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Directional Metric Pseudo Subregularity of Set-valued Mappings: a General Model

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Tribute to Professor Alexander Kruger on his sixty-fifth birthday. With recognition for research achievement and friendship

Received: date / Accepted: date

Abstract In this paper, we introduce a new general pseudo subregularity model which unifies some important nonlinear (sub)regularity models studied recently in the literature. Some slope and abstract coderivative characterizations are established.

Keywords Abstract subdifferential · Metric regularity · Directional metric regularity · Metric subregularity · directional Hölder metric subregularity · Coderivative

Mathematics Subject Classification 49J52 · 49J53 · 90C30

1 Introduction

Over the past decades, mathematical ideas based on the use of advanced techniques of generalized differentiation have allowed to make significant advances in the study of generalized equations, that is inclusions governed by set-valued mappings. These

Research of the last author was supported by the Australian Research Council (ARC) grant DP160100854 and benefited from the support of the FMJH Program PGMO and from the support of EDF.

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inclusions/ generalized equations cover many important problems in various areas of mathematics and applied sciences such as physics, mechanics, and economics: equations; inequality systems; variational inequalities; complementary problems; optimal control, and various other topics. Also, due to their importance, as well as from their theoretical point of view and their broad applicability, variational systems have attracted the interest of many mathematicians.

One of the pillars of study of these generalized equations is the notion of metric regularity. This concept has emerged in the last decades, mainly after the contribution of Borwein [4], even if according to Ioffe [12, 13] in a recent survey, "*the roots of this concept go back to a circle of fundamental regularity ideas of classical analysis embodied in such results as the implicit function theorem, the Banach open mapping theorem, theorems of Lyusternik and Graves, on the one hand, and the Sard theorem and the Thom-Smale transversality theory, on the other*". Nowadays, this concept is commonly regarded as central for studying the existence and behavior of solutions of nonlinear equations under small perturbations of the data. The crucial role of metric regularity in Optimization and Variational Analysis as well as researcher's interest in this field can be found in many seminal works including for instance [4, 7, 9, 11–14, 26], and references therein. By the time and demand of use and applications, variants of this property have emerged suitable to practical problems. Weaker/stronger versions: calmness, (strong) (Hölder) metric sub/regularity, semiregularity or equivalent versions: pseudo Lipschitz, linear openness were studied and have proved to have an important role in various applications in Mathematics, especially in Variational Analysis and Optimization [5, 12, 13, 15–17, 26], ...

Another direction in this line is to build directional models for these objects as recently proposed by Arutyunov-Avakov-Izmailov [1], Gfrerer [8], Ngai-Théra [20], Ngai-Tron-Théra [22], Ngai-Tron-Tinh [23]. Characterizations of these concepts have been established and successfully applied to study optimality conditions for mathematical programs, for calculating tangent cones,...This notion of directional regularity is an extension of an earlier notion used by Bonnans and Shapiro [3] to study sensitivity analysis. Later, Ioffe [10] has introduced and investigated an extension called relative metric regularity which covers many notions of metric regularity in the literature. In [24] Penot studied this property to establish second-order optimality conditions.

The paper concerns a new type of directional metric regularity. In this article, motivated by the works mentioned above, especially those of (co)authors and by Gfrerer, we build a general new model which unifies almost (sub)regularity models in the literature; especially it covers both directional Hölder metric subregularity introduced by Ngai, Tron and Tinh and metric pseudo subregularity explored by Gfrerer. This property is given in Definition 3.2. Our aims are to give characterizations of the property. First, we establish a slope characterization, after that we move to a subdifferential/coderivative or a limit critical set characterization. Section 2 introduces the mathematical notation and basic definitions. In Section 3, we present the motivation of this paper and some slope characterizations of directional pseudo regularity. In the final section, we explore how the use of an abstract subdifferential may derive to characterizations of metric directional pseudo subregularity in terms of coderivatives.

2 Preliminaries and notations

For the convenience of the reader, we include in this section the material concerning set-valued analysis and variational analysis that will be extensively used throughout the sequel. We use the monographs of Mordukhovich [18], Ioffe [14], Rockafellar & Wets [26] and Penot [25] as our desk-copies.

For our purposes, we are going to work in the framework of real Banach spaces. If X is such a space, we denote by $\|\cdot\|$ the associated norm and by $d(x,\Omega)$ the distance from $x \in X$ to the subset Ω of X, that is, $d(x,\Omega) := \inf\{\|x-y\| : y \in \Omega\}$. Given X, we denote the topological dual (continuous dual) by X^*, by $\|\cdot\|^*$ the dual norm of $\|\cdot\|$, by $\mathbb{B}_X = \{x \in X : \|x\| \leq 1\}$ the closed unit ball, by $\mathbf{S}_X = \{x \in X : \|x\| = 1\}$ the unit sphere, by $\mathbb{B}(x,r)$ the closed ball with center x and radius r, respectively.

By a set-valued mapping (also named by some authors multifunction), we mean a mapping T from X into the subsets (possibly empty) of another Banach space Y and we use the notation $T : X \rightrightarrows Y$. The graph of T denoted by $\operatorname{gph} T$ is the set of those points in $X \times Y$ such that $y \in T(x)$, while $T^{-1} : Y \rightrightarrows X$, the inverse of T (always defined), is given by $(x,y) \in \operatorname{gph} T \iff (y,x) \in \operatorname{gph} T^{-1}$. We say that T is closed if its graph is closed with respect to the product topology on $X \times Y$. Given a set $K \subset X$, we use the notation $\operatorname{cone} K$ for the conic hull of K, that is for the set of all conic combinations $\sum_{i=1}^{i=n} \lambda_i x_i$ of points of K where $\lambda_i \geq 0$ for each index i.

Given an extended real-valued function $f : X \to \mathbb{R} \cup \{+\infty\}$ we use the notation $\operatorname{cl} f$ to denote the lower semicontinuous envelope of f defined by $\operatorname{cl} f(x) = \liminf_{u \to x} f(u))$, and $\operatorname{Dom} f$ will refer to the domain of f, that is, the set of those points $x \in X$ such that $f(x)$ is finite. We recall that the convex subdifferential of f at $x \in \operatorname{Dom} f$ is the set

$$\partial f(x) := \{x^\star \in X^* : \langle x^\star, y-x\rangle \leq f(y) - f(x) \quad \text{for all} \quad y \in X\},$$

with the convention that $\partial f(x) = \emptyset$, when $f(x) = +\infty$.

For the purpose of this study, we use the concept of slope $|\nabla f|(x)$ of a function f at $x \in \operatorname{Dom} f$. This is the quantity introduced by De Giovani, Marino & Tosques [6] and defined by

$$|\nabla f|(x) := \begin{cases} 0 & \text{if } x \text{ is a local minimum point of } f \\ \limsup\limits_{y \to x,\, y \neq x} \dfrac{f(x) - f(y)}{d(x,y)} & \text{otherwise.} \end{cases}$$

For $x \notin \operatorname{Dom} f$, we set $|\nabla f|(x) = +\infty$.

When f is a convex function defined on a Banach space and x is not a minimum point, then according to Ioffe [11, Poposition 3.8] (see also Azé & Corvelec [2, Proposition 3.2])

$$|\nabla f|(x) = \sup_{y \neq x} \frac{f(x) - f(y)}{\|x-y\|} \quad \text{and} \quad |\nabla f|(x) = d(0, \partial f(x)).$$

In the following sections, we make use of the notion of abstract subdifferential operator.This operator denoted by ∂ satisfies the following conditions:

(C1) If $f : X \to \mathbb{R}$ is a convex function which is continuous around $\bar{x} \in X$ and $\beta : \mathbb{R} \to \mathbb{R}$ is continuously differentiable at $t = f(x)$, then

$$\partial(\beta \circ f)(x) \subseteq \{\beta'(f(x))x^\star \in X^* : \langle x^\star, y-x\rangle \le f(y) - f(x) \quad \forall y \in X\};$$

(C2) $\partial f(x) = \partial g(x)$ if $f(y) = g(y)$ for all y in a neighborhood of x;

(C3) Let $f_1 : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and $f_2, ..., f_n : X \to \mathbb{R}$ be Lipschitz functions. If $f_1 + f_2 + ... + f_n$ attains a local minimum at x_0, then for any $\varepsilon > 0$, there exist $x_i \in x_0 + \varepsilon\mathbb{B}_X$, $x_i^\star \in \partial f_i(x_i)$, $i \in \overline{1,n}$, such that $|f_i(x_i) - f_i(x_0)| < \varepsilon$, $i \in \overline{1,n}$, and $\|x_1^\star + x_2^\star + ... + x_n^\star\| < \varepsilon$.

We recall that the indicator function δ_C of a closed set C in X is the function defined by $\delta_C(x) = 0$ when $x \in C$ and $\delta_C(x) = +\infty$, otherwise. Given an abstract subdifferential ∂, the set $N(C,x) := \partial\delta_C(x)$ is called the *normal cone* to C at x associated with ∂.

(C4) $N(C,x)$ is assumed to be a cone for any closed subset C of X.

Let $F : X \rightrightarrows Y$ and $(\bar{x},\bar{y}) \in \operatorname{gph} F$. Given an abstract subdifferential ∂ and normal cone associated with ∂, the set-valued mapping $D^*F(\bar{x},\bar{y}) : Y^* \rightrightarrows X^*$ defined by

$$D^*F(\bar{x},\bar{y})(y^\star) = \left\{x^\star \in X^* : (x^\star, -y^\star) \in N\big(\operatorname{gph} F, (\bar{x},\bar{y})\big)\right\}$$

is called the coderivative of F associated with ∂.

We assume further that ∂ satisfies the separable property in the following sense:

(C5) If f is a separable function defined on $X \times Y$, that is, $f(x,y) := f_1(x) + f_2(y)$, $(x,y) \in X \times Y$, where $f_1 : X \to \mathbb{R} \cup \{+\infty\}$, $f_2 : Y \to \mathbb{R} \cup \{+\infty\}$, then

$$\partial f(x,y) = \partial f_1(x) \times \partial f_2(y), \quad \text{for all} \quad (x,y) \in X \times Y.$$

It is well known that the proximal subdifferential in Hilbert spaces, the Fréchet subdifferential in Asplund spaces, the viscosity subdifferentials in smooth spaces as well as the Ioffe and the Clarke-Rockafellar subdifferentials in the setting of general Banach spaces are subdifferentials verifying the conditions (C1)-(C5).

3 Slope characterizations of directional pseudo subregularity

For the understanding of the paper, it would be wise to recall the definitions of metric subregularity and of Hölder metric subregularity. We say that a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces X, Y is called *metrically subregular* at a point $(\bar{x},\bar{y}) \in \operatorname{gph} F$ with constant $\tau > 0$, if there exists a neighbourhood U of $\bar{x}$ such that

$$\tau d(x, F^{-1}(\bar{y})) \leqslant d(\bar{y}, F(x)) \text{ for all } x \in U. \tag{3.1}$$

If in relation (3.1) we replace $d(\bar{y},F(x))$ by $d(\bar{y},F(x))^\gamma$, with $\gamma > 0$, then we say that F is *Hölder metrically subregular* at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ with modulus τ and order γ and (3.1) can be equivalently rewritten as

$$\tau d(x,F^{-1}(\bar{y})) \leqslant d(\bar{y},F(x))^\gamma \text{ for all } x \in U. \tag{3.2}$$

Throughout the rest of the paper, we assume given Banach spaces X and Y_i, $(i = 1,\cdots,m)$, as well as a finite family of set-valued mappings $T_i : X \rightrightarrows Y_i, (i = 1,\cdots,m)$. We note $T := (T_1,\dots,T_m) : X \rightrightarrows Y_1 \times \cdots \times Y_m$, the set-valued mapping defined by $T(x) := T_1(x) \times \dots \times T_m(x)$ and consider $\gamma := (\gamma_1,...,\gamma_m) \in \mathbb{R}^m$ such that $\gamma \geqslant 1$ which means that $\gamma_i \geqslant 1$ for $i = 1,...,m$. To begin with, let us first recall the definition of γ-metric pseudo subregularity (γ-MPSR, for short) w.r.t. a given direction u and order $\gamma \geqslant 1$.

Definition 3.1 (directional pseudo subregularity, [9]) A set-valued mapping $T = (T_1,\dots,T_m) : X \rightrightarrows Y_1 \times \cdots \times Y_m$ is said to be $(\gamma_1,\dots,\gamma_m)$-metrically pseudo subregular $(\gamma_i \geqslant 1)$ in the direction u at $(\bar{x},\bar{y}) \in \operatorname{gph} T$ with modulus $\tau > 0$ iff there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$d\big(x,T^{-1}(\bar{y})\big) \leqslant \tau \sum_{i=1}^{m} \|x-\bar{x}\|^{1-\gamma_i} d\big(\bar{y}_i,T_i(x)\big) \tag{3.3}$$

for $x \neq \bar{x}$ in $\mathbb{B}(\bar{x},\varepsilon) \cap \big(\bar{x} + \operatorname{cone} \mathbb{B}(u,\delta)\big)$.

