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Metric regularity relative to a cone

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Tribute to Professor Alexander Ioffe on his eighty birthday. With recognition for research achievement and friendship

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Abstract n this note, we introduce the concept of metric regularity with respect to a cone. A slope characterization of the relative metric regularity with respect to a cone, as well as a stability result is established. Then, some coderivative characterizations of metric regularity relative to a cone are given.

Keywords Abstract subdifferential · Metric regularity · Directional metric regularity · Metric subregularity · directional Hölder metric subregularity · Coderivative

Mathematics Subject Classification 49J52 · 49J53 · 90C30

1 Introduction and preliminaries

Since the pioneering work of Robinson [25, 26], the study of optimization and complementarity problems, models in game theory, control and design problems, as well as variational inequalities, leads to the study of nclusions of the type:

$$y \in F(x) \quad \text{for} \quad (x,y) \in X \times Y, \tag{1.1}$$

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where, X,Y are metric spaces and $F : X \rightrightarrows Y$ is a set-valued mapping which describes the model under consideration. Undoubtedly, stability of the solutions of (1.1) plays an important role and has attracted over the recent years a large number of contibutions. We refer the reader to the monographs [3, 5, 6, 9, 19, 22, 24, 27], to the recent publications [7, 8, 10] and the references therein.

Before getting further, let us recall several notions from set-valued analysis. By a set-valued mapping (multifunction) $F : X \rightrightarrows Y$, we mean a mapping from X into the (possibly empty) subsets of Y. For such a mapping, the set $\operatorname{gph} F := \{(x,y) \in X \times Y : y \in F(x)\}$ is the graph of T, the domain of T is $\operatorname{dom} F := \{x \in X : F(x) \neq \emptyset\}$, and $F^{-1} : Y \rightrightarrows X$ is the inverse of T defined, for each $y \in Y$, by

$$x \in F^{-1}(y) \iff y \in F(x).$$

In any metric space under consideration, d is the corresponding metric, $B(x,\rho)$ and $\bar{B}(x,\rho)$ are the open and the closed ball with radius $\rho > 0$ around $x \in X$, respectively. We also note $B := B(0,1)$ and $\bar{B} := \bar{B}(0,1)$, the open and closed unit ball when, in addition, the space is a linear vector space. The distance from a point $x \in X$ to a subset Ω of X is $d(x,\Omega) := \inf_{u\in\Omega} d(x,u)$ and $\operatorname{cl}\Omega$ is the closure of Ω. Given a subset V of $X \times Y$ and a point $(x,y) \in X \times Y$, we set

$$V_x := \{z \in Y : \ (x,z) \in V\} \quad \text{and} \quad V_y := \{u \in X : \ (u,y) \in V\}.$$

We begin by recalling the notion of metric regularity relative to a set V introduced by A. Ioffe in [18].

Definition 1.1 Let X and Y be metric spaces, and let $V \subset X \times Y$. We say that a set-valued mapping $F : X \rightrightarrows Y$ is *metrically regular relatively to* V *at* $(\bar{x},\bar{y}) \in V \cap \operatorname{gph} T$ *with a modulus* $\tau > 0$, if there exist $\varepsilon > 0$ such that

$$d\big(x, F^{-1}(y) \cap \operatorname{cl} V_y\big) \leq \tau d(y, F(x)) \tag{1.2}$$

whenever $(x,y) \in \big(B(\bar{x},\varepsilon) \times B(\bar{y},\varepsilon)\big) \cap V$ and $d(y,F(x)) < \varepsilon$. The infinum of $\tau > 0$ such that (1.2) holds for some $\varepsilon > 0$ is called the *exact modulus of the metric regularity relative to V for F at* $(\bar{x},\bar{y})$ and is denoted by $\operatorname{reg}_V F(\bar{x},\bar{y})$.

Choosing various V, one can cover almost every metric regularity model in the literature as examples in [18, p. 343] show. An important subcase is the notion of directional metric regularity introduced by Arutyuanov and Izmailov in [2], and extensively studied in Arutyunov et al [1], Gfrerer [11, 12], Ioffe [18], Ngai-Théra [13]. In this paper, we consider the general version of metric regularity relatively to a cone. Let Y be a normed linear space; given a cone $C \subseteq Y$, and a real $\delta > 0$, we denote by

$$C(\delta) := \{v \in Y : \ d(v,C) \leq \delta \|v\|\}.$$

We will say that F is metrically regular relatively to C, if there exists $\delta > 0$ such that F is metrically regular relatively to $V := V_F(C,\delta)$:

$$V_F(C,\delta) := \{(x,y) \in X \times Y : \ y \in F(x) + C(\delta)\}.$$

The organisation of the paper is as follow: in the first part, we establish a slope characterization of the relative metric regularity, and then by using this characterization, we obtain a stability result; in a second part, some coderivative characterizations of metric regularity relative to a cone will be given. It should be mentioned that in the coderivative characterization result in Section 3, the pseudo-Lipschitz assumption of the multifunction under consideration as in [13] has been removed.

Below we recall some necessary notions and results from Variational Analysis. In order to formulate in this section some coderivative characterizations of directional metric regularity, we require some additional definitions. Let X be a Banach space. Consider now an extended real-valued function $f : X \to \mathbb{R} \cup \{+\infty\}$. Denote by $\operatorname{dom} f = \{x \in X : f(x) < \infty\}$, the *domain* of f. The *Fréchet (regular) subdifferential* of f at $\bar{x} \in \operatorname{dom} f$ is given as

$$\partial f(\bar{x}) = \left\{ x^* \in X^* : \liminf_{x \to \bar{x},\quad x \neq \bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}.$$

For the convenience of the reader, we would like to mention that the terminology *regular subdifferential* instead of Fréchet subdifferential is also popular due to its use in Rockafellar and Wets [27]. The Fréchet subdifferential is always convex and reduces to the classical subdifferential of convex analysis for the case of convex functions. Note also that this subdifferential obviously satisfies the generalized Fermat rule: $0 \in \partial f(x)$ if x is a local minimizer of f. Every element of the Fréchet subdifferential is termed as a Fréchet (regular) subgradient. If $\bar{x}$ is a point where $f(\bar{x}) = \infty$, then we set $\partial f(\bar{x}) = \emptyset$. In fact one can show that an element x^* is a Fréchet subgradient of f at $\bar{x}$ iff

$$f(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x} \rangle + o(\|x - \bar{x}\|) \quad \text{where} \quad \lim_{x \to \bar{x}} \frac{o(\|x - \bar{x}\|)}{\|x - \bar{x}\|} = 0.$$

Some of the results will be proved in the context of Asplund spaces which can be defined as Banach spaces for which every convex continuous function is generically Fréchet differentiable. There is a plethora of equivalent characterizations of Asplundty and many of them can be found, e.g., in [22] and its bibliography and in the well written introduction for beginners by D. Yost [28]. In particular, any space with Fréchet smooth renorming (and hence any reflexive space) is Asplund, as well as each Banach space such that each of its separable subspaces has a separable dual.

It is well known that the Fréchet subdifferential satisfies a fuzzy sum rule on Asplund spaces ([22, Theorem 2.33]). More precisely, if X is an Asplund space and $f_1, f_2 : X \to \mathbb{R} \cup \{\infty\}$ are such that f_1 is Lipschitz continuous around $\bar{x} \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and f_2 is lower semicontinuous around $\bar{x}$, then for any $\gamma > 0$ one has

$$\partial(f_1 + f_2)(\bar{x}) \subset \bigcup \{\partial f_1(x_1) + \partial f_2(x_2) \mid x_i \in \bar{x} + \gamma \overline{B}_X, |f_i(x_i) - f_i(\bar{x})| \leq \gamma, i = 1, 2\} + \gamma B_{X^*}. \tag{1.3}$$

For a nonempty closed set $C \subseteq X$, denote by ι_C the *indicator function* associated to C (i.e. $\iota_C(x) = 0$, when $x \in C$ and $\iota_C(x) = \infty$ otherwise). The *Fréchet (regular) normal cone* to C at $\bar{x}$ is denoted by $N(C, \bar{x})$. It is a closed and convex object in X^* which is defined as $\partial\iota_C(\bar{x})$ if $\bar{x}\in C$, and $\partial\iota_C(\bar{x})=\emptyset$ if $\bar{x}\notin C$. Equivalently a vector $x^*\in X^*$ is a Fréchet normal to C at $\bar{x}$ if

$$\langle x^*, x-\bar{x}\rangle \le o(\|x-\bar{x}\|), \quad \forall x\in C,$$

where $\lim_{x\to\bar{x}} \dfrac{o(\|x-\bar{x}\|)}{\|x-\bar{x}\|}=0$. We will use the following fuzzy intersection formula for Fréchet normal cones (see, e.g., [15]).

