On Large Graphs with Given Degree and Diameter

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Abstract

The degree/diameter problem is to determine the largest possible number of vertices in a graph of given maximum degree and given diameter.

It is well known that the general upper bound, called Moore bound, for the order of such graphs is attainable only for certain special values of degree and diameter. Finding better upper bounds for the maximum possible number of vertices, given the other two parameters remains an enormous fundamental open problem. Constructions producing large graphs of given degree and diameter provide lower bounds for the maximum possible number of vertices, given the other two parameters. Since, in general, the gaps between the upper and lower bounds on the maximum possible order are very big, there seems to be a good chance of finding new largest graphs, both by graph theoretical techniques and by using clever computer searches.

In this paper we give an overview of the current state-of-the-art of the degree/diameter problem for undirected graphs and we present some of the open problems in this area.

Keywords: Degree/diameter problem, Moore graphs, defect and extremal graphs.

1. Introduction

The topology of a network (such as a telecommunication, multiprocessor, or local area network, or a social network, etc.) is usually modelled by a graph in which vertices represent 'nodes' (stations or processors or people) while edges stand for 'links' or other types of connections. In the design of such networks, there are a number of features that must be taken into account. The most common ones, however, seem to be limitations on the vertex degrees and on the diameter of the network. The network interpretation of the two parameters is: The degree of a vertex is the number of connections attached to a node, while the diameter indicates the largest number of links that must be traversed in order to a vertex from any other. We wish to find a network that is in some sense optimal. Usually, this means finding the maximum possible number of nodes in a network, given maximum degree and given the diameter. In graph theoretical terms, we have

- *Degree/Diameter Problem*: Given positive integers Δ and D, find the largest possible number of vertices $n_{\Delta,D}$ in a graph of maximum degree Δ and diameter $\leq D$.

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Research activities related to the degree/diameter problem fall into two main areas. On one hand, there are proofs of the non-existence of graphs of order close to the general upper bounds, known as the Moore bounds. On the other hand, there is a great deal of activity in the constructions of large graphs, which also gives better lower bounds on $n_{\Delta,D}$.

In this paper we give an overview of the problem and we present a short list of some of the interesting open problems in the area. For a more complete reference, see the survey by Miller and Širáň [22].

2. Moore graphs and graphs close to Moore

The Moore bound is a general upper bound on the largest possible order (i.e., the number of vertices) $n_{\Delta,D}$ of a graph G of maximum degree Δ and diameter D. Clearly, if $\Delta = 1$ then $D = 1$ and $n_{1,1} = 2$; so from now on we assume that $\Delta \geq 2$.

Let v be a vertex of the graph G and let n_i, for $0 \leq i \leq D$, be the number of vertices at distance i from v. Since a vertex at distance $i \geq 1$ from v can be adjacent to at most $\Delta - 1$ vertices at distance $i+1$ from v, we have $n_{i+1} \leq (\Delta - 1)n_i$, for all i such that $1 \leq i \leq D-1$. Since $n_1 \leq \Delta$, it follows that $n_i \leq \Delta(\Delta - 1)^{i-1}$, for $1 \leq i \leq D$. Therefore,

$$\begin{aligned} n_{\Delta,D} = \sum_{i=0}^{D} n_i &\leq 1 + \Delta + \Delta(\Delta-1) + \ldots + \Delta(\Delta-1)^{D-1} \\ &= 1 + \Delta(1 + (\Delta-1) + \ldots + (\Delta-1)^{D-1}) \\ &= \begin{cases} 1 + \Delta\,\frac{(\Delta-1)^D - 1}{\Delta - 2} & \text{if } \Delta > 2 \\ 2D+1 & \text{if } \Delta = 2 \end{cases} \end{aligned} \tag{1}$$

The right-hand side of (1) is called the *Moore bound* and is denoted by $M_{\Delta,D}$. A graph whose order is equal to the Moore bound $M_{\Delta,D}$ is called a *Moore graph*. It is easy to see that such a graph is necessarily regular of degree Δ.

The study of Moore graphs was initiated by Hoffman and Singleton [19], who considered Moore graphs of diameter 2 and 3, and proved that if $D = 2$ then Moore graphs exist for $\Delta = 2, 3, 7$ and possibly 57 but for no other degrees. For $D = 3$ they showed that the Moore graph is the heptagon (for $\Delta = 2$). The proofs use the eigenvalues and eigenvectors of the adjacency matrix of graphs.

The fact that Moore graphs do not exist for the parameters $\Delta \geq 3$ and $D \geq 3$ was shown by Damerell [10] and, independently, also by Bannai and Ito [1].