As observed by Gfrerer, γ_i-metric pseudo subregularity $(\gamma_i \geqslant 1)$ in the direction $u = 0$ implies Hölder subregularity of order $\frac{1}{\gamma_i}$. Note also that when $m = 1$, if we note $T = T_1$ and $\gamma = \gamma_1$, then (3.3) becomes:

$$d\big(x,T^{-1}(\bar{y})\big)^\gamma \leqslant \|x-\bar{x}\|^{\gamma-1} d\big(x,T^{-1}(\bar{y})\big) \leqslant \tau d\big(\bar{y},T(x)\big). \tag{3.4}$$

Hence, if a set-valued mapping $T : X \rightrightarrows Y$ is γ-MPSR at $(\bar{x},\bar{y})$ in the direction u, then T is directionally Hölder metrically subregular of order γ at $(\bar{x},\bar{y})$ in the direction u as mentioned in [23].

The new idea in this contribution is to consider a general metric pseudo subregularity model called $(\gamma,h)-$pseudo subregularity associated with a given function $h := (h_1,\dots,h_m) : X \longrightarrow \mathbb{R}^m_+$ and with $\gamma := (\gamma_1,...,\gamma_m) \in \mathbb{R}^m$ with $\gamma_i \geqslant 1$ for $i = 1,...,m$. To facilitate ease of reading, we shall introduce some useful real-valued functions corresponding to $T = (T_1,\dots,T_m) : X \rightrightarrows Y_1 \times \cdots \times Y_m$ and $\gamma = (\gamma_1,\dots,\gamma_m)$. For each j, we define $\rho_{T_j}(\cdot) = d\big(\bar{y}_j,T_j(\cdot)\big)$ and set

$$\varphi_{T_j}(x) := \begin{cases} \frac{\rho_{T_j}(x)}{h_j(x)^{\gamma_j-1}} & \text{if } h_j(x) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now let us denote by φ_T the sum $\varphi_T = \sum_{j=1}^m \varphi_{T_j}$, by $\psi_T = \operatorname{cl} \varphi_T$ the lower semicontinuous envelope of φ_T and by S the sublevel set $[\psi_T \leqslant 0]$. Throughout the rest of the document, we make use of the following assumption:

$(\mathscr{A})$ $h_1,\dots,h_m : X \longrightarrow \mathbb{R}_+$ are continuous functions and satisfy

$$h_i(x) = 0 \Longrightarrow \rho_{T_i}(x) = 0. \tag{3.5}$$

Definition 3.2 **((γ,h)-metric pseudo subregularity)** Let $T=(T_1,\ldots,T_m):X\rightrightarrows Y_1\times\cdots\times Y_m$ be given and let $(\bar{x},\bar{y})\in\operatorname{gph}T$. The mapping T is said to be (γ,h)-metrically pseudo subregular in the direction u at $(\bar{x},\bar{y})\in\operatorname{gph}T$ for $h=(h_1,\ldots,h_m)$ if there exist $\tau>0$, $\varepsilon>0$ and $\delta>0$ such that

$$d\big(x,T^{-1}(\bar{y})\big)\leqslant\tau\sum_{1\leqslant i\leqslant m,h_i(x)>0}\frac{d\big(\bar{y}_i,T_i(x)\big)}{h_i(x)^{\gamma_i-1}} \tag{3.6}$$

for $x\in\mathbb{B}(\bar{x},\varepsilon)\cap\big(\bar{x}+\operatorname{cone}\mathbb{B}(u,\delta)\big)$ with $\sum_{i=1}^m h_i(x)>0$.

When the direction $u:=0$, we will say briefly that T is $(\gamma,h)-$pseudo subregular at $(\bar{x},\bar{y})$ (that is, the inequality (3.6) is satisfied for all x near $\bar{x}$). Let us note that when considering some special mappings h, one recovers the concepts of directional metric subregularity mentioned above. For instance, if we choose the functions $h_i(x)=\|x-\bar{x}\|$, one gets metric pseudo subregularity and if one considers the case $m=1$, $h_1(x):=d(x,T^{-1}(\bar{y}))$ one gets directional Hölder metric subregularity. Especially, when $\gamma_i=1$ $(i=1,\ldots,m)$, one gets the usual metric subregularity.

Throughout the paper, it is convenient to keep in mind the notation used in [20]:

$x\xrightarrow{u}\bar{x}$ is meant to be $x\to\bar{x}$ if $u=0$ and $x\to\bar{x}$ and $\frac{x-\bar{x}}{\|x-\bar{x}\|}\to\frac{u}{\|u\|}$ otherwise, as well.

The rest of this section will be devoted to establish some characterizations for the (γ,h)-metric pseudo subregularity (Definition 3.2). For such a purpose, the next proposition will be useful.

Proposition 3.1 *Suppose that T is not (h,γ)-metrically pseudo subregular in the direction u at $(\bar{x},\bar{y})\in\operatorname{gph}T$. Then, for each nonnegative real sequence $(\tau_k)\downarrow 0$, there exists a sequence $x_k\xrightarrow{u}\bar{x}$ which satisfies for large integer k the following conditions:*

i. $d(x_k,S)>0$;
ii. $\psi_T(x_k)\leqslant\frac{\tau_k}{1-\sqrt{\tau_k}}d(x_k,S)$;
iii. $\big|\nabla\psi_T\big|(x_k)\leqslant\max\big\{\sqrt{\tau_k},\tau_k\big\}$.

Consequently, one has

$$\liminf_{\substack{x\xrightarrow{u}\bar{x},x\notin S,\\ \psi_T(x)/d(x,S)\to 0}}\Big\{|\nabla\psi_T|(x)\Big\}=0. \tag{3.7}$$

Proof According to the definition of metric pseudo subregularity, we can find a sequence $\tilde{x}_k\xrightarrow{u}\bar{x}$ such that

$$\tilde{x}_k\notin S,\quad \tau_k d\big(\tilde{x}_k,S\big)>\sum_{\substack{j=1,\ldots,m\\ h_j(\tilde{x}_k)>0}}\frac{d\big(\bar{y}_j,T_j(\tilde{x}_k)\big)}{h_j(\tilde{x}_k)^{\gamma_j-1}}=\sum_{j=1}^m\varphi_{T_j}(\tilde{x}_k)\geqslant\psi_T(\tilde{x}_k). \tag{3.8}$$

Hence, the lsc function ψ_T inherits the following property:

$$0<\psi_T(\tilde{x}_k)<\tau_k d\big(\tilde{x}_k,S\big),k=1,2,\ldots \tag{3.9}$$

Let $\varepsilon_k:=\psi_T(\tilde{x}_k)>0$ and $\lambda_k:=\min\Big\{\sqrt{\tau_k}d(\tilde{x}_k,S),\frac{\varepsilon_k}{\tau_k}\Big\}>0$. By the Ekeland variational principle, there exists for each k an element $\hat{x}_k$ such that

- $\|\hat{x}_k - \tilde{x}_k\| \leqslant \lambda_k$;
- $\psi_T(\tilde{x}_k) - \frac{\varepsilon_k}{\lambda_k}\|\hat{x}_k - \tilde{x}_k\| \geqslant \psi_T(\hat{x}_k)$;
- the function $f_k : x \longmapsto \psi_T(x) + \frac{\varepsilon_k}{\lambda_k}\|x - \hat{x}_k\|$ attains its minimum at $\hat{x}_k$.

We are going to establish the following facts:

(i) $\hat{x}_k \notin S$;
(ii) $\hat{x}_k \xrightarrow{u} \bar{x}$;
(iii) $\psi_T(\hat{x}_k) \leqslant \frac{\tau_k}{1-\sqrt{\tau_k}} d(\hat{x}_k, S)$;
(iv) $|\nabla \psi_T|(\hat{x}_k) \leqslant \max\left\{\sqrt{\tau_k}, \tau_k\right\}$.

Assuming that (i) does not hold, one has

$$\tau_k d\left(\tilde{x}_k, S\right) \leqslant \tau_k \|\tilde{x}_k - \hat{x}_k\| \leqslant \tau_k \lambda_k \leqslant \varepsilon_k = \psi_T(\tilde{x}_k),$$

in contradiction with (3.9).

To prove (ii), let's note that $\lambda_k = o(\|\tilde{x}_k - \bar{x}\|)$ by virtue of the inequality

$$\lambda_k \leqslant \sqrt{\tau_k} d\left(\tilde{x}_k, S\right) \leqslant \sqrt{\tau_k}\|\tilde{x}_k - \bar{x}\|.$$

Set $\mu_k := \frac{\|\hat{x}_k - \bar{x}\|}{\|\tilde{x}_k - \bar{x}\|}$. After involving the triangle inequality, one deduces

$$\left(1 - \frac{\|\hat{x}_k - \tilde{x}_k\|}{\|\tilde{x}_k - \bar{x}\|}\right)\|\tilde{x}_k - \bar{x}\| \leqslant \|\hat{x}_k - \bar{x}\| \leqslant \left(1 + \frac{\|\hat{x}_k - \tilde{x}_k\|}{\|\tilde{x}_k - \bar{x}\|}\right)\|\tilde{x}_k - \bar{x}\|.$$

Recalling that $\|\hat{x}_k - \tilde{x}_k\| \leqslant \lambda_k$, we conclude that $\mu_k \to 1$ as well as $\|\hat{x}_k - \bar{x}\| \to 0$. If $u \neq 0$, then a few straightforward calculations give us

$$\begin{aligned} \frac{\hat{x}_k - \bar{x}}{\|\hat{x}_k - \bar{x}\|} - \frac{u}{\|u\|} = &\left(\frac{1}{\mu_k}\right)\left(\frac{\tilde{x}_k - \bar{x}}{\|\tilde{x}_k - \bar{x}\|} - \frac{u}{\|u\|}\right) \\ &+ \left(\frac{1}{\mu_k}\right)\frac{\hat{x}_k - \tilde{x}_k}{\|\tilde{x}_k - \bar{x}\|} + \left(\frac{1}{\mu_k} - 1\right)\frac{u}{\|u\|}. \end{aligned} \tag{3.10}$$

Using (3.10) it yields $\left\|\frac{\hat{x}_k - \bar{x}}{\|\hat{x}_k - \bar{x}\|} - \frac{u}{\|u\|}\right\| \to 0$, and therefore, (ii) is proved.

For establishing (iii), we invoke (3.9) and obtain

$$\frac{\psi_T(\hat{x}_k)}{d(\hat{x}, S)} \leqslant \frac{\psi_T(\tilde{x}_k)}{d(\hat{x}_k, S)} \leqslant \frac{\tau_k d\left(\tilde{x}_k, S\right)}{d(\hat{x}_k, S)}.$$

Since

$$|d(\hat{x}_k, S) - d(\tilde{x}_k, S)| \leqslant \|\hat{x}_k - \tilde{x}_k\| \leqslant \lambda_k \leqslant \sqrt{\tau_k} d(\tilde{x}_k, S),$$

for large k we get

$$\frac{\psi_T(\hat{x}_k)}{d(\hat{x}_k, S)} \leqslant \frac{\tau_k d\left(\tilde{x}_k, S\right)}{d\left(\tilde{x}_k, S\right) - \sqrt{\tau_k} d\left(\tilde{x}_k, S\right)} = \frac{\tau_k}{1 - \sqrt{\tau_k}}.$$

Hence, (iii) follows immediately.

In oder to verify (iv), remember that $f_k(\cdot)$ attains a minimum at $\hat{x}_k$. As a result,

$$\psi_T(x) + \frac{\varepsilon_k}{\lambda_k}\|x - \hat{x}_k\| \geqslant \psi_T(\hat{x}_k)$$

for x close to $\hat{x}_k$. Equivalently,

$$\frac{\psi_T(\hat{x}_k) - \psi_T(x)}{\|\hat{x}_k - x\|} \leqslant \frac{\varepsilon_k}{\lambda_k}$$

for all $x \neq \hat{x}_k$ belonging to a neighborhood of $\hat{x}_k$. In summary, we have

$$|\nabla\psi_T|(\hat{x}_k) \leqslant \frac{\varepsilon_k}{\lambda_k} = \max\left\{\frac{\psi_T(\tilde{x}_k)}{\sqrt{\tau_k}d(\tilde{x}_k, S)}, \tau_k\right\} \leqslant \max\left\{\sqrt{\tau_k}, \tau_k\right\}. \tag{3.11}$$

Letting $x_k = \hat{x}_k$, the whole proof is established. □

Based on Proposition 3.1, the next theorem offers a sufficient criterion for metric pseudo subregularity.

Theorem 3.1 *Let T, γ, h and ψ_T be defined as above. Suppose that*

$$\liminf_{\substack{x \xrightarrow{u} \bar{x}, x \notin S, \\ \psi_T(x)/d(x,S) \to 0}} \left\{|\nabla\psi_T|(x)\right\} > 0 \tag{3.12}$$

is fulfilled. Then, the set-valued mapping T is (γ, h)-metrically pseudo subregular w.r.t. the direction u at $(\bar{x}, \bar{y}) \in \operatorname{gph} T$.

Proof When (3.12) is valid, the set-valued mapping T must be (γ, h)-metrically pseudo subregular as a direct consequence of Proposition 3.1. □

Corollary 3.1 *Under the same assumptions of Theorem 3.1, if now we have*

$$\liminf_{\substack{x \xrightarrow{u} \bar{x}, x \notin S, \\ \psi_T(x)/\|x - \bar{x}\| \to 0}} \left\{|\nabla\psi_T|(x)\right\} > 0, \tag{3.13}$$

then, the set-valued mapping T is (γ, h)-metrically pseudo subregular w.r.t. the direction u at $(\bar{x}, \bar{y}) \in \operatorname{gph} T$.