Lemma 1.1 *Let C_i, $i=1,\dots,k$, be nonempty closed subsets of an Asplund space X. For given $\bar{x}\in C:=\bigcap_{i=1}^k C_i$, assume that for any sequences $(x_n^i)\in C_i$, $(x_n^{i*})\subseteq X^*$ with $x_n^{i*}\in N(C_i,x_n^i)$, $x_n^i\to\bar{x}$, $i=1,\dots,k$,*

$$\lim_{n\to\infty}\left\|\sum_{i=1}^k x_n^{i*}\right\|=0 \quad\Rightarrow\quad \lim_{n\to\infty}\|x_n^{i*}\|=0, \textit{ for all } i=1,\dots,k.$$

Then for any x near $\bar{x}$, for every $\varepsilon>0$, one has

$$N(C,x)\subseteq\left\{\sum_{i=1}^k N(C_i,x^i)+\varepsilon B_{X^*}:\ x^i\in C_i\cap B(x,\varepsilon),\ i=1,\dots,k\right\}.$$

The *limiting subdifferential* of f at $\bar{x}\in\operatorname{dom} f$ (also known as the Mordukhovich subdifferential) is defined as

$$\partial_{\mathscr{M}} f(\bar{x})=\{x^*\in X^*:\ \exists x_k\to\bar{x},\ f(x_k)\to f(\bar{x}),\ \text{and } \exists x_k^*\in\partial f(x_k),\ x_k^*\to x^*\}.$$

The concept of *limiting normal cone* $N_{\mathscr{M}}(C,\bar{x})$ to a closed set C can be defined through the indicator function of the set:

$$N_{\mathscr{M}}(C,\bar{x}):=\partial_{\mathscr{M}}\delta_C(\bar{x}).$$

Given a normal cone $\mathbb{N}$, we can associate with a set-valued mapping $F:X\rightrightarrows Y$ a coderivative $D^*_{\mathbb{N}}:Y^\star\rightrightarrows X^*$ through the formula

$$D^*_{\mathbb{N}}F(x,y)(y^*):=\big\{x^*\in X^*\mid (x^*,-y^*)\in\mathbb{N}(\operatorname{gph} F,(x,y))\big\}. \tag{1.4}$$

In variational analysis, this notion is recognized as a powerful tool when applied to problems of optimization and control (e.g., see [21–23], and the references therein). In what follows, when $\mathbb{N}$ is the Fréchet (regular) normal cone, the coderivative of F will be denoted by $D^*_{\mathscr{F}}F$, while when $\mathbb{N}$ is the limiting normal cone, then we will use the notation by $D^*_{\mathscr{M}}F$. When $\mathbb{N}$ is the normal cone to a convex set C, then all the coderivatives coincide and are simply denoted by D^*.

2 Slope criteria for relative metric regularity

In this section (unless clearly indicated otherwise), we suppose that X is a complete metric space, that Y is a metric space, and that $V \subset X \times Y$, $\beta \in (0,1]$, and $F : X \rightrightarrows Y$ are fixed.

Given $a \in \mathbb{R}$, we set $a_+ = \max\{a, 0\}$. Recall from [17], that for an extended real-valued function $f : X \to \mathbb{R} \cup \{+\infty\}$ and a point $x \in X$ with $f(x) < +\infty$, the local and the global strong slope $|\nabla f|(x)$ and $|\Gamma f|(x)$ of f at x are defined by

$$|\nabla f|(x) = \limsup_{x \neq y \to x} \frac{[f(x) - f(y)]_+}{d(x,y)} \quad \text{and} \quad |\Gamma f|(x) = \sup_{y \neq x} \frac{[f(x) - f(y)]_+}{d(x,y)}. \tag{2.1}$$

If $f(x) = +\infty$, then we set $|\nabla f|(x) = |\Gamma f|(x) = +\infty$.

From now on, P will denote a topological space considered in applications as the space of parameters. The following statement is a restatement of Theorem 2 and Corollary 1 in [14].

Proposition 2.1 *Let $f : X \times P \to [0, +\infty]$ be a function and, for each $p \in P$, set*

$$S(p) = \{x \in X : \quad f(x,p) = 0\}.$$

Suppose that $(\bar{x}, \bar{p}) \in X \times P$ is such that $\bar{x} \in S(\bar{p})$, and that, for any p near $\bar{p}$, the function $f(\cdot, p)$ is lower semicontinuous at $\bar{x}$, and $f(\bar{x}, \cdot)$ is continuous at $\bar{p}$. Let $\tau > 0$ be given and consider the following statements:
(i) There exist $\gamma > 0$ and a neighborhood $\mathscr{V} \times \mathscr{W}$ of $(\bar{x}, \bar{p})$ in $X \times P$ such that for any $p \in \mathscr{W}$, we have $\mathscr{V} \cap S(p) \neq \emptyset$ and

$$d(x, S(p)) \leq \tau f(x,p) \quad \textit{for all} \quad (x,p) \in \mathscr{V} \times \mathscr{W} \ \textit{with}\ f(x,p) \in (0, \gamma); \tag{2.2}$$

(ii) There exist a neighborhood $\mathscr{V} \times \mathscr{W}$ of $(\bar{x}, \bar{p})$ in $X \times P$ and $\gamma > 0$ such that for each $(x,p) \in \mathscr{V} \times \mathscr{W}$ with $f(x,p) \in (0, \gamma)$ and for any $\varepsilon > 0$, take $z \in X$ such that

$$0 < d(x,z) < (\tau + \varepsilon)\big(f(x,p) - f(z,p)\big); \tag{2.3}$$

(iii) There exists a neighborhood $\mathscr{V} \times \mathscr{W}$ of $(\bar{x}, \bar{p})$ in $X \times P$ along with positive γ and τ such that $|\nabla f(\cdot, p)|(x) \geq 1/\tau$ for all $(x,p) \in \mathscr{V} \times \mathscr{W}$ with $f(x,p) \in (0, \gamma)$.
Then (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii).

Note that since f has non-negative values only, the continuity of $f(\bar{x}, \cdot)$ at $\bar{p}$ is equivalent to the upper semicontinuity.

For each $y \in Y$, the lower semicontinuous envelope relative to V of the function $x \mapsto d(y, F(x))$ is defined by

$$\varphi_{F,V}(x,y) := \begin{cases} \liminf\limits_{\mathrm{cl}\, V_y \times Y \ni (u,v) \to (x,y)} d(v, F(u)) = \liminf\limits_{\mathrm{cl}\, V_y \ni u \to x} d(y, F(u) & \text{if } x \in \mathrm{cl}\, V_y \\ +\infty & \text{otherwise.} \end{cases} \tag{2.4}$$

Equality in the above definition holds because the function $d(\cdot, F(u))$ is Lipschitz. Observe that $\varphi_{F,V}(x,y) \geq 0$ and $\varphi_{F,V}(x,y) \leq d(y, F(x))$ for every $(x,y) \in \mathrm{cl}\, V_y \times Y$.

Let us start with the following easy observations.

Lemma 2.1 *Suppose that the multifunction $F : X \rightrightarrows Y$ has closed graph. Then*

$$F^{-1}(y) \cap \operatorname{cl} V_y = \{x \in X : \varphi_{F,V}(x,y) = 0\} \quad \textit{whenever} \quad y \in Y,$$

and F is metrically regular relative to V at (x_0, y_0) with a modulus $\tau > 0$ if and only if there exists $\varepsilon > 0$ such that

$$d(x, F^{-1}(y) \cap \operatorname{cl} V_y) \leq \tau \varphi_V(x,y) \quad \textit{for all } (x,y) \in B(x_0, \varepsilon) \times B(y_0, \varepsilon) \textit{ with } d(y, F(x)) < \varepsilon.$$

Proof. Fix any $y \in Y$. Clearly, given $x \in X$, we have $\varphi_{F,V}(x,y) \leq 0$ if and only if $\varphi_{F,V}(x,y) = 0$. If $x \in F^{-1}(y) \cap \operatorname{cl} V_y$, then $y \in F(x)$, and so

$$0 = d(y, F(x)) \geq \varphi_{F,V}(x,y) \geq 0.$$

On the other hand, take an $x \in X$ with $\varphi_{F,V}(x,y) = 0$. By the very definition, $x \in \operatorname{cl} V_y$. Thus, there exists a sequence $(u_n)_{n \in \mathbb{N}}$ in $\operatorname{cl} V_y$ converging to x such that

$$\lim_{n \to \infty} d(y, F(u_n)) = 0.$$

Thus there is a sequence $(y_n)_{n \in \mathbb{N}}$ converging to y such that $y_n \in F(u_n)$ for every $n \in \mathbb{N}$. Since F has closed graph, $y \in F(x)$ which means that $x \in F^{-1}(y)$. □

From Proposition 2.1 and Lemma 2.1, we obtain the following slope characterizations of the relative metric regularity.