To summarise, for diameter $D = 1$ and degree $\Delta \geq 1$ Moore graphs are the complete graphs $K_{\Delta+1}$. For diameter $D = 2$, Moore graphs are the cycle C_5 for degree $\Delta = 2$, the Petersen graph for degree $\Delta = 3$, and the Hoffman-Singleton graph for degree $\Delta = 7$. For diameter $D \geq 3$ and degree $\Delta = 2$, Moore graphs are the cycles on $2D+1$ vertices C_{2D+1}.

Since Moore graphs only exist for few values of degree and diameter, we are interested in studying the existence of large graphs which are somehow 'close' to Moore graphs. Since

we are dealing with the three parameters order, degree and diameter, to get close to Moore graphs, we may consider relaxing each of the parameters in turn.

How to relax the order is simple: the task is to look for graphs of given diameter and maximum degree whose order is 'close' to the Moore bound, that is, graphs of order $M_{\Delta,D} - \delta$, for δ small. The parameter δ is called the *defect*. If $\delta \leq \Delta$ then the defect is referred to as 'small' and the graph is sometimes called an *almost Moore graph*.

Relaxing the degree: This could be dealt with in several ways so that a graph could be considered to be close to a Moore graph if it has $M_{\Delta,D}$ vertices, diameter D and if

1. there is the smallest possible number δ of vertices with degree $\Delta + 1$, while the rest of the vertices all have degree at most Δ; or
2. there is one vertex of degree $\Delta + \delta$, δ as small as possible, while the rest of the vertices all have degree at most Δ; or
3. the average degree of a vertex is $\Delta + \delta$, δ as small as possible.

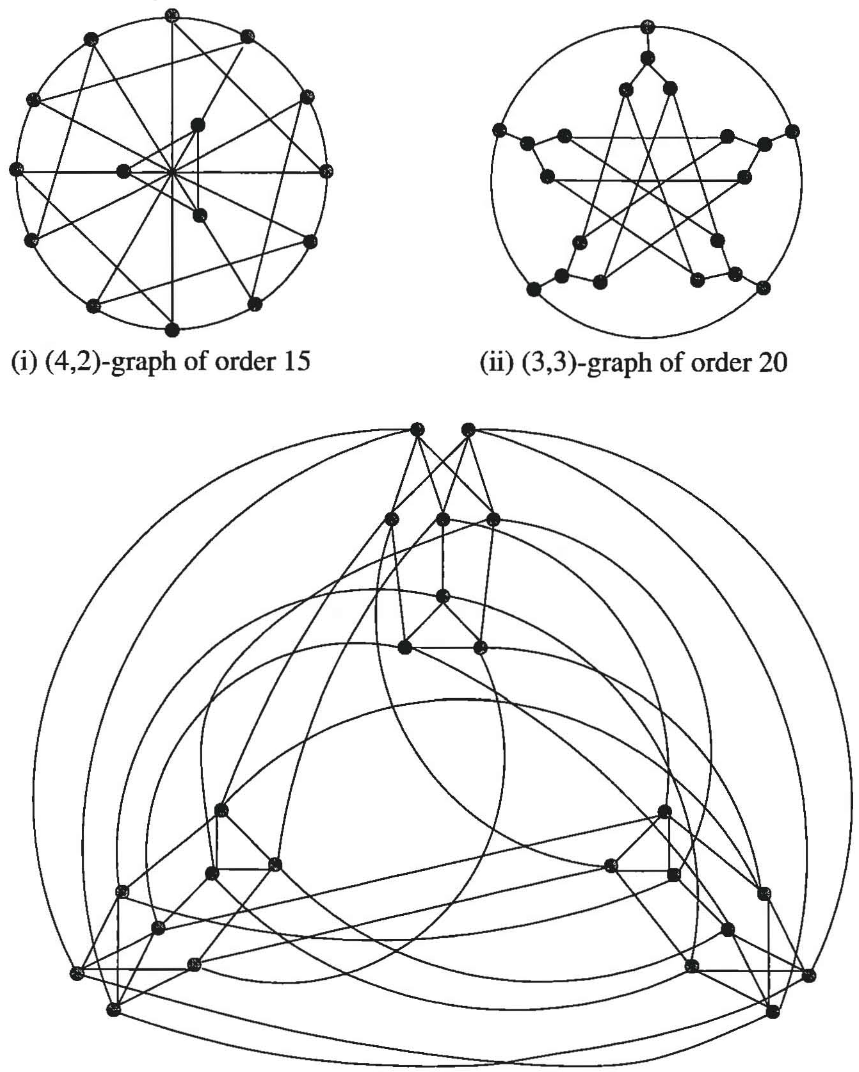
Finally, let us consider relaxing the diameter requirement. Diameter is essentially a coarse measure of the distances between the vertices in a graph. For a finer measure, we could use 'eccentricity'. The *eccentricity* of a vertex v, denoted $e(v)$, is defined as the maximum length of the shortest path between v and any other vertex. Note that the diameter of the graph is equal to the maximum eccentricity over all vertices of the graph. So, relaxing the diameter could mean, for example, that a graph is close to Moore graph if it has $M_{\Delta,D}$ vertices, maximum degree Δ and if