Proof Relation (3.13) implies the one in (3.12). □

Using Theorem 3.1 and taking $h_i(x) = \|x - \bar{x}\|$, one obtains a slope characterization of directional pseudo subregularity of T.

Proposition 3.2 (slope characterization) *If*

$$\liminf_{\substack{x \xrightarrow{u} \bar{x}, x \notin S \\ \|x - \bar{x}\|^{-\gamma}\varphi_{T_i}(x) \to 0}} \left|\nabla \operatorname{cl}\left(\sum_{i=1}^{m} \frac{\rho_{T_i}(\cdot)}{\|\cdot - \bar{x}\|^{\gamma_i - 1}}\right)\right|(x) > 0 \tag{3.14}$$

then T is (γ, h)-MPSR in the direction u at $(\bar{x}, \bar{y})$.

Similarly, considering the case $m=1$, $h_1(x) := d\big(x, T^{-1}(\bar{y})\big)$, we derive a characterization of directional Hölder metric subregularity of T.

Proposition 3.3 (slope characterization) *Suppose that the following condition*

$$\liminf_{\substack{x \xrightarrow{u} \bar{x}, x \notin S, \\ \frac{d(\bar{y}, T(x))}{\|x-\bar{x}\| d(x, T^{-1}(\bar{y}))^{\gamma-1}} \to 0}} \left| \nabla \operatorname{cl} \frac{d(\bar{y}, T(\cdot))}{d(\cdot, T^{-1}(\bar{y}))^{\gamma-1}} \right|(x) > 0 \tag{3.15}$$

does hold. Then T is directional Hölder γ-metric subregular in the direction u at $(\bar{x}, \bar{y})$.

In many applications, it is sufficient to focus on the case $m=1$. For such a situation, when applying Theorem 3.1 we have to deal with the slope of the quotient of two functions. The next lemma will be useful to the computation of the slope of such a function.

Lemma 3.1 *Let $f, g : X \longrightarrow \mathbb{R} \cup \{\infty\}$ be lsc extended real-valued functions and let $x \in \operatorname{Dom} f \cap \operatorname{Dom} g$ satisfying $f(x) \geqslant 0$ and $g(x) > 0$. Suppose in addition that g is continuous at x. Under these assumptions, one has*

$$\left| \nabla \left(\frac{f}{g} \right) \right|(x) \geqslant \frac{|\nabla f|(x)}{g(x)} - \frac{f(x)}{g(x)^2} |\nabla g|(x). \tag{3.16}$$

Proof Let us denote $\Theta := \frac{f}{g}$. If x is an isolated point of $\operatorname{Dom}\Theta$, then x is a local minimum of both Θ and f, so $|\nabla \Theta|(x) = |\nabla f|(x) = 0$, and the conclusion is trivial. On the contrary, we fix a sequence of nonnegative reals $(\varepsilon_k) \downarrow 0$. Then, there is a sequence $(\delta_k) \downarrow 0$ for which the following property holds true:

$$\|z - x\| \leqslant \delta_k \Longrightarrow g(x) - g(z) \leqslant \big(|\nabla g|(x) + \varepsilon_k\big) \|x - z\|. \tag{3.17}$$

For each k, we may select $z_k \in \mathbb{B}(x, \delta_k) \setminus \{x\}$ such that

$$f(x) - f(z_k) \geqslant \big(|\nabla f|(x) - \varepsilon_k\big) \|x - z_k\|. \tag{3.18}$$

Due to the continuity of g, it is possible to assume $g(z_k) > 0$. We have

$$\begin{aligned}
\frac{\Theta(x) - \Theta(z_k)}{\|x - z_k\|} &= \frac{\dfrac{f(x)}{g(x)} - \dfrac{f(z_k)}{g(z_k)}}{\|x - z_k\|} \\
&= \frac{1}{g(z_k)} \cdot \frac{f(x) - f(z_k)}{\|x - z_k\|} - \frac{f(x)}{g(x) g(z_k)} \cdot \frac{g(x) - g(z_k)}{\|x - z_k\|} \\
&\geqslant \frac{1}{g(z_k)} \big(|\nabla f|(x) - \varepsilon_k\big) - \frac{f(x)}{g(x) g(z_k)} \big(|\nabla g|(x) + \varepsilon_k\big).
\end{aligned}$$

From the last inequality we derive

$$\begin{aligned}
&\limsup_{k\to\infty}\frac{\Theta(x)-\Theta(z_k)}{\|x-z_k\|}\\
&\quad\geqslant \limsup_{k\to\infty}\left(\frac{1}{g(z_k)}\left(|\nabla f|(x)-\varepsilon_k\right)-\frac{f(x)}{g(x)g(z_k)}\left(|\nabla g|(x)+\varepsilon_k\right)\right)\\
&\quad=\frac{1}{g(x)}|\nabla f|(x)-\frac{f(x)}{g(x)^2}|\nabla g|(x).
\end{aligned}$$

Since $|\nabla\Theta|(x)=\limsup_{z\to x}\frac{\Theta(x)-\Theta(z)}{\|x-z\|}$, we obtain

$$|\nabla\Theta|(x)\geqslant\limsup_{k\to\infty}\frac{\Theta(x)-\Theta(z_k)}{\|x-z_k\|}\geqslant\frac{1}{g(x)}|\nabla f|(x)-\frac{f(x)}{g(x)^2}|\nabla g|(x).$$

□

Invoking Lemma 3.1, we obtain Lemma 3.2 used in the sequel for proving Proposition 3.4.

Lemma 3.2 *Let $T:X\rightrightarrows Y$ be a given set-valued mapping and let $(\bar{x},\bar{y})\in\operatorname{gph}T$. Suppose that $h:X\longrightarrow\mathbb{R}_+$ is locally Lipschitz around $\bar{x}$ and that the subsequent condition is valid as well*

$$\limsup_{x\xrightarrow{u}\bar{x},x\notin S}\frac{d(x,S)}{h(x)}<+\infty \tag{3.19}$$

for some $u\in X$. Then,

$$\liminf_{\substack{x\xrightarrow{u}\bar{x},x\notin S,\\ \frac{h(x)^{1-\gamma}\rho_T(x)}{d(x,S)}\to 0}}\left|\nabla\left(\frac{\operatorname{cl}\rho_T}{h^{\gamma-1}}\right)\right|(x)\geqslant\liminf_{\substack{x\xrightarrow{u}\bar{x},x\notin S,\\ h(x)^{-\gamma}\rho_T(x)\to 0}}\left\{\frac{|\nabla\operatorname{cl}\rho_T|(x)}{h(x)^{\gamma-1}}\right\}. \tag{3.20}$$

Proof Applying Lemma 3.1 with $f=\operatorname{cl}\rho_T$ and $g(\cdot)=h(\cdot)^{\gamma-1}$ we deduce

$$\left|\nabla\left(\frac{\operatorname{cl}\rho_T}{h^{\gamma-1}}\right)\right|(x)\geqslant\frac{|\nabla\operatorname{cl}\rho_T|(x)}{h(x)^{\gamma-1}}-\frac{\operatorname{cl}\rho_T(x)}{h(x)^{\gamma-1}}\frac{\left|\nabla\left(h^{\gamma-1}\right)\right|(x)}{h(x)^{\gamma-1}} \tag{3.21}$$

for each $x\notin S$. According to [19], we have

$$\left|\nabla\left(h^{\gamma-1}\right)\right|(x)=(\gamma-1)h(x)^{\gamma-2}\left|\nabla h\right|(x). \tag{3.22}$$

But since h is locally Lipschitz, it holds that $\kappa(x):=\left|\nabla h\right|(x)$ is locally bounded around $\bar{x}$. Let's re-write (3.21) as follows

$$\begin{aligned}
\left|\nabla\left(\frac{\operatorname{cl}\rho_T}{h^{\gamma-1}}\right)\right|(x)&\geqslant\frac{|\nabla\operatorname{cl}\rho_T|(x)}{h(x)^{\gamma-1}}-(\gamma-1)\kappa(x)\frac{\operatorname{cl}\rho_T(x)}{h(x)}\\
&\geqslant\frac{|\nabla\operatorname{cl}\rho_T|(x)}{h(x)^{\gamma-1}}-(\gamma-1)\kappa(x)h(x)^{\gamma-1}\frac{\rho_T(x)}{h(x)^{\gamma-1}d(x,S)}\cdot\frac{d(x,S)}{h(x)}.
\end{aligned} \tag{3.23}$$

Combining (3.19), (3.20) with (3.23) we infer

$$\liminf_{\substack{x \xrightarrow{u} \bar{x}, x \notin S, \\ \frac{h(x)^{1-\gamma}\rho_T(x)}{d(x,S)} \to 0}} \left|\nabla\left(\frac{\operatorname{cl}\rho_T}{h^{\gamma-1}}\right)\right|(x) \geqslant \liminf_{\substack{x \xrightarrow{u} \bar{x}, x \notin S, \\ \frac{h(x)^{1-\gamma}\rho_T(x)}{d(x,S)} \to 0}} \frac{|\nabla \operatorname{cl}\rho_T|(x)}{h(x)^{\gamma-1}}. \tag{3.24}$$

Assume that (x_k) is a sequence in X such that

$$x_k \xrightarrow{u} \bar{x},\ x_k \notin S,\ \frac{h(x_k)^{1-\gamma}\rho_T(x_k)}{d(x_k,S)} \to 0. \tag{3.25}$$

By virtue of (3.19), we have

$$\limsup_{k\to\infty} \frac{d(x_k,S)}{h(x_k)} < +\infty,$$

which yields

$$\limsup_{k\to\infty}\left\{h(x_k)^{-\gamma}\rho_T(x_k)\right\} = \limsup_{k\to\infty}\left\{\frac{h(x_k)^{1-\gamma}\rho_T(x_k)}{d(x_k,S)}\cdot\frac{d(x_k,S)}{h(x_k)}\right\} = 0. \tag{3.26}$$

As a result, we get

$$\liminf_{k\to\infty}\left\{\frac{|\nabla \operatorname{cl}\rho_T|(x_k)}{h(x_k)^{\gamma-1}}\right\} \geqslant \liminf_{\substack{x \xrightarrow{u} \bar{x}, x \notin S, \\ h(x)^{-\gamma}\rho_T(x)\to 0}} \left\{\frac{|\nabla \operatorname{cl}\rho_T|(x)}{h(x)^{\gamma-1}}\right\}. \tag{3.27}$$

Combining (3.25) with (3.27), it holds that

$$\liminf_{\substack{x \xrightarrow{u} \bar{x}, x \notin S, \\ \frac{h(x)^{1-\gamma}\rho_T(x)}{d(x,S)} \to 0}} \frac{|\nabla \operatorname{cl}\rho_T|(x)}{h(x)^{\gamma-1}} \geqslant \liminf_{\substack{x \xrightarrow{u} \bar{x}, x \notin S, \\ h(x)^{-\gamma}\rho_T(x)\to 0}} \left\{\frac{|\nabla \operatorname{cl}\rho_T|(x)}{h(x)^{\gamma-1}}\right\}. \tag{3.28}$$

In summary, (3.24) and (3.28) give us

$$\liminf_{\substack{x \xrightarrow{u} \bar{x}, x \notin S, \\ \frac{h(x)^{1-\gamma}\rho_T(x)}{d(x,S)} \to 0}} \left|\nabla\left(\frac{\operatorname{cl}\rho_T}{h^{\gamma-1}}\right)\right|(x) \geqslant \liminf_{\substack{x \xrightarrow{u} \bar{x}, x \notin S, \\ h(x)^{-\gamma}\rho_T(x)\to 0}} \left\{\frac{|\nabla \operatorname{cl}\rho_T|(x)}{h(x)^{\gamma-1}}\right\}. \tag{3.29}$$

This completes the proof of Lemma 3.2. □

Based on the previous results, we present a robust version of Theorem 3.1 for the case $m = 1$ which might be more comfortable in practice.

Proposition 3.4 *Let T, $\bar{x}$, $\bar{y}$ and h satisfy all assumptions of Lemma 3.2. Then, under the following condition*

$$\liminf_{\substack{x \xrightarrow{u} \bar{x}, x \notin S, \\ h(x)^{-\gamma}\rho_T(x)\to 0}} \frac{|\nabla \operatorname{cl}\rho_T|(x)}{h(x)^{\gamma-1}} > 0, \tag{3.30}$$

the set-valued mapping T is γ-metrically pseudo subregular at $\bar{x}$ for $\bar{y}$ in the direction u.

Proof The proof follows by combining Theorem 3.1 with Lemma 3.2. □

4 Coderivative characterization of directional metric pseudo subregularity

Theorem 3.1 provides two sufficient conditions ensuring the validity of directional pseudo subregularity through the slopes corresponding to suitable functions. Gfrerer in his work [9] dealt with such a property using the notion of coderivatives. In order to study the directional metric regularity property, the authors of [21], introduced the notion of limiting critical set around a reference point $(\bar{x},\bar{y}) \in \operatorname{gph} T$. Following a similar trend, we shall develop an infinitesimal criterion for directional metric pseudo subregularity using the coderivative associated with an abstract subdifferential ∂. Hereinafter, we assume that the abstract subdifferential ∂ satisfies the following quotient fuzzy rule:

(C_5): Let $f_1, f_2 : X \to \mathbb{R} \cup \{+\infty\}$ be two locally Lipschitz functions around $x_0 \in X$ with $f_1(x_0) \geq 0$, $f_2(x_0) > 0$. For any $\varepsilon > 0$ one has

$$\partial\left(\frac{f_1}{f_2}\right)(x_0) \subseteq \bigcup_{x_1,x_2 \in B(x_0,\varepsilon)} \left\{ \frac{f_2(x_0)\partial f_1(x_1) - f_1(x_0)\partial f_2(x_2)}{f_2(x_0)^2} + \varepsilon B_{X^\star} \right\}.$$

Note that (C_5) is valid for all usual subdifferentials in the literature of variational analysis.