Theorem 2.1 *Let X be a complete metric space, Y be a metric space. Let $F : X \rightrightarrows Y$ be a set-valued mapping and let $(\bar{x}, \bar{y}) \in \operatorname{gph} F \cap V$, $V \subset X \times Y$; $\tau \in (0, +\infty)$ be given. Suppose that the set-valued mapping F has a closed graph. Then, among the following statements, one has (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii).*

(i) F is metrically regular relative to V at $(\bar{x}, \bar{y})$;
(ii) There exist $\delta, \gamma > 0$ such that

$$|\Gamma \varphi_{F,V}(\cdot, y)|(x) \geq \tau^{-1} \textit{ for all } (x,y) \in \mathbb{B}(\bar{x}, \delta) \times B(\bar{y}, \delta) \textit{ with } \varphi_{F,V}(x,y) \in (0, \gamma);$$

(iii) There exist $\delta, \gamma > 0$ such that

$$|\nabla \varphi_{F,V}(\cdot, y)|(x) \geq \tau^{-1} \textit{ for all } (x,y) \in \mathbb{B}(\bar{x}, \delta) \times B(\bar{y}, \delta) \textit{ with } \varphi_{F,V}(x,y) \in (0, \gamma).$$

3 Stability of metric regularity relative to a cone

We establish the stability of metric regularity relative to a cone under a sufficiently small Lipschitz perturbation. For a given multifunction$F : X \rightrightarrows Y$ from a complete metric space X to a normed linear space Y, a cone $C \subseteq Y$, and a positive real δ, denote by

$$V_F(\delta) := \{(x,y) : \ y \in F(x) + C(\delta)\};$$

for $y \in Y$,

$$V_{F,y}(\delta) := \{x \in X : \ y \in F(x) + C(\delta)\},$$

and $\varphi_{V_F(\delta)}(x,y)$, the lower semicontinuous envelope relative to $V_F(C, \delta)$ of $d(y, F(\cdot))$.

Theorem 3.1 *Let X be a complete metric space and Y be a normed space. Let $C \subseteq Y$ be a nonempty cone in Y. Let $F : X \rightrightarrows Y$ be a closed multifunction and $(x_0, y_0) \in$* gph F*. Suppose that F is metrically regular with a modulus $\tau > 0$ relatively to C, i.e., there exist reals $\varepsilon > 0$ and $\delta > 0$ such that*

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x)) \text{ for all } (x,y) \in B((x_0,y_0),\varepsilon) \cap V_F(C,\delta) \text{ with } d(y,F(x)) < \varepsilon. \quad (3.1)$$

Let $g : X \to Y$ be locally Lipschitz around x_0 with a Lipschitz constant $L > 0$. Then $F + g$ is metrically regular relatively to C at $(x_0, y_0 + g(x_0))$ with modulus

$$\mathrm{reg}_C(F+g)(x_0, y_0 + g(x_0)) \leq \left(\frac{1-\alpha}{\tau(1+\alpha)} - L\right)^{-1},$$

provided

$$\alpha \in (0,1), \text{ and } L < \frac{\delta\alpha}{\tau(1+\alpha)(1+\delta(1-\alpha))}.$$

Proof. Let $\varepsilon, \delta, \alpha, L$ be as in the theorem, and $g : X \to Y$ be Lipschitz with constant L on $B(x_0, \varepsilon)$. First note that any $\rho > 0$,

$$\varphi_{V_{F+g}(\rho)}(x,y) = \varphi_{V_F(\rho)}(x, y - g(x)), \text{ for all } (x,y) \in X \times Y.$$

According to Theorem 2.1, it suffices to prove that

$$|\Gamma \varphi_{V_{F+g}}(\cdot, y)|(x) \geq \left(\frac{1-\alpha}{\tau(1+\alpha)} - L\right), \quad (3.2)$$

whenever

$$(x,y) \in B((x_0, y_0 + g(x_0)), \eta) \text{ satisfies } x \in \mathrm{cl}\, V_{F+g,y}(\rho);\ 0 < \varphi_{V_{F+g}}(x,y) < \eta, \quad (3.3)$$

where $0 < \rho < \delta(1-\alpha)$ and $\eta = \min\{\varepsilon/(L+2), \varepsilon/(8\tau)\}$.

Let x, y be as in (3.3) and take sequences $(\lambda_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}, (x_n)_{n\in\mathbb{N}}$ satisfying $\lambda_n > 0, z_n \in B_X, (x_n) \to x$ and such that $d(z_n, C) \leq \rho$, and

$$y - g(x_n) \in F(x_n) + \lambda_n z_n, \quad \lim_{n\to\infty} d(y, F(x_n) + g(x_n)) = \varphi_{V_{F+g}(\rho)}(x,y). \quad (3.4)$$

As $d(z_n, C) \leq \rho$, there is $v_n \in C$ such that $\|z_n - v_n\| \leq$
$rho + 1/n$. Note that since (x_n) tends to x and $x \in B(x_0, \eta)$, then for n large we have

$$d(y, F(x_n) + g(x_n)) < \eta. \quad (3.5)$$

Setting

$$t_n := \alpha\varphi_{V_{F+g}(\rho)}(x_n, y)/(\rho+1), \quad (3.6)$$

then as

$$\|v_n\| \leq \|z_n\| + \|z_n - v_n\| \leq 1 + \rho + 1/n$$

$$t_n\|v_n\| < \varphi_{V_{F+g}(\rho)}(x_n,y)(1+\rho+1/n)/(1+\rho) < \eta, \quad (3.7)$$

for n sufficiently large, without loss of generality, say, this holds for all n, and $d(y, F(x_n) + g(x_n)) \le \lambda_n$. This yields,

$$t_n(\rho+1)/\alpha \le \varphi_{V_{F+g}(\rho)}(x_n, y) \le d(y, F(x_n)+g(x_n)) \le \lambda_n.$$

Consequently,

$$t_n/\lambda_n \le \frac{\alpha}{\rho+1}. \tag{3.8}$$

According to this and by noticing that

$$\|\lambda_n z_n - t_n v_n\| = \|(\lambda_n - t_n)z_n + t_n(z_n - v_n)\| \ge \lambda_n - t_n(1+\rho+1/n),$$

$\lambda_n - t_n > 0$ and C is a cone, one obtains for n large enough, the estimate:

$$\begin{aligned} d(\lambda_n z_n - t_n v_n, C) &\le \|\lambda_n z_n - t_n v_n - (\lambda_n - t_n)v_n\| = \lambda_n\|z_n - v_n\| \le \lambda_n(\rho+1/n) \\ &= \frac{\lambda_n(\rho+1/n)}{\|\lambda_n z_n - t_n v_n\|}\|\lambda_n z_n - t_n v_n\| \\ &\le \frac{\lambda_n(\rho+1/n)}{\lambda_n - t_n(1+\rho+1/n)}\|\lambda_n z_n - t_n v_n\| \\ &\le \frac{\rho+1/n}{1-\alpha(1+\rho+1/n)/(\rho+1)}\|\lambda_n z_n - t_n v_n\| \le \delta\|\lambda_n z_n - t_n v_n\|, \end{aligned}$$

