$  with any  $\delta'_1 \in ]0, \delta_1]$  and  $\delta'_2 > 0$  satisfying  $\varphi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2$ .

*Remark 5.2* In the linear case, i.e. when  $\varphi(t) := \alpha t$  for some  $\alpha > 0$  and all  $t \geq 0$ , in view of Corollaries 5.6(ii)(a) and 5.7(iii), the properties in parts (ii) and (iii) of Definition 5.2 reduce, respectively, to the ones in [28, Definitions 7.11 and 7.8]. The property in part (i) is new.

The set-valued mapping (41), crucial for establishing equivalences between transversality properties of collections of sets and the corresponding regularity properties of set-valued mappings, in the setting considered here translates into the mapping  $G : X \times Y \rightrightarrows (X \times Y) \times (X \times Y)$  of the following form:

$$G(x, y) := (\text{gph} F - (x, y)) \times (X \times (S - y)), \quad (x, y) \in X \times Y. \quad (46)$$

Observe that  $G^{-1}(x_1, y_1, x_2, y_2) = (\text{gph} F - (x_1, y_1)) \cap (X \times (S - y_2))$  for all  $x_1, x_2 \in X$ ,  $y_1, y_2 \in Y$  and, if  $(\bar{x}, \bar{y}) \in \text{gph} F$ ,  $\bar{y} \in S$ , then  $((0, 0), (0, 0)) \in G(\bar{x}, \bar{y})$ .

The relationships between the nonlinear transversality and regularity properties in the next statement are direct consequences of Theorem 5.1.

**Theorem 5.4** *Let  $G$  be defined by (46).*

- (i)  $F$  is  $\varphi$ -semitransversal to  $S$  at  $(\bar{x}, \bar{y})$  with some  $\delta > 0$  if and only if  $G$  is  $\varphi$ -semiregular at  $((\bar{x}, \bar{y}), (0, 0), (0, 0))$  with  $\delta$ .
- (ii)  $F$  is  $\varphi$ -subtransversal to  $S$  at  $(\bar{x}, \bar{y})$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if and only if  $G$  is  $\varphi$ -subregular at  $((\bar{x}, \bar{y}), (0, 0), (0, 0))$  with  $\delta_1$  and  $\delta_2$ .
- (iii) If  $F$  is  $\varphi$ -transversal to  $S$  at  $(\bar{x}, \bar{y})$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then  $G$  is  $\varphi$ -regular at  $((\bar{x}, \bar{y}), (0, 0), (0, 0))$  with any  $\delta'_1 \in ]0, \delta_1]$  and  $\delta'_2 > 0$  satisfying  $\delta'_2 + \varphi^{-1}(\delta'_1) \leq \delta_2$ .  
Conversely, if  $G$  is  $\varphi$ -regular at  $((\bar{x}, \bar{y}), (0, 0), (0, 0))$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then  $F$  is  $\varphi$ -transversal to  $S$  at  $(\bar{x}, \bar{y})$  with any  $\delta'_1 \in ]0, \delta_1]$  and  $\delta'_2 > 0$  satisfying  $\delta'_2 + \varphi^{-1}(\delta'_1) \leq \delta_2$ .

*Remark 5.3* It is easy to see that the set-valued mapping (46) can be replaced in our considerations by the truncated mapping  $\mathcal{G} : X \times Y \rightrightarrows X \times Y \times Y$  defined by

$$\mathcal{G}(x, y) := (\text{gph} F - (x, y)) \times (S - y), \quad (x, y) \in X \times Y.$$

The last mapping admits a simple representation  $\mathcal{G}(x, y) = \text{gph} \mathcal{F} - (x, y, y)$ , where the set-valued mapping  $\mathcal{F} : X \rightrightarrows Y \times Y$  is defined by

$$\mathcal{F}(x) := F(x) \times S, \quad x \in X.$$

It was shown in [28, Theorems 7.12 and 7.9] that in the linear case the subtransversality and transversality of  $F$  to  $S$  at  $(\bar{x}, \bar{y})$  are equivalent to the metric subregularity and regularity, respectively, of the mapping  $(x, y) \mapsto \mathcal{F}(x) - (y, y)$  at  $((\bar{x}, \bar{y}), 0)$ .

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