Proposition 4.1 *Let $T : X \rightrightarrows Y = Y_1 \times \cdots \times Y_m$ be a closed set-valued mapping between two Banach spaces X and Y and $(\bar{x},\bar{y}) \in \operatorname{gph} T$. Suppose given functions $h_i : X \longrightarrow \mathbb{R}_+, (i = 1, \cdots, m)$ locally Lipschitz around $\bar{x}$. Let $u \in X$ and $\gamma \geqslant 1$ be given for which the following condition is fulfilled*

$$\limsup_{x \xrightarrow{u} \bar{x}, x \notin S} \max_{i=1,\dots,m} \frac{d(x,S)}{h_i(x)} < +\infty. \tag{4.1}$$

If T is not metrically pseudo subregular in the direction u at $(\bar{x},\bar{y})$, then there exist some real sequence $(t_k) \downarrow 0$ together with $(u_k, v_k) \in \mathbf{S}_{X\times Y}$, $u_k^\star \in X^*$ and $v_k^\star \in Y^*$ such that

(a). $\lim_{k\to\infty} \|u_k\| = 1$, $\lim_{k\to\infty} \big\| \|u\|u_k - u \big\| = 0$;

(b). $\lim_{k\to\infty} \max_{i=1,\dots,m} \left\{ (t_k)^{1-\gamma_i} \|v_{ki}\| \right\} = 0$;

(c). the vector $(u_k^\star, -\tilde{v}_k^\star)$ *belongs to* $N\big(\operatorname{gph} T, (\bar{x} + t_k u_k, \bar{y} + t_k v_k)\big)$, *where* $\tilde{v}_k^\star \in Y^*$ *is given by* $\tilde{v}_{ki}^\star = h_i(\bar{x} + t_k u_k)^{1-\gamma_i} v_{ki}^\star$;

(d). $\lim_{k\to\infty} \left\{ \dfrac{\sum_{i=1}^m \langle v_{ki}^\star, h_i(\bar{x}+t_k u_k)^{1-\gamma_i} v_{ki}\rangle}{\big\| \big(h_1(\bar{x}+t_k u_k)^{1-\gamma_1} v_{k1}, \dots, h_m(\bar{x}+t_k u_k)^{1-\gamma_m} v_{km}\big) \big\|} \right\} = 1$;

(e). $\lim_{k\to\infty} \left\{ \|u_k^\star\| \right\} = 0$, $\lim_{k\to\infty} \left\{ \|v_{ki}^\star\| \right\} = 1$.

Proof For convenience, we assume that the distance on Y is given by

$$\|y\|_Y := \|y_1\|_{Y_1} + \cdots + \|y_m\|_{Y_m}.$$

Further, with the aim of simplifying the notation, let us use $\|\cdot\|$ to indicate the norm in any Banach space. By Proposition 3.1, we can find some sequences $x_k \xrightarrow{u} \bar{x}$ such that

(i) $x_k \notin S$;

(ii) $\lim\limits_{k\to\infty} \dfrac{\psi_T(x_k)}{d(x_k,S)} = 0$;

(iii) $\lim\limits_{k\to\infty} \left\{ \left|\nabla\psi_T\right|(x_k) \right\} = 0$.

Denoting $\alpha_k := \|x_k - \bar{x}\| > 0$ and assuming $\psi_T(x_k) = \beta_k d(x_k,S)$ where $\beta_k \downarrow 0$, for each k, choose some positive parameters σ_k and η_k satisfying $\sigma_k = o(\psi_T(x_k))$ and $\eta_k < \sigma_k$. Making σ_k and η_k smaller if necessary, we may suppose

- $2\eta_k + \sigma_k < \psi_T(x_k)$;
- $\varphi_T(z) > \psi_T(x_k) - \sigma_k$ whenever $\|z - x_k\| \leqslant \eta_k$.

Let $\tau_k := \left|\nabla\psi_T\right|(x_k)$, then according to the definition, x_k is a local minimum to the function $\psi_T(\cdot) + (\tau_k + \sigma_k)\|\cdot - x_k\|$. Hence, there exists a radius $r_k \leqslant \min\limits_{j=1,\dots,m} \left\{ h_j(x_k)^{\gamma_j}\eta_k \right\}$ such that

$$\psi_T(x_k) = \min_{\|x-x_k\|\leqslant 2r_k} \left\{ \psi_T(x) + (\tau_k+\sigma_k)\|x - x_k\| \right\}. \tag{4.2}$$

By the definition of a lsc envelope, we may select some $(\hat{x}_k,\hat{y}_k) \in \operatorname{gph} T$ which fulfills the two properties below:

$$\|\hat{x}_k - x_k\| \leqslant \sigma_k r_k \quad \text{and} \quad \sum_{j=1}^{m} \frac{\|\bar{y}_j - \hat{y}_{kj}\|}{h_j(\hat{x}_k)^{\gamma_j - 1}} \leqslant \psi_T(x_k) + \sigma_k r_k. \tag{4.3}$$

Using (4.2), we deduce

$$\begin{aligned} \sum_{j=1}^{m} \frac{\|\bar{y}_j - \hat{y}_{kj}\|}{h_j(\hat{x}_k)^{\gamma_j-1}} &\leqslant \psi_T(x) + (\tau_k+\sigma_k)\|x - x_k\| + \sigma_k r_k \\ &\leqslant \delta_{\operatorname{gph} T}(x,y) + \sum_{j=1}^{m} \frac{\|\bar{y}_j - y_j\|}{h_j(x)^{\gamma_j-1}} + (\tau_k+\sigma_k)\|x - \hat{x}_k\| \\ &\quad + (\tau_k+\sigma_k)\|x_k - \hat{x}_k\| + \sigma_k r_k. \end{aligned} \tag{4.4}$$

Define a lsc function $f_k : X \times Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ by the formula

$$f_k(x,y) := \delta_{\operatorname{gph} T}(x,y) + \sum_{j=1}^{m} \frac{\|\bar{y}_j - y_j\|}{h_j(x)^{\gamma_j-1}} + (\tau_k+\sigma_k)\|x - \hat{x}_k\|. \tag{4.5}$$

Observe that $f_k(\hat{x}_k,\hat{y}_k) = \sum_{j=1}^{m} \frac{\|\bar{y}_j - \hat{y}_{kj}\|}{h_j(\hat{x}_k)^{\gamma_j-1}}$ and that

$$f_k(\hat{x}_k,\hat{y}_k) \leqslant \inf_{\|x-x_k\|\leqslant 2r_k} f(x,y) + \varepsilon_k,$$

where $\varepsilon_k := (\tau_k+\sigma_k)\|x_k - \hat{x}_k\| + \sigma_k r_k > 0$. Setting $\lambda_k := r_k\sqrt{\sigma_k} > 0$ and applying the Ekeland variational principle, take $(\tilde{x}_k,\tilde{y}_k) \in X \times Y$ such that

- $\|(\hat{x}_k,\hat{y}_k) - (\tilde{x}_k,\tilde{y}_k)\| \leqslant \lambda_k$;
- $f_k(\hat{x}_k,\hat{y}_k) - \frac{\varepsilon_k}{\lambda_k}\|(\hat{x}_k,\hat{y}_k) - (\tilde{x}_k,\tilde{y}_k)\| \geqslant f_k(\tilde{x}_k,\tilde{y}_k)$;
- $(\tilde{x}_k,\tilde{y}_k)$ is a minimum to the function $(x,y) \in X \times Y \longmapsto f_k(x,y) + \frac{\varepsilon_k}{\lambda_k}\|(x,y) - (\tilde{x}_k,\tilde{y}_k)\|$ subject to the constraint $\|x - x_k\| \leqslant 2r_k$.

Observe that $\lambda_k = o(r_k)$, so $\tilde{x}_k \neq \bar{x}$, which allows for writing $\tilde{y}_{kj} \neq \bar{y}_j$. The last condition along with the properties $(C_1)-(C_5)$ of the subdifferential operator show that, there are some elements $\tilde{x}_k^i, \tilde{x}_{kj} \in X$, $\tilde{y}_k^i \in Y$, $\tilde{y}_{kj} \in Y_j$ and also $\tilde{x}_k^{i\star}, \tilde{x}_{kj}^\star \in X^*$, $\tilde{y}_k^{i\star} \in Y^*$, $\tilde{y}_{kj}^\star \in Y_j^\star$ which fulfill the following conditions:

(i) $\max\left\{\|\tilde{x}_k - \tilde{x}_k^i\|, \|\tilde{x}_k - \tilde{x}_{kj}\|, \|\tilde{y}_k - \tilde{y}_k^i\|, \|\tilde{y}_k - \tilde{y}_{kj}\|\right\} \leqslant \nu_k$, where $\nu_k \leqslant \min\left\{\lambda_k, \beta_k\|\bar{y}_1 - \tilde{y}_{k1}\|, \ldots, \beta_k\|\bar{y}_m - \tilde{y}_{km}\|\right\}$;

(ii) $(\tilde{x}_k^{1\star}, \tilde{y}_k^{1\star}) \in \partial_{X\times Y}\delta_{\operatorname{gph} T}(\tilde{x}_k^1, \tilde{y}_k^1) = N\big(\operatorname{gph} T, (\tilde{x}_k^1, \tilde{y}_k^1)\big)$;

(iii) $\tilde{x}_k^{2\star} \in \partial_X\|\cdot - \hat{x}_k\|(\tilde{x}_k^2)$;

(iv) $(\tilde{x}_k^{3\star}, \tilde{y}_k^{3\star}) \in \partial_{X\times Y}\|(\cdot,\cdot) - (\tilde{x}_k, \tilde{y}_k)\|(\tilde{x}_k^3, \tilde{y}_k^3)$;

(v) $\tilde{y}_{kj}^{4\star} \in \partial_{Y_j}\|\bar{y}_j - \cdot\|(\tilde{y}_{kj}^4)$;

(vi) $\tilde{x}_{kj}^{4\star} \in \partial_X\left(h_j^{1-\gamma_j}\right)(\tilde{x}_{kj}^4) = (1-\gamma_j)h_j(\tilde{x}_{kj}^4)^{-\gamma_j}\left(\partial_X h_j\right)(\tilde{x}_{kj}^4)$;

(vii) $\left\|\tilde{x}_k^{1\star} + (\tau_k + \sigma_k)\tilde{x}_k^{2\star} + \frac{\varepsilon_k}{\lambda_k}\tilde{x}_k^{3\star} + \sum_{j=1}^m \|\bar{y}_j - \tilde{y}_{kj}^4\|\tilde{x}_{kj}^{4\star}\right\| \leqslant \sigma_k$;

(viii) $\left\|\tilde{y}_{kj}^{1\star} + \frac{\varepsilon_k}{\lambda_k}\tilde{y}_{kj}^{3\star} + h_j(x_{kj}^4)^{1-\gamma_j}\tilde{y}_{kj}^{4\star}\right\| \leqslant \sigma_k$.

We shall establish the conclusion of Proposition 4.1 step-by-step through several auxiliary facts.

Fact 1 *It holds that*

$$\lim_{k\to\infty}\frac{\|\hat{x}_k - x_k\|}{\|x_k - \bar{x}\|} = \lim_{k\to\infty}\frac{\|\tilde{x}_k^i - x_k\|}{\|x_k - \bar{x}\|} = \lim_{k\to\infty}\frac{\|\tilde{x}_{kj}^4 - x_k\|}{\|x_k - \bar{x}\|} = 0; \tag{4.6a}$$

$$\lim_{k\to\infty}\frac{h_j(\hat{x}_k)}{h_j(x_k)} = \lim_{k\to\infty}\frac{h_j(\tilde{x}_k^i)}{h_j(x_k)} = \lim_{k\to\infty}\frac{h_j(\tilde{x}_{kj}^4)}{h_j(x_k)} = 1; \tag{4.6b}$$

$$\lim_{k\to\infty}\left\{\frac{1}{\psi_T(x_k)}\sum_{j=1}^m\frac{\|\bar{y}_j - \hat{y}_{kj}\|}{h_j(\hat{x}_k)^{\gamma_j - 1}}\right\} = 1; \tag{4.6c}$$

$$\lim_{k\to\infty}\left\{\frac{1}{\psi_T(x_k)}\sum_{j=1}^m\frac{\|\bar{y}_j - \tilde{y}_{kj}^i\|}{h_j(\tilde{x}_k^i)^{\gamma_j - 1}}\right\} = 1, i = 1,2,3; \tag{4.6d}$$

$$\lim_{k\to\infty}\left\{\frac{1}{\psi_T(x_k)}\sum_{j=1}^m\frac{\|\bar{y}_j - \tilde{y}_{kj}^4\|}{h_j(\tilde{x}_{kj}^4)^{\gamma_j - 1}}\right\} = 1. \tag{4.6e}$$

Proof of Fact 1 Firstly, we establish (4.6a). Equality $\lim_{k\to\infty}\frac{\|\hat{x}_k - x_k\|}{\|x_k - \bar{x}\|} = 0$ is trivial due to the choice of $\hat{x}_k$. Further, since

$$\max\left\{\|\tilde{x}_k^i - \tilde{x}_k\|, \|\tilde{x}_{kj}^4 - \tilde{x}_k\|\right\} \leqslant \lambda_k = o(\|x_k - \bar{x}\|),$$

whereas $\|\tilde{x}_k - \hat{x}_k\| \leqslant \lambda_k = o(\|x_k - \bar{x}\|)$, one gets

$$\max\left\{\|\tilde{x}_k^i - x_k\|, \|\tilde{x}_{kj}^4 - x_k\|\right\} = o(\|x_k - \bar{x}\|),$$

which implies (4.6a).