Therefore, we may assume it holds for all $n \in \mathbb{N}$. Hence, $\lambda_n(\bar{y} + \rho z_n) - t_n\bar{y} \in C(\delta)$ and thanks to (3.4), this yields

$$y - g(x_n) - t_n v_n \in F(x_n) + C(\delta). \tag{3.9}$$

Moreover,

$$\|y - g(x_n) - t_n v_n - y_0\| \le \|y - g(x_0) - y_0\| + \|g(x_n) - g(x_0)\| + t_n\|v_n\| < (2+L)\eta \le \varepsilon, \tag{3.10}$$

and combining (3.5) and (3.7) we also have

$$d(y - g(x_n) - t_n\bar{y}, F(x_n)) \le d(y - g(x_n), F(x_n)) + t_n\|v_n\| < 2\eta < \frac{2\varepsilon}{L+2} < \varepsilon. \tag{3.11}$$

From (3.10) and (3.11) we deduce that

$$y - g(x_n) - t_n v_n \in B(y_0, \varepsilon);\; d(y - g(x_n) - t_n v_n, F(x_n)) < \varepsilon,$$

and

$$(x_n, y - g(x_n) - t_n v_n) \in V_F(\delta).$$

Hence according to Lemma 3, we have

$$\begin{aligned} & d(x_n, F^{-1}(y - g(x_n) - t_n v_n)) \\ & \qquad < \tau\varphi_{V_{F(\rho)}}(x_n, y - g(x_n) - t_n v_n) \\ & \qquad \le \tau(\varphi_{V_{F+g}(\rho)}(x_n, y) + t_n\|v_n\|) \\ & \qquad = \tau t_n \frac{(1+\alpha)(1+\rho)+\alpha/n}{\alpha} \qquad \text{thanks to (3.6).} \end{aligned} \tag{3.12}$$

Using the fact that $t_n\|v_n\| < \eta$ and $\varphi_{V_{F+g}(\rho)}(x_n,y) \le d(y-g(x_n),F(x_n)) < \eta$, we obtain

$$d(x_n,F^{-1}(y-g(x_n)-t_nv_n)) < 2\tau\eta.$$

By the choice of η, we derive $d(x_n,F^{-1}(y-g(x_n)-t_nv_n)) < \varepsilon/2$, and therefore for any $r \in (0,1)$, the existence of some $u_n \in F^{-1}(y-g(x_n)-t_nv_n)$ such that for n sufficiently large,

$$d(x_n,u_n) < \tau(1+r)t_n\frac{(1+\alpha)(1+\rho)+\alpha/n}{\alpha} < \varepsilon/2.$$

Since $(x_n) \to x \in B(x_0,\eta)$, for n sufficiently large we have

$$d(x_n,x_0) \le d(x_n,x)+d(x,x_0) < \varepsilon/2+\eta < \varepsilon,$$

so that $u_n \in B(x_0,\varepsilon)$. Since $u_n \in F^{-1}(y-g(x_n)-t_nv_n)\cap B(x_0,\varepsilon)$ and $\|g(u_n)-g(x_n)\| \le Ld(u_n,x_n)$, (by the Lipschitz property of g on $B(x_0,\varepsilon)$), then

$$y \in F(u_n)+g(x_n)+t_nv_n \subseteq F(u_n)+g(u_n)+t_n\Big(v_n+L\frac{d(u_n,x_n)}{t_n}B_Y\Big).$$

By the definition of L, for r sufficiently small, one obtains

$$y \in F(u_n)+g(u_n)+C(\rho).$$

Therefore,

$$\varphi_{V_{F+g}(\rho)}(u_n,y) \le d(y-g(u_n),F(u_n)) \le t_n\|v_n\|+Ld(x_n,u_n). \tag{3.13}$$

As

$$t_n\|v_n\| \le \alpha\varphi_{V_{F+g}(\rho)}(x_n,y)(1+\rho+1/n)/(1+\rho)$$

with $\alpha \in (0,1)$, it follows that $\liminf_{n\to\infty} d(x_n,u_n) > 0$. Therefore, one has

$$\begin{aligned}
&\liminf_{n\to\infty}\frac{\varphi_{V_{F+g}(\rho)}(x,y)-\varphi_{V_{F+g}(\rho)}(u_n,y)}{d(x,u_n)}\\
&=\liminf_{n\to\infty}\frac{\varphi_{V_{F+g}(\rho)}(x_n,y)-\varphi_{V_{F+g}(\rho)}(u_n,y)}{d(x_n,u_n)}\\
&\ge \liminf_{n\to\infty}\frac{t_n(1+\rho)/\alpha-t_n\|v_n\|}{t_n(1+r)[(1+\rho)(1+\alpha)+\alpha/n]/\alpha}-L\\
&\ge \frac{1-\alpha}{\tau(1+r)(1+\alpha)}-L,
\end{aligned}$$

where we make use of $\|v_n\| \le 1+\rho+1/n$. As $r>0$ is arbitrarily small, one obtains

$$|\Gamma\varphi_{V_{F+g}(\rho)}(\cdot,y)|(x) \ge \frac{1-\alpha}{\tau(1+\alpha)}-L,$$

which completes the proof. □

4 Coderivative characterizations of directional metric regularity

For the usual metric regularity, sufficient conditions in terms of coderivatives have been given by various authors, for instance, in [4,16,20,22]. In this section, we establish a characterization of relative metric regularity using the Fréchet subdifferential in Asplund spaces.

Associated to the multifunction F, for a given $\varepsilon > 0$ and $(x_0, y_0) \in \mathrm{gph}\, F$, we define the *localization* of F by

$$F_{(x_0,y_0,\varepsilon)}(x) := \begin{cases} F(x) \cap \bar{B}(y_0, \delta_0) & \text{if} \quad x \in \bar{B}(x_0, \varepsilon) \\ \emptyset & \text{otherwise.} \end{cases} \tag{4.1}$$

Note that, by definition, one has

$$D^*_{\mathscr{F}} F(x,y) = D^*_{\mathscr{F}} F_{(x_0,y_0,\varepsilon)}(x,y) \quad \forall (x,y) \in \mathrm{gph}\, F \cap (B(x_0,\varepsilon) \times B(y_0,\varepsilon)). \tag{4.2}$$

The following proposition gives a connection between metric regularity relative to a cone for a multifunction with convex values and metric regularity relative to the same cone for its localizations.

Proposition 4.1 *Suppose given a multifunction $F : X \rightrightarrows Y$ with convex values for x near x_0 and $(x_0, y_0) \in \mathrm{gph}\, F$. Then F is metrically regular relatively to a cone $C \subseteq Y$ if and only if $F_{(x_0,y_0,\varepsilon)}$ is metrically regular relatively to the same cone C for any $\varepsilon > 0$.*

Proposition 4.1 follows immediately from the following lemma.

Lemma 4.1 *Let $F : X \rightrightarrows Y$ be a multifunction with convex values for x near x_0 and $(x_0, y_0) \in \mathrm{gph}\, F$. Then for given a nonempty $C \subseteq Y$, for any $\delta_1, \delta_2 > 0$, there exist $\eta, \delta > 0$ such that for all $x \in B(x_0, \eta)$, one has*

$$\big(F(x) + C(\delta)\big) \cap B(y_0, \eta) \cap \{y \in Y : \ d(y, F(x)) < \eta\} \subseteq F(x) \cap B(y_0, \delta_1) + C(\delta_2). \tag{4.3}$$

Proof. For δ_1, δ_2, take $\delta = \delta_2/2$. Let $\eta \in (0, \delta_1/4)$ such that $F(x)$ is convex for all $x \in B(x_0, \eta)$ and select $x \in B(x_0, \eta)$ and $y \in (F(x) + C(\delta)) \cap B(y_0, \eta)$ with $d(y, F(x)) < \eta$. Then, there exist $z, v \in F(x)$ such that

$$y = z + \lambda u, \ \text{ for } \lambda \geq 0, \ u \in Y, \ \ \|u\| = 1; \ \ d(u, C) \leq \delta, \ \ \|y - v\| < \eta.$$