1. there is the smallest possible number δ of vertices with eccentricity equal to $D+1$, while the rest of the vertices all have eccentricity D; or
2. there is one vertex of eccentricity $D + \delta$, δ as small as possible, while the rest of the vertices all have eccentricity at most D; or
3. the average eccentricity of a vertex is $D + \delta$, δ as small as possible.

In this paper we will concentrate only on the first meaning of 'closeness' of a graph to a Moore graph. From now on, for convenience, by a (Δ, D)-graph we will understand any graph of maximum degree Δ and of diameter at most D; if such a graph has order $M_{\Delta,D} - \delta$ then it will be referred to as a (Δ, D)-graph of defect δ.

Erdős, Fajtlowitcz and Hoffman [13] proved that, apart from the cycle C_4, there are no graphs of degree Δ, diameter 2 and defect 1, that is, of order one less than the Moore bound. This was later generalized by Bannai and Ito [2], and also independently by Kurosawa and Tsujii [21], to all diameters. Hence, for all $\Delta \geq 3$, there are no (Δ, D)-graphs of defect 1, and for $\Delta = 2$ the only such graphs are the cycles C_{2D}. It follows that, for $\Delta \geq 3$, we have $n_{\Delta,D} \leq M_{\Delta,D} - 2$.

Next, we will discuss the case of defect $\delta = 2$. When $\Delta = 2$, the (Δ, D)-graphs of defect 2 are the cycles C_{2D-1}. For $\Delta \geq 3$, there are only five (Δ, D)-graphs of defect 2 known at present: two $(3,2)$-graphs of order 8, one $(4,2)$-graph of order 15, one $(5,2)$-graph of order 24 and one $(3,3)$-graph of order 20. The last three graphs were found by Elspas [12]; and are depicted in Fig. 1. Thus we have $n_{4,2} = 15$, $n_{5,2} = 24$, and $n_{3,3} = 20$.



(i) (4,2)-graph of order 15 (ii) (3,3)-graph of order 20

(iii) (5,2)-graph of order 24

Figure 1: Examples of graphs of order $M_{d,k} - 2$.

In [24, 25], Nguyen and Miller proved that graphs of diameter 2 and defect two do not exist if $\Delta \geq 8$ and even and $\Delta \equiv 2$ (mod3); and that graphs of diameter 2 and defect 2 also do not exist for odd $\Delta \geq 7$ such that $(\Delta^2 - 1)[1/2(\Delta - 3)(\Delta^2 + \Delta + 4) + \Delta + 2]$ is not a multiple of 5. They conjecture

Conjecture (Miller and Nguyen, 2005). For $\Delta \geq 6$ there are no graphs of maximum degree Δ, diameter 2 and defect 2.

Little is known about graphs with defects larger than 2. Jorgensen [20] proved that a graph with maximum degree 3 and diameter $D \geq 4$ cannot have defect 2, which shows that $n_{3,D} \leq M_{3,D} - 3$ if $D \geq 4$. Miller and Simanjuntak [23] proved that a graph with maximum degree 4 and diameter $D \geq 3$ cannot have defect 2 which shows that $n_{4,D} \leq M_{4,D} - 3$ if $D \geq 3$. Nguyen and Miller [25] consider graphs with diameter 2 and defect 3. They prove that such graphs must contain a certain induced subgraph, which in turn leads to the proof that, for degree 6 and diameter 2, the largest order of a vertex-transitive graph is 32.

3. Constructions of large graphs

In an effort to improve the bound on the maximum possible order of graphs for given D and Δ, we try to construct graphs with order close to the Moore bound.