For (4.6b), let us involve the Lipschitz property to each function h_j. Indeed, for each $j = 1,2,\ldots,m$ there is $L_j > 0$ and $r_j > 0$ for which one has

$$\left.\begin{array}{c}\|x - \bar{x}\| \leqslant r_j, \|x' - \bar{x}\| \leqslant r_j \\ \|x - x'\| \leqslant r_j\end{array}\right\} \Longrightarrow |h_j(x) - h_j(x')| \leqslant L_j\|x - x'\|. \tag{4.7}$$

Thus, when k is large, it holds that

$$|h_j(\hat{x}_k) - h_j(x_k)| \leqslant L_j \|\hat{x}_k - x_k\| \leqslant L_j \sigma_k r_k. \tag{4.8}$$

This allows us to write

$$\Big|\frac{h_j(\hat{x}_k)}{h_j(x_k)} - 1\Big| \leqslant L_j \frac{r_k}{h_j(x_k)} \sigma_k. \tag{4.9}$$

By interchanging in turn the role between $\hat{x}_k$ with $\tilde{x}_{kj}$, $\tilde{x}_k^i$ and repeating the arguments above, we deduce

$$\max\Big\{\Big|\frac{h_j(\tilde{x}^4_{kj})}{h_j(x_k)} - 1\Big|, \Big|\frac{h_j(\tilde{x}^i_k)}{h_j(x_k)} - 1\Big|\Big\} \leqslant L_j \frac{r_k}{h_j(x_k)} (\sigma_k + 2\sqrt{\sigma_k}). \tag{4.10}$$

Since $r_k \leqslant h_j(x_k)^\gamma \eta_k$, (4.6b) follows from (4.9) and (4.10).

In the next step, we prove (4.6c). According to the choice of $\hat{x}_k$, it is possible to derive

$$h_j(\hat{x}_k)^{1-\gamma_j} \|\bar{y}_j - \hat{y}_{kj}\| \geqslant h_j(\hat{x}_k)^{1-\gamma_j} d\big(\bar{y}_j, T_j(\hat{x}_k)\big) = \varphi_{T_j}(x_k).$$

As a result

$$\sum_{j=1}^m \frac{\|\bar{y}_j - \hat{y}_{kj}\|}{h_j(\hat{x}_k)^{\gamma_j - 1}} \geqslant \varphi_T(x_k) > \psi_T(x_k) - \sigma_k.$$

Further, we have $\sum_{j=1}^m \frac{\|\bar{y}_j - \hat{y}_{kj}\|}{h_j(\hat{x}_k)^{\gamma_j-1}} < \psi_T(x_k) + \sigma_k r_k$ by the choice of $\hat{y}_{kj}$. Thus, the limit (4.6c) is a direct consequence of the fact that $\sigma_k = o(\psi_T(x_k))$.

Based on (4.6b), we can derive that the proof of (4.6e) is similar to the one of (4.6d). Thus, it is sufficient to verify (4.6d) only. Indeed, fix an index $i \in \{1,2,3\}$. We infer from (4.6b) that

$$\lim_{k\to\infty} \left\{ \frac{1}{\psi_T(x_k)} \sum_{j=1}^m \frac{\|\bar{y}_j - \hat{y}_{kj}\|}{h_j(\tilde{x}^i_k)^{\gamma_j - 1}} \right\} = 1.$$

According to the choice of $\tilde{y}^i_{kj}$ we get

$$\|\tilde{y}^i_{kj} - \hat{y}_{kj}\| \leqslant \|\tilde{y}_{kj} - \hat{y}_{kj}\| + \|\tilde{y}^i_{kj} - \hat{y}_{kj}\| \leqslant 2\lambda_k = 2r_k\sqrt{\sigma_k},$$

which yields

$$\begin{aligned} \frac{\|\tilde{y}^i_{kj} - \hat{y}_{kj}\|}{\psi_T(x_k) h_j(\tilde{x}^i_k)^{\gamma_j - 1}} &\leqslant 2\sqrt{\sigma_k} \frac{r_k}{\psi_T(x_k) h_j(\tilde{x}^i_k)^{\gamma_j - 1}} \\ &\leqslant 2\sqrt{\sigma_k} \frac{h(x_k)^{\gamma_j} \eta_k}{\psi_T(x_k) h_j(\tilde{x}^i_k)^{\gamma_j - 1}}. \end{aligned}$$

Recalling that $\eta_k < \sigma_k = o(\psi_T(x_k))$, we obtain (4.6d) by applying (4.6b). □

Fact 2 *We have $\bar{y} \notin T(\tilde{x}_k^1)$.*

Let us define

$$t_k := \left\| \left(\tilde{x}_k^1 - \bar{x}, \tilde{y}_k^1 - \bar{y}\right) \right\|, \tag{4.11a}$$

$$u_k := (t_k)^{-1}(\tilde{x}_k^1 - \bar{x}), \tag{4.11b}$$

$$v_{ki} := (t_k)^{-1}(\tilde{y}_{ki}^1 - \bar{y}_i),\ i = 1, \dots, m. \tag{4.11c}$$

Then the following relations are valid as well

$$\|(u_k, v_k)\| = 1, \tag{4.12a}$$

$$\lim_{k\to\infty} \|u_k\| = 1, \tag{4.12b}$$

$$\lim_{k\to\infty} \left\| \|u\| u_k - u \right\| = 0, \tag{4.12c}$$

$$\lim_{k\to\infty} \left\{ (t_k)^{1-\gamma_i} \|v_{ki}\| \right\} = 0. \tag{4.12d}$$

Proof of Fact 2 Firstly, by the triangle inequality:

$$\|x_k - \tilde{x}_k^1\| \leqslant \|x_k - \hat{x}_k\| + \|\hat{x}_k - \tilde{x}_k\| + \|\tilde{x}_k - \tilde{x}_k^1\| \leqslant \sigma_k r_k + \lambda_k + \nu_k = o(\eta_k).$$

Hence, for k large $\|x_k - \tilde{x}_k^1\| < \eta_k$. Thus, we infer from the choice of η_k that

$$\varphi_T(\tilde{x}_k^1) > \psi_T(\tilde{x}_k^1) - \sigma_k > \eta_k.$$

This shows that $\bar{y} \notin T(\tilde{x}_k^1)$. Particularly, one has $\tilde{x}_k^1 \neq \bar{x}$, which implies $t_k > 0$. Hence, the elements u_k, v_{ki} are well-defined. The equality (4.12a) is trivial by the definitions (4.11a)–(4.11c). To prove (4.12b), we note that $\|\tilde{x}_k^1 - \bar{x}\| \sim \|x_k - \bar{x}\| = \alpha_k$ according to (4.6a). It follows from the choice of $\hat{y}_k$ that (see (4.3))

$$\|\hat{y}_{ki} - \bar{y}_i\| \leqslant h_i(\hat{x}_k)^{\gamma_i - 1}\left(\psi_T(x_k) + \sigma_k r_k\right) = o(\alpha_k),$$

which allows for writing

$$\begin{aligned} \|\tilde{y}_{ki}^1 - \bar{y}_i\| &\leqslant \|\tilde{y}_{ki}^1 - \tilde{y}_{ki}\| + \|\tilde{y}_{ki} - \hat{y}_{ki}\| + \|\hat{y}_{ki} - \bar{y}_i\| \\ &\leqslant \|\tilde{y}_k^1 - \tilde{y}_k\| + \|\tilde{y}_k - \hat{y}_k\| + \|\hat{y}_{ki} - \bar{y}_i\| \\ &\leqslant \nu_k + \lambda_k + \|\hat{y}_{ki} - \bar{y}_i\| = o(\alpha_k). \end{aligned}$$

Thus, the limit in (4.12b) is obtained directly from (4.11a) and (4.11b).

With the aim of verifying (4.12c), let us define the auxiliary element $\hat{u}_k := \frac{\|u\|}{\|x_k - \bar{x}\|}(x_k - \bar{x}) - u$. For such a notation, we have

$$\begin{aligned} \|u\| u_k - u &= (t_k)^{-1} \|u\| (\tilde{x}_k^1 - \bar{u}) - u \\ &= \|u_k\| \left(\hat{u}_k + u\right) - u + \|u\| \|u_k\| \frac{\tilde{x}_k^1 - x_k}{\|x_k - \bar{x}\|} \\ &= \|u_k\| \hat{u}_k + (\|u_k\| - 1)u + \|u\| \|u_k\| \frac{\tilde{x}_k^1 - x_k}{\|x_k - \bar{x}\|}. \end{aligned}$$

Observe that $x_k \xrightarrow{u} \bar{x}$; we get $\|\hat{u}_k\| \to 0$, and hence, (4.12c) follows from (4.12b) and (4.6a).

Finally, we establish (4.12d). Due to (4.11a), (4.11c) we may write

$$(t_k)^{1-\gamma_i}\|v_{ki}\| = \frac{\|\tilde{y}^1_k - \bar{y}\|}{\|(\tilde{x}^1_k - \bar{x}, \tilde{y}^1_k - \bar{y})\|^{\gamma_i}} \leqslant \frac{\|\tilde{y}^1_k - \bar{y}\|}{\|\tilde{x}^1_k - \bar{x}\|^{\gamma_i}} \overset{k\to\infty}{\sim} \frac{\|\tilde{y}^1_k - \bar{y}\|}{\|x_k - \bar{x}\|^{\gamma_i}}.$$

Recalling the estimation for $\|\tilde{y}^1_k - \bar{y}\|$ as above, we find

$$\begin{aligned}\|\tilde{y}^1_{ki} - \bar{y}_i\| &\leqslant \nu_k + \lambda_k + \|\hat{y}_{ki} - \bar{y}_i\| \\ &\leqslant \nu_k + \lambda_k + h_i(\hat{x}_k)^{\gamma_i - 1}\big(\psi_T(x_k) + \sigma_k r_k\big).\end{aligned}$$

We know that $\limsup\limits_{k\to\infty} \dfrac{h_i(x_k)}{\|x_k - \bar{x}\|} < +\infty$, $\lim\limits_{k\to\infty} \dfrac{\psi_T(x_k)}{d(x_k, S)} = 0$. Therefore, (4.6a) and (4.6b) imply $\limsup\limits_{k\to\infty} \dfrac{h_i(\hat{x}_k)^{\gamma_i-1}\psi_T(x_k)}{\|x_k - \bar{x}\|^{\gamma_i}} = 0$. Since $r_k \leqslant h_i(x_k)^{\gamma_i}\eta_k$ and $\lambda_k = o(r_k)$, $\nu_k = o(r_k)$, we conclude that $\|\tilde{y}^1_{ki} - \bar{y}_i\| = o\big(\|x_k - \bar{x}\|^{\gamma_i}\big)$. Thus, (4.12d) is thereby proved. □

Fact 3 *For each $i = 1, 2, \ldots, m$ it holds that*

$$\lim_{k\to\infty} \Big\{[h_i(x_k)]^{\gamma_i - 1}\|\tilde{y}^{1\star}_{ki}\|\Big\} = 1. \tag{4.13}$$

Proof of Fact 3 Invoking (viii), there exists $w^\star_{ki} \in \mathbb{B}_{Y_i^*}$ such that

$$\sigma_k w^\star_{ki} = \tilde{y}^{1\star}_{ki} + \frac{\varepsilon_k}{\lambda_k}\tilde{y}^{3\star}_{ki} + h(x^4_{ki})^{1-\gamma_i}\tilde{y}^{4\star}_{ki}.$$

By virtue of (iv), the sequence $(\tilde{y}^{3\star}_{ki})$ is bounded in norm, so

$$\lim_{k\to\infty} \big\|\tilde{y}^{1\star}_{ki} + h(x^4_{ki})^{1-\gamma_i}\tilde{y}^{4\star}_{ki}\big\| = 0. \tag{4.14}$$

By using (4.6b), it follows that $h_i(\tilde{x}^4_{ki}) > 0$ as k is sufficiently large (because of $h_i(\tilde{x}^4_{ki}) \sim h_i(x_k)$). This implies $\tilde{y}^4_{ki} \neq \bar{y}_i$ unless a finite many of indexes k. Taking into account (v) and since the function $\|\bar{y}_i - \cdot\|$ is convex continuous on Y_j, we can say that $\tilde{y}^4_{ki}$ is a minimum to the function $\|\bar{y}_i - \cdot\| - \langle \tilde{y}^{4\star}_{ki}, \cdot - \tilde{y}^4_{ki}\rangle$. Thus,

$$\|\tilde{y}^{4\star}_{ki}\| = 1, \quad \|\bar{y}_i - \tilde{y}^4_{ki}\| = -\langle \tilde{y}^{4\star}_{ki}, \bar{y}_i - \tilde{y}^4_{ki}\rangle. \tag{4.15}$$