If $z \in B(y_0, \delta_1)$ then (4.3) holds trivially. Otherwise, one has

$$\lambda \geq \|y - z\| \geq \|z - y_0\| - \|y - y_0\| \geq \delta_1 - \eta.$$

Setting

$$t := \frac{\eta(1 + \delta_2)}{\delta_2(\delta_1 - \eta)/2 + \eta(1 + \delta_2)}, \quad w := tz + (1 - t)v \in F(x), \tag{4.4}$$

and by taking η sufficiently small such as $t < 1/2$, one has

$$\|w - y_0\| \leq t\|z - v\| + \|v - y_0\| \leq t\lambda\|u\| + t\|y - v\| + \|v - y_0\| < t(\delta_1 - \eta) + t\eta + 2\eta < \delta_1.$$

and,

$$\|y-w\| = \|t\lambda u + (1-t)(y-v)\| \geq t\lambda - (1-t)\eta.$$

Thus

$$\begin{aligned} d(y-w,C) &= d(t\lambda u + (1-t)(y-v), C) \leq (1-t)\|y-v\| + d(t\lambda u, C) \\ &\leq (1-t)\eta + t\lambda\delta \leq \frac{(1-t)\eta + t\lambda\delta}{t\lambda - (1-t)\eta}\|y-w\| = \delta_2\|y-w\|, \end{aligned}$$

the last equality follows from the definition of t (4.4). That is, $y \in F(x) \cap B(y_0, \delta_1) + C(\delta_2)$. □

Denote by S_{Y^*} the unit sphere in the continuous dual Y^* of Y, and by d_* the metric associated with the dual norm on X^*. For given $\bar{y} \in Y$ and $\delta > 0$, let us define the set

$$T(C,\delta) := \{(y_1^*, y_2^*) \in Y^* \times Y^* : \ \exists a \in C \cap S_{Y^*}, \max\{\langle y_1^*, a\rangle, |\langle y_2^*, a\rangle|\} \leq \delta, \ \|y_1^* + y_2^*\| = 1\}. \quad (4.5)$$

To a given multifunction $F : X \rightrightarrows Y$, we associate the multifunction $G : X \rightrightarrows Y \times Y$ defined by

$$G(x) = F(x) \times F(x), \quad x \in X.$$

A coderivative characterization of relative metric regularity is stated in the following theorem.

Theorem 4.1 *Let X, Y be Asplund spaces and let $F : X \rightrightarrows Y$ be a closed multifunction. Let $(x_0, y_0) \in \operatorname{gph} F$ and a nonempty cone $C \subseteq Y$ be given. Assume that F has convex values around x_0, i.e., $F(x)$ is convex for all x near x_0. If*

$$\liminf_{\substack{(x,y_1,y_2) \xrightarrow{G} (x_0,y_0,y_0) \\ \delta \downarrow 0^+}} d_*(0, D^*_{\mathscr{F}} G(x, y_1, y_2)(T(C,\delta))) > m > 0, \quad (4.6)$$

then F is metrically regular relatively to C with modulus $\tau \leq m^{-1}$ at (x_0, y_0).The notation $(x, y_1, y_2) \xrightarrow{G} (x_0, y_0, y_0)$ means that $(x, y_1, y_2) \to (x_0, y_0, y_0)$ with $(x, y_1, y_2) \in$ gph G.

Proof. By the assumption, there is $\delta_0 \in (0,1)$ such that

$$\inf_{(x,y_1,y_2) \in \operatorname{gph} G \cap B((x_0,y_0,y_0), 2\delta_0)} d_*(0, D^*_{\mathscr{F}} G(x, y_1, y_2)(T(\bar{y}, \delta_0))) \geq m + \delta_0. \quad (4.7)$$

According to Proposition 4.1 and relation (4.2), by considering the localization $F_{(x_0,y_0,\delta_0)}$ instead of F, without any loss of generality, we may assume that

$$F(x) \subseteq \bar{B}(y_0, \delta_0) \quad \text{for all} \ \ x \in \bar{B}(x_0, \delta_0). \quad (4.8)$$

Denote by $\varphi_\delta(\cdot, y) := \varphi_{V(C,\delta)}(\cdot, y)$, the lower semicontinuous envelope of $d(y, F(\cdot))$ relative to $V(C,\delta)$. By virtue of Theorem 2.1, it suffices to show that one has $|\nabla \varphi_\delta(\cdot, y)|(x) > m$ for any $(x,y) \in \big(B(x_0,\delta) \times B(y_0,\delta)\big)$, $x \in \operatorname{cl} V_y(C,\delta)$ with $\varphi_\delta(x,y) \in (0,\delta)$. Let $(x,y) \in B(x_0,\delta) \times B(y_0,\delta)$, $x \in \operatorname{cl} V(\bar{y},\delta)$ with $\varphi_\delta(x,y) \in (0,\delta)$ be given.

Set $|\nabla\varphi_\delta(\cdot,y)|(x) := \alpha$. By the definition of the strong slope, for each $\varepsilon \in (0, \min\{\delta, 1/2\})$, there is $\eta \in (0,\varepsilon)$ with

$$2\eta + \varepsilon < \gamma/2,\ \ 2\eta < \varepsilon\varphi_\delta(x,y) \text{ and } 1-(\alpha+\varepsilon+2)\eta > 0$$

such that

$$d(y, F(x')) \geq (1-\varepsilon)\varphi_\delta(x,y)\ \forall x' \in B(x, 4\eta) \tag{4.9}$$

and

$$\varphi_\delta(x,y) \leq \varphi_\delta(x',y) + (m+\varepsilon)\|x'-x\| \quad \text{for all } x' \in \bar{B}(x,3\eta)\cap \operatorname{cl} V_y(C,\delta). \tag{4.10}$$

Take $u \in B(x,\eta^2/4)\cap V_y(C,\delta)$, $v \in F(u)$ such that $\|y-v\| \leq \varphi_\delta(x,y)+\eta^2/4$. Then,

$$\|y-v\| \leq d(y,F(x')) + (\alpha+\varepsilon)\|x'-x\| + \eta^2/4\ \forall x' \in \bar{B}(u,2\eta)\cap \operatorname{cl} V_y(C,\delta).$$

Consequently, for every $(x',z') \in \big(\bar{B}(u,2\eta)\times Y\big)\cap V(C,\delta)$ we have

$$\|y-v\| \leq d(y,F(x')) + (\alpha+\varepsilon)\|x'-u\| + (\alpha+\varepsilon+1)\eta^2/4. \tag{4.11}$$

Let $z \in C(\delta)$ such that $y-z \in F(u)$. Then,

$$\|z\| \geq d(y,F(u)) \geq (1-\varepsilon)\varphi_\delta(x,y)) > \eta/\varepsilon. \tag{4.12}$$

Setting

$$W := \{(x,w_1,w_2,z) \in X\times Y\times Y\times Y :\ \ (x,w_1,w_2) \in \operatorname{gph} G,\ y = w_2+z,\ z \in C(\delta)\},$$

we derive

$$\begin{aligned}\|y-v\| &\leq \|y-w_1\| + (\alpha+\varepsilon)\|x'-u\| + \iota_W(x',w_1,w_2,z') + (\alpha+\varepsilon+1)\eta^2/4 \\ &\text{for all}(x',w_1,w_2,z') \in \bar{B}(u,\eta)\times Y\times Y\times \bar{B}(z,\eta).\end{aligned}$$

Next, applying the Ekeland variational principle to the function

$$(x',w_1,w_2,z') \mapsto \psi(x',w_1,w_2,z') := \|y-w_1\| + (\alpha+\varepsilon)\|x'-u\| + \iota_W(x',w_1,w_2,z')$$

on $\bar{B}(u,\eta)\times Y\times Y\times \bar{B}(z,\eta)$, we select $(u_1,v_1,v_2,z_1) \in (u,v,y-z,z) + \frac{\eta}{4}B_{X\times Y\times Y\times Y}$ with $(u_1,v_1,v_2,z_1) \in W$, such that

$$\|y-v_1\| \leq \|y-v\|(\leq d(y,F(x)) + \eta^2/4) \tag{4.13}$$

and

$$\psi(u_1,v_1,v_2,z_1) \leq \psi(x',w_1,w_2,z') + (\alpha+\varepsilon+1)\eta\|(x',w_1,w_2,z') - (u_1,v_1,v_2,z_1)\|$$

for all $(x',w_1,w_2,z') \in \bar{B}(u,\eta)\times Y\times Y\times \bar{B}(z,\eta)$. Thus,

$$0 \in \partial(\psi + (\alpha+\varepsilon+1)\eta\|\cdot - (u_1,v_1,v_2,z_1)\|)(u_1,v_1,v_2,z_1). \tag{4.14}$$

We need the following claim in order to make use the fuzzy sum rule.