The *undirected de Bruijn graph* of type (t, k) has vertex set V formed by all sequences of length k, the entries of which are taken from a fixed 'alphabet' consisting of t distinct letters. In the graph, two vertices a and b -say $a = (a_1, a_2, \ldots, a_k)$ and $b = (b_1, b_2, \ldots, b_k)$- are joined by an edge if either $a_i = b_{i+1}$ for $1 \leq i \leq k-1$, or if $a_{i+1} = b_i$, for $1 \leq i \leq k-1$. Obviously, the undirected de Bruijn graph of type (t, k) has order t^k and degree $2t$. These graphs give, for any Δ and D, the lower bound

$$n_{\Delta,D} \geq \left(\frac{\Delta}{2}\right)^D.$$

Some improvements on this bound have been obtained. For example, ignoring directions in the digraph construction of Baskoro and Miller [3] produces graphs of even maximum degree Δ and diameter at most D whose order is equal to

$$\left(\frac{\Delta}{2}\right)^D + \left(\frac{\Delta}{2}\right)^{D-1}.$$

A substantial improvement was achieved by Canale and Gómez [9] by exhibiting, for an infinite set of values of Δ, families of graphs which show that

$$n_{\Delta,D} \geq \left(\frac{\Delta}{1.57}\right)^D$$

for D congruent with -1, 0, or 1 (mod 6).

For small values of D we can obtain much better results than in the general case. The best result is given by Brown [8] for diameter 2, using finite projective geometries: for each Δ such that $\Delta - 1$ is a prime power,

$$n_{\Delta,2} \geq \Delta^2 - \Delta + 1. \qquad (2)$$

Erdős, Fajtlowicz and Hoffman [13], and Delorme [11], improved this bound to

$$n_{\Delta,2} \geq \Delta^2 - \Delta + 2 \qquad (3)$$

if $\Delta - 1$ is a power of 2.
Several techniques have been introduced for the construction of large graphs of given degree and diameter. The most important earlier techniques seem to be the star product of Bermond,

Delorme and Farhi [4, 5] and compounding of graphs introduced by Bermond, Delorme and Quisquater [6].

More recent results include a technique by Gómez, Pelayo and Balbuena [18] who produce large graphs of diameter six by replacing some vertices of a Moore bipartite graph (see [16]) of diameter six with complete graphs which are joined to each other and to the rest of the graph using a special graph of diameter two. The degree of the constructed graph remains the same as the degree of the original graph. In an extension to this work, Gómez and Miller [17] presented two new generalizations of two large compound graphs.

Other methods have been designed in an effort to construct large (Δ, D)-graphs for relatively small values of Δ and D. Possibly the most promising is the voltage assignment technique, introduced by Brankovic *et al.* [7] (see also [15]).

Additionally, in many cases the largest currently known (Δ, D)-graphs have been found with the assistance of computers. See, for example, the recent paper by Exoo [14].

Table of the current largest known graphs for degree $\Delta \leq 16$ and diameter $D \leq 10$ can be found on the website "http://maite71.upc.es/grup_de_grafs/grafs/taula_delta_d.html" which is updated regularly by Francesc Comellas.

To highlight the size of the problem, we tabulate below, in Tables 1,2 and 3, the outstanding potential values of orders larger than those obtained so far, for diameter up to 10, and for maximum degree $\Delta = 3, 4$ and 5, in turn. The 'Largest Known' column gives the order of the current largest known graph of the given maximum degree Δ and diameter D. Graphs with larger number of vertices may possibly exist; the possible larger orders are tabulated under the heading 'Possible Larger Values'. Note that every value listed in this column represents an open problem. Furthermore, similar tables could of course be made also for degrees $\Delta > 5$.

D	Largest Known	Possible Larger Values
2	10	–
3	20	–
4	38	39 - 44
5	70	71 - 92
6	132	133 - 188
7	190	191 - 380
8	330	331 - 764
9	570	571 - 1,532
10	950	951 - 3,068

Table 1: Possible values of largest orders for degree $\Delta = 3$.

Since the gaps between the best current lower and upper bounds for the maximum order of graphs with given degree and diameter are huge, we believe that there is a great scope for improving the lower bound by finding better construction techniques as well as by using clever

D	Largest Known	Possible Larger Values
2	15	–
3	41	42 - 51
4	96	97 - 159
5	364	365 - 483
6	740	741 - 1,435
7	1,155	1,156 - 4,371
8	3,080	3,081 - 13,119
9	7,550	7,551 - 39,363
10	17,604	17,605 - 118,095

Table 2: Possible values of largest orders for degree $\Delta = 4$.

D	Largest Known	Possible Larger Values
2	24	–
3	72	73 - 104
4	210	211 - 424
5	558	559 - 1,704
6	2,760	2,761 - 6,824
7	5,500	5,501 - 27,304
8	16,956	16,957 - 109,224
9	53,020	53,021 - 436,904
10	164,700	164,701 - 1,747,626

Table 3: Possible values of largest orders for degree $\Delta = 5$.

search algorithms.

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