The latter permits us to obtain

$$\begin{aligned}h(x^4_{ki})^{1-\gamma_i} - \big\|\tilde{y}^{1\star}_{ki} + h(x^4_{ki})^{1-\gamma_i}\tilde{y}^{4\star}_{ki}\big\| &\leqslant \|\tilde{y}^{1\star}_{ki}\| \\ &\leqslant h(x^4_{ki})^{1-\gamma_i} + \big\|\tilde{y}^{1\star}_{ki} + h(x^4_{ki})^{1-\gamma_i}\tilde{y}^{4\star}_{ki}\big\|\end{aligned} \tag{4.16}$$

Combining (4.6b), (4.14) with (4.16) we obtain (4.13). □

Fact 4 Let us define with respect to each k the elements below:

$$u_k^\star := \tilde{x}_k^{1\star}, \tag{4.17a}$$
$$v_{ki}^\star := -h_i(\bar{x}+t_k u_k)^{\gamma_i - 1}\tilde{y}_{ki}^{1\star}, \tag{4.17b}$$
$$\tilde{v}_{ki}^\star := h_i(\bar{x}+t_k u_k)^{1-\gamma_i} v_{ki}^\star. \tag{4.17c}$$

For such elements, we have

$$\lim_{k\to\infty} \|v_{ki}^\star\| = 1, \tag{4.18a}$$
$$\left(u_k^\star, -\tilde{v}_{k1}^\star, \ldots, -\tilde{v}_{km}^\star\right) \in N\left(\operatorname{gph} T, (\bar{x}+t_k u_k, \bar{y}+t_k v_k)\right), \tag{4.18b}$$
$$\lim_{k\to\infty} \|u_k^\star\| = 0, \tag{4.18c}$$
$$\lim_{k\to\infty} \frac{\sum_{i=1}^m \left\langle v_{ki}^\star, h_i(\bar{x}+t_k u_k)^{1-\gamma_i} v_{ki}\right\rangle}{\left\|\left(h_1(\bar{x}+t_k u_k)^{1-\gamma_1} v_{k1}, \ldots, h_m(\bar{x}+t_k u_k)^{1-\gamma_m} v_{km}\right)\right\|} = 1. \tag{4.18d}$$

Proof of Fact 4 The relations (4.18a) is derived from (4.17b) and (4.13), while (4.18b) is a consequence of (ii). In order to obtain (4.18c), we invoke (iii), (iv), (vi), (vii), (4.6e) and (4.13). Indeed, thanks to (vii), there is $\hat{u}_k^\star \in \mathbb{B}_{X^*}$ such that

$$\sigma_k \hat{u}_k^\star = \tilde{x}_k^{1\star} + (\tau_k + \sigma_k)\tilde{x}_k^{2\star} + \frac{\varepsilon_k}{\lambda_k}\tilde{x}_k^{3\star} + \sum_{i=1}^m (1-\gamma_i)\frac{\|\bar{y}_i - \tilde{y}_{ki}^4\|}{h_i(\tilde{x}_{ki}^4)^{\gamma_i}}\tilde{u}_{ki}^\star, \tag{4.19}$$

where $\tilde{u}_{ki}^\star \in \partial_X h_i(\tilde{x}_{ki}^4)$ satisfies $\tilde{x}_{ki}^{4\star} = (1-\gamma_i)h_i(\tilde{x}_{ki}^4)^{-\gamma_i}\tilde{u}_{ki}^\star$. According to (iii), (iv) and by the Lipschitz property of each function h_i, it is possible to check that

$$\limsup_{k\to\infty} \left\{ \max\left\{ \|\tilde{x}_k^{2\star}\|, \|\tilde{x}_k^{3\star}\|, \|\tilde{u}_{k1}^\star\|, \ldots, \|\tilde{u}_{km}^\star\| \right\}\right\} < +\infty. \tag{4.20}$$

Denoting $\gamma^* = \max\{\gamma_1, \ldots, \gamma_m\}$, (4.19) yields

$$\begin{aligned} \|\tilde{x}_k^{1\star}\| &\leqslant \sigma_k \|\hat{u}_k^\star\| + (\tau_k + \sigma_k)\|\tilde{x}_k^{2\star}\| + \frac{\varepsilon_k}{\lambda_k}\|\tilde{x}_k^{3\star}\| \\ &\quad + (\gamma^* - 1)\left(\max_{i=1,\ldots,m} \frac{1}{h_i(\tilde{x}_{ki}^4)}\right)\left(\max_{i=1,\ldots,m} \|\tilde{u}_{ki}^\star\|\right)\sum_{i=1}^m \frac{\|\bar{y}_i - \tilde{y}_{ki}^4\|}{h_i(\tilde{x}_{ki}^4)^{\gamma_i - 1}}. \end{aligned} \tag{4.21}$$

Using (4.6b) and observing that $\limsup_{k\to\infty} \frac{d(x_k, S)}{h_i(x_k)} < +\infty$, we deduce

$$\limsup_{k\to\infty} \frac{d(x_k, S)}{h_i(\tilde{x}_{ki}^4)} < +\infty.$$

As a result, (4.18c) is obtained under the combination of (4.20), (4.21), (4.6e) and the assumption that $\frac{\psi_T(x_k)}{d(x_k,S)} \to 0$.

With the aim of establishing (4.18d), we define some quantities

$$\begin{cases} a_{ki} := h_i(\bar{x}+t_k u_k)^{1-\gamma_i}\langle v_{ki}^\star, v_{ki}\rangle, a_k := \sum_{i=1}^m a_{ki}, \\ b_{ki} := h_i(\bar{x}+t_k u_k)^{1-\gamma_i}\|v_{ki}\|, b_k := \sum_{i=1}^m b_{ki}. \end{cases}$$

Then, we have

$$\begin{cases} -a_{ki} = (t_k)^{-1}\langle \tilde{y}_{ki}^{1\star}, \tilde{y}_{ki}^1 - \bar{y}_i\rangle, \\ b_{ki} = (t_k)^{-1}h_i(h_i(\tilde{x}_k^1))^{1-\gamma_i}\|\tilde{y}_{ki}^1 - \bar{y}_i\|. \end{cases} \tag{4.22}$$

Since $\tilde{y}_{ki}^{1\star} + \frac{\varepsilon_k}{\lambda_k}\tilde{y}_{ki}^{3\star} + h_i(\tilde{x}_{ki}^4)^{1-\gamma_i}\tilde{y}_{ki}^{4\star} = \sigma_k w_{ki}^\star$ (see in the proof of Fact 3), it follows that

$$\begin{aligned} -t_k h_i(\tilde{x}_{ki}^4)^{\gamma_i-1}a_{ki} &= \langle h_i(\tilde{x}_{ki}^4)^{\gamma_i-1}\tilde{y}_{ki}^{1\star}, \tilde{y}_{ki}^1 - \bar{y}_i\rangle = \sigma_k h_i(\tilde{x}_{ki}^4)^{\gamma_i-1}\langle \tilde{w}_{ki}^\star, \tilde{y}_{ki}^1 - \bar{y}_i\rangle \\ &- \frac{\varepsilon_k}{\lambda_k}h_i(\tilde{x}_{ki}^4)^{\gamma_i-1}\langle \tilde{y}_{ki}^{3\star}, \tilde{y}_{ki}^1 - \bar{y}_i\rangle - \langle \tilde{y}_{ki}^{4\star}, \tilde{y}_{ki}^1 - \bar{y}_i\rangle. \end{aligned}$$

Consequently, we find

$$\begin{aligned} & t_k h_i(\bar{x} + t_k u_k)^{\gamma_i-1}\Big\{b_{ki} - \Big[\frac{h_i(\tilde{x}_{ki}^4)}{h_i(\tilde{x}_k^1)}\Big]^{\gamma_i-1}a_{ki}\Big\} \\ &= \|\bar{y}_i - \tilde{y}_{ki}^1\|\left\{\sigma_k h_i(\tilde{x}_{ki}^4)^{\gamma_i-1}\frac{\langle \tilde{w}_{ki}^\star, \tilde{y}_{ki}^1 - \bar{y}_i\rangle}{\|\bar{y}_i - \tilde{y}_{ki}^1\|}\right\} \\ &\quad - \|\bar{y}_i - \tilde{y}_{ki}^1\|\left\{\frac{\varepsilon_k}{\lambda_k}h_i(\tilde{x}_{ki}^4)^{\gamma_i-1}\frac{\langle \tilde{y}_{ki}^{3\star}, \tilde{y}_{ki}^1 - \bar{y}_i\rangle}{\|\bar{y}_i - \tilde{y}_{ki}^1\|}\right\} \\ &\quad + \|\bar{y}_i - \tilde{y}_{ki}^1\|\left\{1 - \frac{\langle \tilde{y}_{ki}^{4\star}, \tilde{y}_{ki}^1 - \bar{y}_i\rangle}{\|\bar{y}_i - \tilde{y}_{ki}^1\|}\right\}. \end{aligned} \tag{4.23}$$

As in the proof of Fact 3 (see (4.15))

$$\|\tilde{y}_{ki}^{4\star}\| = 1,\ \|\bar{y}_i - \tilde{y}_{ki}^4\| + \langle \tilde{y}_{ki}^{4\star}, \bar{y}_i - \tilde{y}_{ki}^4\rangle = 0,$$

and the latter implies

$$\langle \tilde{y}_{ki}^{4\star}, \tilde{y}_{ki}^1 - \bar{y}_i\rangle = \|\bar{y}_i - \tilde{y}_{ki}^4\| + \langle \tilde{y}_{ki}^{4\star}, \tilde{y}_{ki}^1 - \tilde{y}_{ki}^4\rangle. \tag{4.24}$$

Moreover, recall that $\nu_k \leqslant \beta_k\|\bar{y}_i - \tilde{y}_{ki}\|$ (cf. (i)). Therefore, from the fact $\max\{\|\tilde{y}_{ki}^1 - \tilde{y}_{ki}\|, \|\tilde{y}_{ki}^4 - \tilde{y}_{ki}\|\} \leqslant \nu_k$ (by virtue of (i)) one has

$$\lim_{k\to\infty}\frac{\|\tilde{y}_{ki}^1 - \tilde{y}_{ki}\|}{\|\bar{y}_i - \tilde{y}_{ki}\|} = \lim_{k\to\infty}\frac{\|\tilde{y}_{ki}^4 - \tilde{y}_{ki}\|}{\|\bar{y}_i - \tilde{y}_{ki}\|} = 0, \tag{4.25a}$$

$$\lim_{k\to\infty}\frac{\|\bar{y}_i - \tilde{y}_{ki}^1\|}{\|\bar{y}_i - \tilde{y}_{ki}\|} = \lim_{k\to\infty}\frac{\|\bar{y}_i - \tilde{y}_{ki}^4\|}{\|\bar{y}_i - \tilde{y}_{ki}\|} = 1. \tag{4.25b}$$

Combining (4.25a) with (4.25b), we obtain

$$\limsup_{k\to\infty}\left(\frac{|\langle \tilde{y}_{ki}^{4\star}, \tilde{y}_{ki}^1 - \tilde{y}_{ki}^4\rangle|}{\|\bar{y}_i - \tilde{y}_{ki}^1\|}\right) \leqslant \limsup_{k\to\infty}\left(\frac{\|\tilde{y}_{ki}^{4\star}\|\|\tilde{y}_{ki}^1 - \tilde{y}_{ki}^4\|}{\|\bar{y}_i - \tilde{y}_{ki}^1\|}\right) = 0.$$

Taking into account the estimation

$$\left|1 - \frac{\langle \tilde{y}_{ki}^{4\star}, \tilde{y}_{ki}^1 - \bar{y}_i\rangle}{\|\bar{y}_i - \tilde{y}_{ki}^1\|}\right| \leqslant \left|1 - \frac{\|\bar{y}_i - \tilde{y}_{ki}^4\|}{\|\bar{y}_i - \tilde{y}_{ki}\|}\right| + \frac{|\langle \tilde{y}_{ki}^{4\star}, \tilde{y}_{ki}^1 - \tilde{y}_{ki}^4\rangle|}{\|\bar{y}_i - \tilde{y}_{ki}^1\|},$$

we find

$$\lim_{k\to\infty}\left(1-\frac{\langle \tilde{y}_{ki}^{4\star},\tilde{y}_{ki}^{1}-\bar{y}_i\rangle}{\|\bar{y}_i-\tilde{y}_{ki}^{1}\|}\right)=0.$$

Hence, we can infer from (4.23) that

$$t_k h_i(\bar{x}+t_k u_k)^{\gamma_i-1}\left\{b_{ki}-\left[\frac{h_i(\tilde{x}_{ki}^{4})}{h_i(\tilde{x}_k^{1})}\right]^{\gamma_i-1}a_{ki}\right\}=o(\|\bar{y}_i-\tilde{y}_{ki}^{1}\|),\tag{4.26}$$

which implies

$$b_{ki}-\left[\frac{h_i(\tilde{x}_{ki}^{4})}{h_i(\tilde{x}_k^{1})}\right]^{\gamma_i-1}a_{ki}=o(b_{ki}).\tag{4.27}$$