Claim. For each $(u,w_1,w_2,z_1) \in W$ near $(u,v,y-z,z)$, for every $\varepsilon > 0$, one has

$$N(W,(u,w_1,w_2,z_1)) \subseteq \left\{ \begin{array}{l} (x^*,w_1^*,w_2^*,z^*)+\varepsilon B^*_{X\times Y\times Y\times Y}: \ (x^*,w_1^*,w_2^*)\in N(\text{gph}\, G,(u',w_1',w_2')),\ z'\in N(C(\delta),z'), \\ \qquad\qquad \|w_2^*+z^*\|\le\varepsilon,\ \|(u',w_1',w_2',z')-(u,w_1,w_2,z_1)\|<\varepsilon. \end{array} \right\}.$$

Proof of the claim. Observe that $W = W_1\cap W_2\cap W_3,$ where

$$W_1 := \{(x,w_1,w_2,z)\in X\times Y\times Y\times Y:\ w_2+z=y\};$$

$$W_2 := \{(x,w_1,w_2,z)\in X\times Y\times Y\times Y:\ (x,w_1,w_2)\in \text{gph}\, G\};$$

$$W_3 := \{(x,w_1,w_2,z)\in X\times Y\times Y\times Y:\ z\in C(\delta)\}.$$

It suffices to check that the condition of Lemma 1.1 is satisfied. Indeed, pick any sequences $w_n^i := (u_n^i, w_{1,n}^i, w_{2,n}^i, z_n^i)\in W_i$, converging to $(u,v,y-z,z)$ and $w_n^{i*} := (u_n^{i*}, w_{1,n}^{i*}, w_{2,n}^{i*}, z_n^{i*})\in N(W_i,(u_n^i, w_{1,n}^i, w_{2,n}^i, z_n^i))$ $(i=1,2,3)$, such that

$$\|w_n^{1*}+w_n^{2*}+w_n^{3*}\|\to 0.$$

Then, by the definition of W_i, $(i=1,2,3)$,

$$u_n^{1*}=0,\ \ w_{1,n}^{1*}=0,\ \ w_{2,n}^{1*}=-z_n^{1*};$$

$$u_n^{2*}\in D^*_{\mathscr{F}}G((u_n^1,w_{1,n}^2,w_{2,n}^2))(-(w_{1,n}^{2*},w_{2,n}^{2*}));$$

$$u_n^{3*}=0,\ \ w_{1,n}^{3*}=0,\ \ w_{2,n}^{3*}=0,\ \ z_n^{3*}\in N(C(\delta),z_n^3).$$

Hence,

$$\|u_n^{2*}\|\to 0,\ \ \|w_{1,n}^{2*}\|\to 0,\ \ \text{and}\ \ \|w_{2,n}^{2*}+w_{2,n}^{1*}\|\to 0,\ \|z_n^{1*}+z_n^{3*}\|\to 0,\ \ \text{as}\ n\to\infty.$$

As $w_{2,n}^{1*}=-z_n^{1*}$, the latter relations imply $\|w_{2,n}^{2*}+z_n^{3*}\|\to 0$ as $n\to\infty$. Since $z_n^{3*}\in N(C(\delta),z_n^3)$, $\langle z_n^{3*},z_n^3\rangle=0$. As $z\neq 0$, $z_n^3\neq 0$ when n is sufficiently large, therefore for each n (sufficiently large) there is $a\in C\cap S_{Y^*}$ such that $\|z_n^3/\|z_n^3\|-a\|\le(\delta+1/n)<\delta_0$. Hence when n is sufficiently large,

$$\langle w_{2,n}^{2*},a\rangle\le\|w_{2,n}^{2*}\|\delta_0.$$

By (4.8), $(u,v,y-z)\in B((x_0,y_0,y_0),\delta_0)$; therefore, in view of relation (4.7), the latter relation implies that the sequences (w_n^{i*}) $(i=1,2,3)$ converge to 0, and the assumption from Lemma 1.1 is satisfied, and the claim is proved.

Now using the claim and applying the fuzzy sum rule, applied to (4.14), we derive the existence of

$$v_3\in B(v_1,\eta),\ z_2\in B(z,\eta);$$

$$(u_2,w_1,w_2)\in B(u_1,\eta)\times B(v_1,\eta)\times B(v_2,\eta)\cap\text{gph}\, G;$$

$$v_3^*\in\partial\|y-\cdot\|(v_3);\ (u_2^*,-w_1^*,-w_2^*)\in N(\text{gph}\, G,(u_2,w_1,w_2)),\ z_2^*\in N(C(\delta),z_2),$$

such that

$$\|v_3^*-w_1^*\|<(\alpha+\varepsilon+2)\eta;\ \ \|w_2^*+z_2^*\|<(\alpha+\varepsilon+2)\eta,\ \ \|u_2^*\|\le\alpha+\varepsilon+(\alpha+\varepsilon+2)\eta. \quad (4.15)$$

Since $v_3^* \in \partial\|y-\cdot\|(v_3)$ (note that $\|y-v_3\| \geq \|y-v\| - \|v_3 - v\| \geq d(y,F(x)) - \varepsilon - 2\eta > 0$), then $\|v_3^*\| = 1$ and $\langle v_3^*, v_3 - y\rangle = \|y - v_3\|$. Thus, $\|w_1^*\| \leq 1 + (\alpha+\varepsilon+2)\eta$, and from the first relation of (4.15) it follows that

$$\langle w_1^*, w_1 - y\rangle \geq \langle v_3^*, w_1 - y\rangle - (\alpha+\varepsilon+2)\eta\|w_1 - y\| \geq (1-(\alpha+\varepsilon+2)\eta)\|w_1 - y\| - 2\eta.$$

As $\eta \leq \varepsilon d(y,F(x)) \leq \varepsilon d(y,F(u))/(1-\varepsilon)$ for all $u \in B(x,4\eta)$, then $\eta \leq \varepsilon\|w_1 - y\|/(1-\varepsilon)$, therefore one obtains

$$\langle w_1^*, w_1 - y\rangle \geq (1-\varepsilon_1)\|w_1 - y\|, \tag{4.16}$$

where

$$\varepsilon_1 := (\alpha+\varepsilon+2)\eta - 2\varepsilon(1-\varepsilon)^{-1}.$$

Since $w_2 \in B(v_2,\eta)$ and $v_2 \in B(y-z,\eta)$, $w_2 \in B(y-z,2\eta)$. As $F(u_2)$ is convex, $w_2 \in F(u_2)$, and $w_1^* \in -N(F(u_2),w_1)$, one has

$$\langle w_1^*, y - w_2\rangle = \langle w_1^*, y - w_1\rangle + \langle w_1^*, w_1 - w_2\rangle \leq 0.$$

Therefore,

$$\begin{aligned}\langle w_1^*, z\rangle &= \langle w_1^*, y - w_2\rangle + \langle w_1^*, z - (y - w_2)\rangle \\ &\leq 2\eta\|w_1^*\| \leq 2\eta[1 + (\alpha+\varepsilon+2)\eta],\end{aligned}$$

and by (4.12), $\|z\| \geq \eta/\varepsilon$,

$$\left\langle w_1^*, \frac{z}{\|z\|}\right\rangle \leq 2\varepsilon[1 + (\alpha+\varepsilon+2)\eta].$$

As $z \in C(\delta)$, there is $d \in C$ such that $\|z/\|z\| - d\| \leq 2\delta$. Then $\|d\| \geq 1 - 2\delta$, and by $\|w_1^*\| \leq 1 + (\alpha+\varepsilon+2)\eta$, one obtains

$$\begin{aligned}\langle w_1^*, d\rangle &\leq \langle w_1^*, z/\|z\|\rangle + 2\delta\|w_1^*\| \\ &\leq 2\varepsilon[1 + (\alpha+\varepsilon+2)\eta] + 2\delta[1 + (\alpha+\varepsilon+2)\eta].\end{aligned}$$

Hence for $a := d/\|d\| \in C \cap S_{Y^*}$, one has

$$\langle w_1^*, a\rangle \leq (2\varepsilon[1 + (\alpha+\varepsilon+2)\eta] + 2\delta[1 + (\alpha+\varepsilon+2)\eta])(1-2\delta)^{-1} := \varepsilon_2. \tag{4.17}$$

As $z_2^* \in N(C(\delta), z_2)$, with $z_2 \neq 0$, then $\langle z_2^*, z_2\rangle = 0$. Therefore, by $\|w_2^* + z_2^*\| < (\alpha+\varepsilon+2)\eta$,

$$|\langle w_2^*, z_2\rangle| \leq (\alpha+\varepsilon+2)\eta\|z_2\|.$$

As $z_2 \in B(z,\eta)$ and $\|z\| \geq \eta/\varepsilon$, one has $\|z_2\| \leq (1+\varepsilon)\|z\|$, and therefore,

$$\begin{aligned}|\langle w_2^*, z\rangle| &\leq \langle w_2^*, z_2\rangle + \eta\|w_2^*\| \\ &[(\alpha+\varepsilon+2)\eta(1+\varepsilon) + \varepsilon\|w_2^*\|]\|z\|,\end{aligned}$$

which implies

$$|\langle w_2^*, a\rangle| \leq [(\alpha+\varepsilon+2)\eta(1+\varepsilon) + (2\delta+\varepsilon)\|w_2^*\|](1-2\delta)^{-1}. \tag{4.18}$$

We consider the following two cases:

Case 1. $\|w_2^*\| \leq 1+2\|w_1^*\|(\leq 1+2(1+(\alpha+\varepsilon+2)\eta))$. Then

$$\langle w_2^*,a\rangle| \leq (\alpha+\varepsilon+2)\eta(1+\varepsilon)+(2\delta+\varepsilon)(1+2(1+(\alpha+\varepsilon+2)\eta))(1-2\delta)^{-1} := \varepsilon_3. \quad (4.19)$$

Moreover, remind that $\langle z_2^*, z_2\rangle = 0$,

$$|\langle w_2^*, w_2-y\rangle| \leq |\langle z_2^*, w_2-(y-z_2)\rangle| + |\langle w_2^*+z_2^*, w_2-y\rangle| \leq \varepsilon_4\|w_1-y\|,$$

where

$$\varepsilon_4 := \begin{pmatrix} 3[1+2(1+(\alpha+\varepsilon+2)\eta)) \; +(\alpha+\varepsilon+2)\eta]\eta) \\ \qquad\qquad +2(\alpha+\varepsilon+2)(\|y_0\|+2\delta_0+2\eta) \end{pmatrix} \varepsilon(1-\varepsilon)^{-1}.$$

The second inequality of the preceding relation follows from (4.15), as well as

$$\eta \leq \varepsilon\|w_1-y\|/(1-\varepsilon);$$

$$\|z_2^*\| \leq \|w_2^*\| + \|z_2^*+w_2^*\| \leq 1+2(1+(\alpha+\varepsilon+2)\eta)) + (\alpha+\varepsilon+2)\eta;$$

$$\|w_2-y-z_2\| \leq \|w_2-v_2\| + \|v_2-(y-z)\| + \|z_2-z\| \leq 3\eta$$

and

$$\|w_2-y\| \leq \|w_2-v_2\| + \|v_2-(y-z)\| + \|z\| < 2\eta+2\delta_0+\|y_0\|.$$

Hence, using the convexity of $F(u_2)$, and the fact that $w_2^* \in -N(F(u_2),w_2)$ we have

$$\langle w_2^*, w_1-y\rangle = \langle w_2^*, w_1-w_2\rangle + \langle w_2^*, w_2-y\rangle \geq -\varepsilon_4\|w_1-y\|. \quad (4.20)$$

From relations (4.16) and (4.20), one derives that

$$\langle w_1^*+w_2^*, w_1-y\rangle \geq (1-\varepsilon_1-\varepsilon_4)\|w_1-y\|. \quad (4.21)$$

Consequently, $\|w_1^*+w_2^*\| \geq 1-\varepsilon_1-\varepsilon_4$.

Set

$$y_1^* = \frac{w_1^*}{\|w_1^*+w_2^*\|}; \;\; y_2^* = \frac{w_2^*}{\|w_1^*+w_2^*\|} \;\text{ and }\; x^* = \frac{u_2^*}{\|w_1^*+w_2^*\|}.$$

From relations (4.17), (4.18), (4.21), one has

$$\langle y_1^*, a\rangle \leq \varepsilon_2(1-\varepsilon_1-\varepsilon_4)^{-1};$$

$$|\langle y_2^*, a\rangle| \leq \varepsilon_3(1-\varepsilon_1-\varepsilon_4)^{-1},$$

and

$$x^* \in D^*_{\mathscr{F}}G(u_2,w_1,w_2)(y_1^*,y_2^*); \;\; \|y_1^*+y_2^*\| = 1.$$

As $\varepsilon_1,\varepsilon_2,\varepsilon_3,\delta,\varepsilon,\eta \to 0$, then $(y_1^*, y_2^* \in T(C,\delta)$. Since $(u_2,w_1,w_2) \in B((x_0,y_0,y_0),\delta_0)$, according to (4.15), one obtains

$$m+\delta_0 \leq \|x^*\| = \|u_2^*\|/\|w_1^*+w_2^*\| \leq \frac{\alpha+\varepsilon+(\alpha+\varepsilon+2)\eta}{1-\varepsilon_1-\varepsilon_4}. \quad (4.22)$$

As $\varepsilon,\eta,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4$ are arbitrarily small, we obtain $m+\delta_0 \leq \alpha$.

Case 2. $\|w_2^*\| > 1+2\|w_1^*\|$. For this case,

$$\|w_1^* + w_2^*\| \geq \|w_2^*\| - \|w_1^*\| \geq (\|w_2^*\|+1)/2 > 1.$$

Therefore,

$$\langle y_1^*, a\rangle \leq \varepsilon_2,$$

and by (4.18),

$$\begin{aligned}|\langle y_2^*, a\rangle| &\leq [(\alpha+\varepsilon+2)\eta(1+\varepsilon)+(2\delta+\varepsilon)\|w_2^*\|](1-2\delta)^{-1}\|w_1^*+w_2^*\|^{-1}\\ &\leq [(\alpha+\varepsilon+2)\eta(1+\varepsilon)+2(2\delta+\varepsilon)](1-2\delta)^{-1}.\end{aligned}$$

Thus we also get $(y_1^*, y_2^*) \in T(C,\delta_0)$. Similarly to the first case, one has $m+\delta_0 \leq \alpha$, and the proof is complete. □

The following proposition shows that Condition (4.6) is also a necessary condition for metric regularity relative to a cone in Banach spaces for the cases of either F is a multifunction with a convex graph or $F : X \to Y$ is a continuous single-valued mapping.

Proposition 4.2 *Let X, Y be Banach spaces, let $C \subseteq Y$ be a nonempty cone. Suppose that $F : X \rightrightarrows Y$ is either a closed convex multifunction or $F : X \to Y$ is a continuous mapping. For a given $(x_0, y_0) \in \operatorname{gph} F$, if F is metrically regular relatively to C at (x_0, y_0), then*

$$\liminf_{\substack{(x,y_1,y_2)\xrightarrow{G}(x_0,y_0,y_0)\\ \delta\downarrow 0^+}} d_*(0, D^*_{\mathscr{F}}G(x,y_1,y_2)(T(C,\delta))) > 0.$$

Proof. Assuming that F is metrically regular relatively to C, there exist $\tau > 0, \delta > 0, \varepsilon > \delta$ such that

$$d(x,F^{-1}(y)) \leq \tau d(y,F(x)) \quad \text{for all } (x,y) \in B(x_0,\varepsilon)\times B(y_0,\varepsilon);\ d(y,F(x)) < \varepsilon;\ y \in F(x)+C(\delta). \tag{4.23}$$

For $\gamma \in (0,\delta)$, $\eta \in (0,\varepsilon-\delta)$, let $(x,y_1,y_2) \in \operatorname{gph} G \cap [B(x_0,\varepsilon/2)\times B(y_0,\varepsilon/2)\times B(y_0,\varepsilon/2)]$, $(y_1^*, y_2^*) \in T(C,\gamma)$ and $x^* \in D^*_{\mathscr{F}}G(x,y_1,y_2)(y_1^*,y_2^*)$.

Case 1. F is a convex multifunction. As $x^* \in D^*_{\mathscr{F}}G(x,y_1,y_2)(y_1^*,y_2^*)$, one has

$$\langle x^*, u-x\rangle + \langle y_1^*, v_1-y_1\rangle - \langle y_2^*, v_2-y_2\rangle \leq 0 \tag{4.24}$$

for all $(u,v_1,v_2) \in \operatorname{gph} G$.