Taking the sum over the index i in (4.27), we reach the conclusion

$$b_k-\sum_{i=1}^{m}\left\{\left[\frac{h_i(\tilde{x}_{ki}^{4})}{h_i(\tilde{x}_k^{1})}\right]^{\gamma_i-1}a_{ki}\right\}=o(b_k).\tag{4.28}$$

Observe that

$$\begin{aligned}\limsup_{k\to\infty}\frac{|a_{ki}|}{b_{ki}}&=\limsup_{k\to\infty}\left\{h_i(\tilde{x}_k^{1})^{\gamma_i-1}\frac{|\langle \tilde{y}_{ki}^{1\star},\bar{y}_i-\tilde{y}_{ki}^{1}\rangle|}{\|\bar{y}_i-\tilde{y}_{ki}^{1}\|}\right\}\\&\leqslant\limsup_{k\to\infty}\left\{h_i(\tilde{x}_k^{1})^{\gamma_i-1}\|\tilde{y}_{ki}^{1\star}\|\right\}=1,\end{aligned}\tag{4.29}$$

we deduce

$$\begin{aligned}b_k-a_k=b_k&-\sum_{i=1}^{m}\left\{\left[\frac{h_i(\tilde{x}_{ki}^{4})}{h_i(\tilde{x}_k^{1})}\right]^{\gamma_i-1}a_{ki}\right\}\\&+\sum_{i=1}^{m}\left\{\left\{\left[\frac{h_i(\tilde{x}_{ki}^{4})}{h_i(\tilde{x}_k^{1})}\right]^{\gamma_i-1}-1\right\}a_{ki}\right\}=o(b_k).\end{aligned}\tag{4.30}$$

This shows that $\lim_{k\to\infty}\left(1-\frac{a_k}{b_k}\right)=0$, which is equivalent to (4.18d). □

Combining Facts 1, 2, 3, 4 we obtain a full proof for Proposition 4.1. □

Definition 4.1 **((γ,h)-limiting critical set)** Let T, γ and h as similar as in Definition 3.2 and let $(\bar{x},\bar{y})\in\operatorname{gph}T$. For some fixed element $u\in X$, we define the limiting critical set $\mathrm{SCr}_{\gamma,h}T(\bar{x},\bar{y})(u)$ with respect to T, γ, h and u at the reference point $(\bar{x},\bar{y})$ as follows. A pair $(v,u^\star)\in Y\times X^*$ lies in $\mathrm{SCr}_{\gamma,h}T(\bar{x},\bar{y})(u)$ if it is possible to find some sequences $(t_k)\downarrow 0$, $(v_k,u_k^\star)\xrightarrow{Y\times X^*}(v,u^\star)$ and $(u_k,v_k^\star)\in S_X\times S_{Y^*}$ which fulfill simultaneously the following conditions:

i. $\lim_{k\to\infty}\big\|\|u\|u_k-u\big\|=0$;
ii. the pair (x_k,y_k) with $x_k:=\bar{x}+t_ku_k$, $y_{ki}:=\bar{y}_i+(t_k)^{\gamma_i}v_{ki}$ is in $\operatorname{gph}T$ but $\bar{y}\notin T(x_k)$;
iii. $\big(u_k^\star,-h_1(x_k)^{1-\gamma_1}v_{k1}^\star,\ldots,-h_m(x_k)^{1-\gamma_m}v_{km}^\star\big)\in N\big(\operatorname{gph}T,(x_k,y_k)\big)$ and one has

$$\lim_{k\to\infty}\left\{\frac{\sum_{i=1}^{m}\langle v_{ki}^\star,h_i(x_k)^{1-\gamma_i}(y_{ki}-\bar{y}_i)\rangle}{\big\|\big(h_1(x_k)^{1-\gamma_1}(y_{k1}-\bar{y}_1),\ldots,h_m(x_k)^{1-\gamma_m}(y_{km}-\bar{y}_m)\big)\big\|}\right\}=1.$$

Using this new notion, we are now ready to present the infinitesimal characterization for the property of (γ,h)-metric pseudo subregularity. The next theorem is in this sense.

Theorem 4.1 *Let T, γ, h and $(\bar{x},\bar{y})$ as in Definition 4.1. Suppose that each function h_i is locally Lipschitz around $\bar{x}$ and that*

$$\limsup_{x\to\bar{x},x\notin S}\frac{d(x,S)}{h_i(x)}<+\infty,\ i=1,\ldots,m. \tag{4.31}$$

If $(0,0)\notin \mathrm{SCr}_{\gamma,h}T(\bar{x},\bar{y})(u)$, then the set-valued mapping T is (γ,h)-metrically pseudo subregular in the direction $u\in X$ at $(\bar{x},\bar{y})$.

Proof The proof is almost based on Proposition 4.1. Assume that the norm in $Y^*=Y_1^*\times\cdots\times Y_m^*$ coincides with the maximum

$$\|y^\star\|_{Y^*}=\left\|\left(y_1^\star,\ldots,y_m^\star\right)\right\|_{Y^*}=\max\left\{\|y_1^\star\|_{Y_1^*},\ldots,\|y_m^\star\|_{Y_m^*}\right\}.$$

Suppose on the contrary that T is not (γ,h)-metrically pseudo subregular in the direction u at $(\bar{x},\bar{y})$. Let $t_k>0$, $(u_k,v_k)\in S_{X\times Y}$, and $(u_k^\star,v_k^\star)\in X^*\times Y^*$ be the sequences in the conclusion of Proposition 4.1. Let us define

$$x_k:=\bar{x}+t_ku_k, y_k:=\bar{y}+t_kv_k; \tag{4.32}$$

then it is clear that

$$\left(u_k^\star,-h_1(x_k)^{1-\gamma_1}v_{k1}^\star,\ldots,-h_m(x_k)^{1-\gamma_m}v_{km}^\star\right)\in N\left(\mathrm{gph}\,T,(x_k,y_k)\right), \tag{4.33a}$$

$$\lim_{k\to\infty}\|u_k^\star\|=0, \lim_{k\to\infty}\|v_{ki}^\star\|=1, \tag{4.33b}$$

$$\lim_{k\to\infty}\|u_k\|=1,\ \lim_{k\to\infty}\left\|\|u\|u_k-u\right\|=0, \tag{4.33c}$$

$$\lim_{k\to\infty}\left\{(t_k)^{1-\gamma_1}\|v_{k1}\|\right\}=\cdots=\lim_{k\to\infty}\left\{(t_k)^{1-\gamma_m}\|v_{km}\|\right\}=0, \tag{4.33d}$$

$$\lim_{k\to\infty}\left\{\frac{\sum_{i=1}^m\langle v_{ki}^\star,h_i(x_k)^{1-\gamma_i}v_{ki}\rangle}{\left\|\left(h_1(x_k)^{1-\gamma_1}v_{k1},\ldots,h_m(x_k)^{1-\gamma_m}v_{km}\right)\right\|}\right\}=1. \tag{4.33e}$$

Further, as proved in Fact 2, we also have $\bar{y}\notin T(\bar{x}+t_ku_k)=T(x_k)$. Setting

$$\hat{t}_k:=\|u_k\|t_k, \tag{4.34a}$$

$$\hat{u}_k:=\|u_k\|^{-1}u_k,\ \hat{v}_{ki}:=(t_k)^{1-\gamma_i}\|u_k\|^{-\gamma_i}v_{ki}, \tag{4.34b}$$

$$\hat{u}_k^\star:=\|v_k^\star\|^{-1}u_k^\star,\ \hat{v}_k^\star:=\|v_k^\star\|^{-1}v_k^\star, \tag{4.34c}$$

we obtain

$$x_k=\bar{x}+\hat{t}_k\hat{u}_k,\ y_{ki}=\bar{y}_i+(\hat{t}_k)^{\gamma_i}\hat{v}_{ki}, \tag{4.35a}$$

$$\left(\hat{u}_k^\star,-h_1(x_k)^{1-\gamma_i}\hat{v}_{k1}^\star,\ldots,-h_m(x_k)^{1-\gamma_m}\hat{v}_{km}^\star\right)\in N\left(\mathrm{gph}\,T,(x_k,y_k)\right), \tag{4.35b}$$

$$\|\hat{u}_k\|=1,\ \|\hat{v}_k^\star\|=1, \tag{4.35c}$$

$$\lim_{k\to\infty}\|\hat{v}_{ki}\|=0,\ \lim_{k\to\infty}\|\hat{u}_k^\star\|=0. \tag{4.35d}$$

On the other hand, combining (4.34c) with (4.33b) and (4.33e), we find

$$\begin{aligned}
&\lim_{k\to\infty}\left\{\frac{\sum_{i=1}^m \langle \hat{v}^\star_{ki}, h_i(x_k)^{1-\gamma_i}(y_{ki}-\bar{y}_i)\rangle}{\left\|\left(h_1(x_k)^{1-\gamma_1}(y_{k1}-\bar{y}_1),\dots,h_m(x_k)^{1-\gamma_m}(y_{km}-\bar{y}_m)\right)\right\|}\right\}\\
&=\lim_{k\to\infty}\left\{\|v^\star_k\|^{-1}\frac{\sum_{i=1}^m \langle v^\star_{ki}, h_i(x_k)^{1-\gamma_i}v_{ki}\rangle}{\left\|\left(h_1(x_k)^{1-\gamma_1}v_{k1},\dots,h_m(x_k)^{1-\gamma_m}v_{km}\right)\right\|}\right\}\\
&=\lim_{k\to\infty}\left\{\left(\max_{i=1,\dots,m}\|v^\star_{ki}\|\right)^{-1}\frac{\sum_{i=1}^m \langle v^\star_{ki}, h_i(x_k)^{1-\gamma_i}v_{ki}\rangle}{\left\|\left(h_1(x_k)^{1-\gamma_1}v_{k1},\dots,h_m(x_k)^{1-\gamma_m}v_{km}\right)\right\|}\right\}\\
&=1.
\end{aligned}$$

In addition, taking into account the representation

$$\|u\|\hat{u}_k-u=\|u_k\|^{-1}(\|u\|u_k-u)+\left(\|u_k\|^{-1}-1\right)u,$$

(4.33c) implies $\lim_{k\to\infty}\big\|\|u\|\hat{u}_k-u\big\|=0$. In summary, it follows that $(0,0)\in \mathrm{SCr}_{\gamma,h}T(\bar{x},\bar{y})(u)$, which contradicts the assumption of Theorem 4.1. □

Remark 4.1 By letting $h_i(x)=\|x-\bar{x}\|$, Theorem 4.1 subsumes to somewhat studied in [9, Theorem 1]. Taking $m=1$, $h_1=d\big(x,T^{-1}(\bar{y})\big)$, then Theorem 4.1 recovers the results presented in the works [21,23].

Note that certain applications concern set-valued mapping having a convex (and closed) graph. For such situations, the counterpart of Theorem 4.1 might be also fulfilled. The next result is in this sense.

Proposition 4.2 (Set-valued mapping with convex graph) *Suppose that the set-valued mapping $T:X\rightrightarrows Y$ has closed convex graph. Fix some given $(\bar{x},\bar{y})\in \operatorname{gph}T$ and $u\in X$. Let γ and h be the same as in Theorem 4.1. If T is (γ,h)-metrically pseudo subregular in the direction u at $(\bar{x},\bar{y})$, then one has $(0,0)\notin \mathrm{SCr}_{\gamma,h}T(\bar{x},\bar{y})(u)$.*

Proof Replacing T by $\tilde{T}(\cdot):=T(\cdot+\bar{x})-\bar{y}$ if necessary, we may assume $\bar{x}=0$ and $\bar{y}=0$. Let T satisfy the assumptions of Proposition 4.2 but $(0,0)$ be in the set $\mathrm{SCr}_{\gamma,h}T(0,0)(u)$. The definition of $\mathrm{SCr}_{\gamma,h}T(0,0)(u)$ shows that, there exist a real sequence $(s_k)\downarrow 0$ together with some sequences $(u_k,v^\star_k)\in \mathbf{S}_X\times\mathbf{S}_{Y^*}$ and $(v_k,u^\star_k)\xrightarrow{Y\times X^*}(0,0)$ which fulfill the conditions below:

- $\lim_{k\to\infty}\big\|\|u\|u_k-u\big\|=0$;
- we have $y_k\in T(x_k)$ but $0=\bar{y}\notin T(x_k)$, in which $x_k=s_ku_k\in X$ and $y_{ki}=(s_k)^{\gamma_i}v_{ki}$;
- $\left(u^\star_k,-h_1(x_k)^{1-\gamma_1}v^\star_{k1},\dots,-h_m(x_k)^{1-\gamma_m}v^\star_{km}\right)$ is an element of the normal cone $N\big(\operatorname{gph}T,(x_k,y_k)\big)$;
- $\lim_{k\to\infty}\left\{\frac{\sum_{i=1}^m\langle v^\star_{ki},h_i(x_k)^{1-\gamma_i}y_{ki}\rangle}{\left\|\left(h_1(x_k)^{1-\gamma_1}y_{k1},\dots,h_m(x_k)^{1-\gamma_m}y_{km}\right)\right\|}\right\}=1.$

Let τ, δ and r be positive real parameters such that

$$d\big(x,T^{-1}(0)\big)\leqslant\tau\sum_{i:h_i(x)>0}h_i(x)^{1-\gamma_i}d\big(0,T_i(x)\big)\tag{4.36}$$

whenever $x \in \operatorname{cone}\mathbb{B}(u,\delta)$ with $0 < \|x\| < r$. Since $\lim_{k\to\infty} \big\| \|u\| u_k - u \big\| = 0$, x_k will be in $\operatorname{cone}\mathbb{B}(u,\delta)$ after skipping a few first indexes k. Hence, it is possible to apply (4.36) at $x = x_k$

$$\begin{aligned} d\big(x_k, T^{-1}(0)\big) &\leqslant \tau \sum_{i=1}^{m} h_i(x_k)^{1-\gamma_i} d\big(0, T_i(x_k)\big) \\ &\leqslant \tau \sum_{i=1}^{m} h_i(x_k)^{1-\gamma_i} \|y_{ki}\|. \end{aligned} \tag{4.37}$$

Recall that $x_k \notin T^{-1}(0)$. By virtue of (4.37), for some $0 < \sigma_k < 1$ there exists $z_k \in T^{-1}(0)$ which fulfills the inequalities

$$0 < \|x_k - z_k\| \leqslant \tau(1+\sigma_k) \sum_{i=1}^{m} h_i(x_k)^{1-\gamma_i} \|y_{ki}\|. \tag{4.38}$$

Because the set $\operatorname{gph} T$ is convex, we may derive from the choice of $u_k^\star$ and $v_k^\star$ that

$$\langle u_k^\star, z_k - x_k \rangle + \sum_{i=1}^{m} \langle -h_i(x_k)^{1-\gamma_i} v_{ki}^\star, -y_{ki} \rangle \leqslant 0. \tag{4.39}$$

As a result, we obtain

$$\begin{aligned} &\frac{\sum_{i=1}^{m} h_i(x_k)^{1-\gamma_i} \|y_{ki}\|}{\|z_k - x_k\|} \frac{\sum_{i=1}^{m} \langle v_{ki}^\star, h_i(x_k)^{1-\gamma_i} y_{ki} \rangle}{\big\| \big(h_1(x_k)^{1-\gamma_1} y_{k1}, \ldots, h_m(x_k)^{1-\gamma_m} y_{km}\big) \big\|} \\ &\qquad \leqslant \frac{\langle u_k^\star, x_k - z_k \rangle}{\|z_k - x_k\|}. \end{aligned} \tag{4.40}$$

Combining (4.38) with (4.40), we deduce

$$\begin{aligned} &\frac{1}{\tau(1+\sigma_k)} \frac{\sum_{i=1}^{m} \langle v_{ki}^\star, h_i(x_k)^{1-\gamma_i} y_{ki} \rangle}{\big\| \big(h_1(x_k)^{1-\gamma_1} y_{k1}, \ldots, h_m(x_k)^{1-\gamma_m} y_{km}\big) \big\|} \\ &\qquad \leqslant \frac{\langle u_k^\star, x_k - z_k \rangle}{\|z_k - x_k\|} \leqslant \|u_k^\star\|. \end{aligned} \tag{4.41}$$

Passing to the limit w.r.t k in both sides of (4.41), we obtain the following estimate

$$\liminf_{k\to\infty} \left\{ \frac{1}{\tau(1+\sigma_k)} \frac{\sum_{i=1}^{m} \langle v_{ki}^\star, h_i(x_k)^{1-\gamma_i} y_{ki} \rangle}{\big\| \big(h_1(x_k)^{1-\gamma_1} y_{k1}, \ldots, h_m(x_k)^{1-\gamma_m} y_{km}\big) \big\|} \right\} \leqslant 0. \tag{4.42}$$

Since the left-hand side of (4.42) takes a positive value, we reach a contradiction. This completes our proof. □

As arising from many applications, for instance, in generalized equations, we restrict our consideration to the case $m = 1$ and $T = f + F$ where $f : X \longrightarrow Y = Y_1$ is C^1 and $F : X \rightrightarrows Y$ has a closed and convex graph. Under some robust condition imposed on the given data f and the abstract subdifferential ∂, the next proposition has the advantage of offering a necessary and sufficient condition for (γ,h)-metric pseudo subregularity.

Proposition 4.3 () *Let $\gamma = \gamma_1 \in [1,2)$ and $h = h_1 : X \longrightarrow \mathbb{R}$ be the same as in statement of Theorem 4.1. Suppose in addition that the Jacobian map ∇f is Lipschitz around $\bar{x}$ while the coderivative associated with ∂ obeys the sum rule*

$$D^*(f+F)(x, f(x)+z) = \nabla f(x)^* + D^*F(x,z) \tag{4.43}$$

for every $(x,z) \in \operatorname{gph} F$ near $(\bar{x},\bar{z}) \in \operatorname{gph} F$. Then, $T = f + F$ is (γ,h)-metrically pseudo subregular in the direction u at $(\bar{x},\bar{y}) \in \operatorname{gph} T$ if and only if $(0,0) \notin \mathrm{SCr}_{\gamma,h} T(\bar{x},\bar{y})(u)$.

Proof It is sufficient to prove only the necessary part. Without any loss of generality, we may assume $\bar{x} = 0$, $f(\bar{x}) = \bar{y} = 0$. Suppose T is (γ,h)-metrically pseudo subregular in the direction u at $(0,0)$. Let $\kappa, \delta, \varepsilon$ be positive real numbers under which the following estimation

$$d\big(x, T^{-1}(0)\big) \leqslant \tau h(x)^{1-\gamma} d\big(0, T(x)\big) \tag{4.44}$$

holds whenever $x \neq 0 \in \operatorname{cone} \mathbb{B}(u,\delta) \cap \varepsilon\mathbb{B}$. If $(0,0) \in \mathrm{SCr}_{\gamma,h} T(0,0)(u)$, we may select for each k, some elements $s_k > 0$, $u_k \in \mathbf{S}_X$, $v_k \in Y$, $u_k^\star \in X^*$ and $v_k^\star \in \mathbf{S}_{Y^*}$ such that

(a). the pair (x_k, y_k) is in $\operatorname{gph} T$ with $x_k = s_k u_k$ and $y_k = f(x_k) + z_k = (s_k)^\gamma v_k$;
(b). $\lim_{k\to\infty} \big\| \|u\| u_k - u \big\| = \lim_{k\to\infty} \|v_k\| = \lim_{k\to\infty} \|u_k^\star\| = 0$;
(c). $\big(u_k^\star, -h(x_k)^{1-\gamma} v_k^\star\big) \in N(\operatorname{gph} T, (x_k, y_k))$;
(d). $\lim_{k\to\infty} \left\{ \dfrac{\langle v_k^\star, h(x_k)^{1-\gamma} y_k \rangle}{\|h(x_k)^{1-\gamma} y_k\|} \right\} = 1.$

In view of (c), one has

$$h(x_k)^{\gamma-1} u_k^\star \in D^*T(x_k, y_k)(v_k^\star).$$

Applying the sum rule formula to $T = f + F$ at (x_k, z_k), the latter inclusion yields

$$h(x_k)^{\gamma-1} u_k^\star - \nabla f(x_k)^* v_k^\star \in D^*F(x_k, z_k)(v_k^\star). \tag{4.45}$$

According to the definition of a coderivative, and noticing that $\operatorname{gph} F$ is closed and convex, we deduce

$$\big\langle h(x_k)^{\gamma-1} u_k^\star - \nabla f(x_k)^* v_k^\star, x - x_k \big\rangle + \big\langle -v_k^\star, z - z_k \big\rangle \leqslant 0 \tag{4.46}$$

when $(x,z) \in \operatorname{gph} F$. Taking (b) into account, we may apply (4.44) with $x = x_k$ and get

$$d\big(x_k, T^{-1}(0)\big) \leqslant \tau h(x_k)^{1-\gamma} d\big(0, T(x_k)\big) \leqslant \tau h(x_k)^{1-\gamma} \|y_k\|. \tag{4.47}$$

Recall that $S = T^{-1}(0)$. Let $\sigma_k > 0$ be such that $\limsup_{k\to\infty} \dfrac{\sigma_k}{d(x_k,S)} = 0$ and let $\hat{x}_k \in S \setminus \{x_k\}$ satisfy

$$\|x_k - \hat{x}_k\| \leqslant d(x_k, S) + \min\left\{ \sigma_k, \frac{1}{k^2} h(x_k)^{1-\gamma} \|y_k\| \right\}. \tag{4.48}$$

Substituting $x = \hat{x}_k$ and $z = -f(\hat{x}_k)$ in (4.46), we get

$$\big\langle h(x_k)^{\gamma-1} u_k^\star - \nabla f(x_k)^* v_k^\star, \hat{x}_k - x_k \big\rangle + \big\langle -v_k^\star, -f(\hat{x}_k) - z_k \big\rangle \leqslant 0. \tag{4.49}$$

Consequently, after replacing z_k by $y_k - f(x_k)$, (4.49) reads

$$\begin{aligned} &\left\langle h(x_k)^{\gamma-1} u_k^\star, \hat{x}_k - x_k \right\rangle + \left\langle v_k^\star, y_k \right\rangle \\ &\quad \leqslant \left\langle v_k^\star, -f(\hat{x}_k) + f(x_k) + \nabla f(x_k)(\hat{x}_k - x_k) \right\rangle. \end{aligned} \tag{4.50}$$

Combining inequalities (4.47), (4.48) and (4.50), yields

$$\begin{aligned} &h(x_k)^{\gamma-1} \frac{\left\langle u_k^\star, \hat{x}_k - x_k \right\rangle}{\|\hat{x}_k - x_k\|} + h(x_k)^{\gamma-1} \left(\tau + \frac{1}{k^2}\right)^{-1} \frac{\left\langle v_k^\star, y_k \right\rangle}{\|y_k\|} \\ &\quad \leqslant \frac{1}{\|\hat{x}_k - x_k\|} \left\langle v_k^\star, -f(\hat{x}_k) + f(x_k) + \nabla f(x_k)(\hat{x}_k - x_k) \right\rangle. \end{aligned} \tag{4.51}$$

Setting $\hat{u}_k := \hat{x}_k - x_k$ and applying the Taylor expansion to f gives:

$$f(\tilde{x}_k) = f(x_k) + \nabla f(x_k)(\hat{u}_k) + \int_0^1 [\nabla f(x_k + t\hat{u}_k) - \nabla f(x_k)](\hat{u}_k)\, dt.$$

From the Lipschitz continuity of the Jacobian ∇f, we have

$$\limsup_{k\to\infty} \frac{\sup_{t\in[0,1]} \left\{ \|\nabla f(x_k + t\hat{u}_k) - \nabla f(x_k)\| \right\}}{\|\hat{u}_k\|} < +\infty,$$

and therefore, we obtain

$$\limsup_{k\to\infty} \frac{\|f(\tilde{x}_k) - f(x_k) - \nabla f(x_k)(\hat{u}_k)\|}{\|\hat{u}_k\|^2} < +\infty. \tag{4.52}$$

This leads to the estimation

$$\begin{aligned} &\frac{\left\langle u_k^\star, \hat{x}_k - x_k \right\rangle}{\|\hat{x}_k - x_k\|} + \left(\tau + \frac{1}{k^2}\right)^{-1} \frac{\left\langle v_k^\star, y_k \right\rangle}{\|y_k\|} \\ &\quad \leqslant \theta_k h(x_k)^{1-\gamma} \|\hat{x}_k - x_k\|, \end{aligned} \tag{4.53}$$

in which the real sequence (θ_k) is bounded. Due to the choice of $\hat{x}_k$, it is possible to write

$$d\big(x_k, S\big) \leqslant \|\hat{x}_k - x_k\| \leqslant d\big(x_k, S\big) + \sigma_k,$$

which permits to deduce that

$$\limsup_{k\to\infty} \frac{\|\hat{x}_k - x_k\|}{h(x_k)} = \limsup_{k\to\infty} \left(\frac{\|\hat{x}_k - x_k\|}{d(x_k, S)} \cdot \frac{d(x_k, S)}{h(x_k)} \right) < +\infty. \tag{4.54}$$

Taking into account that $\|u_k^\star\| \to 0$, and passing to the limit as $k \to \infty$ in (4.53), we obtain that

$$\liminf_{k\to\infty} \left\{ \left(\tau + \frac{1}{k^2}\right)^{-1} \frac{\left\langle v_k^\star, y_k \right\rangle}{\|y_k\|} \right\} \leqslant 0.$$

However, the latter relation obviously contradicts (d). Thus, the proof is complete.

□

Remark 4.2 The conclusion of the preceding proposition is still valid for $\gamma = 2$ if we choose $h(x) = \|x - \bar{x}\|$. Indeed, following the proof above, we have

$$\|y_k\| = (s_k)^\gamma \|v_k\| = \|x_k\|^\gamma \|v_k\| = h(x_k)^\gamma \|v_k\|. \tag{4.55}$$

Reminding (4.47), this yields $d(x_k, S) = o(h(x_k))$, and hence, from the choice of $\hat{x}_k$, this implies

$$\|\hat{x}_k - x_k\| = o(h(x_k)). \tag{4.56}$$

Combining this relation with (4.53), we reach a contradiction as in the previous proof.

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