For $\delta_1 \in (0,\delta)$, since $(y_1^*, y_2^*) \in T(C,\gamma)$, there are $a \in C\cap S_{Y^*}$ and $w \in B_Y$ such that $\langle y_1^*+y_2^*, a+\delta w\rangle \leq 2\gamma-\delta_1$. Since (4.23), for $t := \varepsilon-\eta-\delta_1$, then $y_2+t(a+\delta w) \in B(y_0,\varepsilon)$, $d(y_2+t(a+\delta w),F(x)) \leq t(1+\delta)$, and therefore we may find $u \in F^{-1}(y_2+t(a+\delta w))$ such that

$$\|x-u\| \leq (1+\alpha)\tau d(y_2+t(a+\delta w),F(x)) \leq (1+\alpha)t\|a+\delta w\|.$$

By taking $v_1 = v_2 = y_2 + t(\bar{y} + \delta w)$ into account in (4.24), one obtains

$$\begin{aligned}(1+\alpha)\tau\|ta+\delta w\|\|x^*\| & \\ &\geq \langle x^*, x-u\rangle \\ &\geq -\langle y_1^*+y_2^*, v-y_2\rangle - \langle y_1^*, y_2-y_1\rangle \\ &\geq t(\delta_1-\gamma) - 2\eta\|y_1^*\|. \end{aligned} \tag{4.25}$$

As $\alpha > 0$, $\delta_1 \in (0,\delta)$, $\eta \in (0, \varepsilon - \delta)$ are arbitrarily, one has

$$\|x^*\| \geq \frac{\delta-\gamma}{\tau\|a+\delta w\|} \geq \frac{\delta-\gamma}{\tau(1+\delta)}.$$

Thus,

$$\liminf_{\substack{(x,y_1,y_2)\overset{G}{\to}(x_0,y_0,y_0)\\ \delta\downarrow 0^+}} d_*(0, D^*_{\mathscr{F}}G(x,y_1,y_2)(T(C,\gamma))) \geq \frac{\delta}{\tau(1+\delta)} > 0.$$

Case 2. $F := f$ **is a continuous single-valued maping around** x_0**.** For this case, $y_1 = y_2 = f(x)$, by setting $g := (f,f) : X \to Y \times Y$ and using the usual notation: $D^*_{\mathscr{F}}g(x)(y^*) := D^*_{\mathscr{F}}f(x,f(x))(y^*)$, one has that for any $\alpha \in (0,1)$, there exists $\beta \in (0,\varepsilon/2)$ such that

$$\langle x^*, u-x\rangle - \langle y_1^*+y_2^*, f(u)-f(x)\rangle \leq \alpha(\|u-x\|+\|f(u)-f(x)\|), \tag{4.26}$$

for all $u \in B(x,\beta)$.

As in the first case, for $\delta_1 \in (0,\delta)$, take $w \in B_Y$ such that $\langle y_1^*+y_2^*, \bar{y}+\delta w\rangle \leq \gamma - \delta_1$. Since (4.23), for all sufficiently small $t > 0$, we may find $u \in f^{-1}(f(x) + t(a+\delta w))$ such that

$$\|x-u\| \leq (1+\alpha)\tau\|f(x)+t(\bar{y}+\delta w)-f(x))\| = \tau(1+\alpha)t\|a+\delta w\| < \beta.$$

Therefore, by (4.26), one obtains

$$\begin{aligned}(1+\alpha)\tau t\|a+\delta w\|\|x^*\| & \\ &\geq \langle x^*, x-u\rangle \\ &\geq -\langle y_1^*+y_2^*, f(u)-f(x)\rangle - \alpha(\|u-x\|+\|f(u)-f(x)\|) \\ &\geq t(\delta_1-\gamma) - \alpha t\|a+\delta w\|[(1+\alpha)\tau+1]. \end{aligned} \tag{4.27}$$

As $\alpha > 0$, $\delta_1 \in (0,\delta)$ are arbitrary, one has

$$\|x^*\| \geq \frac{\delta-\gamma}{\tau(1+\delta)} \geq \frac{\delta-\gamma}{\tau(1+\delta)}.$$

Thus,

$$\liminf_{\substack{(x,y_1,y_2)\overset{G}{\to}(x_0,y_0,y_0)\\ \gamma\to 0^+}} d_*(0, D^*_{\mathscr{F}}G(x,y_1,y_2)(T(C,\gamma))) \geq \frac{\delta}{\tau(1+\delta)} > 0.$$

The proof is complete. □

Let us now recall the notion of *partial sequential normal compactness* (PSNC, in short, [22, page 76]). A multifunction $F : X \rightrightarrows Y$ with nonempty graph is *partially sequentially normally compact* at $(\bar{x},\bar{y}) \in \operatorname{gph} F$, if for any sequence of quadruples $\{(x_k,y_k,x_k^\star,y_k^\star)\}_{n\in\mathbb{N}} \subset \operatorname{gph} F \times X^\star \times Y^\star$ satisfying

$$(x_k,y_k) \to (\bar{x},\bar{y}), x_k^\star \in D_{\mathscr{F}}^\star F(x_k,y_k)(y_k^\star), y_k^\star \xrightarrow{w^\star} 0, \|x_k^\star\| \to 0,$$

one has $\|y_k^\star\| \to 0$ as $k \to \infty$.

Remark 4.1 Note that condition (PSNC) at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ is satisfied if Y is finite dimensional.

The next corollary that follows directly from the preceding corollary, gives a point-based condition for the relative metric regularity.

Corollary 4.1 *Under the assumptions of Theorem 4.1, suppose further that G^{-1} is PSNC at (x_0,y_0,y_0). Then F is metrically regular relatively to C at (x_0,y_0) provided*

$$d_*(0,D_{\mathscr{M}}^* G(x_0,y_0,y_0)(T(C,0)) > 0.$$

Next, let us consider the special case of $F(x) := f(x) - K$, where, $K \subseteq Y$ is a nonempty closed convex subset, $f : X \to Y$ is a continuous mapping around a given point $x_0 \in X$ with $f(x_0) \in K$. Defining $g := (f,f) : X \to Y \times Y$ and using the usual notation: $D_{\mathscr{F}}^* f(x)(y^*) := D_{\mathscr{F}}^* f(x,f(x))(y^*)$, one has

$$D_{\mathscr{F}}^* G(x,y_1,y_2)((y_1^*,y_2^*)) = \begin{cases} D_{\mathscr{F}}^* g(x)((y_1^*,y_2^*)) & \text{if } f(x)-y_i \in K,\ y_i^* \in N(K,f(x)-y_i),\ i=1,2 \\ \emptyset & \text{otherwise.} \end{cases}$$

From Theorem 4.1 we may deduce the following result.

Corollary 4.2 *Let X,Y be Asplund spaces and let $C \subseteq$ be a nonempty cone. Let $K \subseteq Y$ be a nonempty closed convex subset and let $f : X \to Y$ be a continuous mapping around $x_0 \in X$ with $k_0 := f(x_0) \in K$. If*

$$\liminf_{\substack{(x,k_1,k_2)\to(x_0,k_0,k_0)\\ \delta\downarrow 0^+}} d_*(0,D_{\mathscr{F}}^* f(x)(T(C,\delta)) \cap N(K,k_1) \times N(K,k_2))) > m > 0, \quad (4.28)$$

then the mapping $F(x) := f(x) - K$, $x \in X$ is metrically regular relatively to C with modulus $\tau = m^{-1}$ at x_0.

Remark 4.2 Note that if K is *sequentially normally compact at* $\bar{k}$, i.e., for all sequences $(k_n)_{n\in\mathbb{N}} \subseteq K$, $(k_n^*)_{n\in\mathbb{N}}$ with $k_n^* \in N(K,k_n)$,

$$k_n \to \bar{k} \text{ and } k_n^* \xrightarrow{w^*} 0 \quad \Longleftrightarrow \|k_n^*\| \to 0,$$

then instead of (4.28), the following point-based condition

$$d_*(0,D_{\mathscr{L}}^* f(x_0)[T(C,0) \cap (N(K,k_0) \times N(K,k_0))]) > 0 \quad (4.29)$$

is also sufficient for metric regularity relatively to C of $F(x) := f(x) - K$ at x_0.

Corollary 4.3 *Under the assumptions of Corollary 4.2, suppose further that f is Fréchet differential with respect to x near x_0, and its derivative is continuous at x_0. Then, the mapping $F(x) := f(x) - K$, $x \in X$ is metrically regular relatively to C if and only if*

$$\liminf_{\substack{(k_1,k_2)\to k_0\\ \delta\downarrow 0^+}} d_*(0, g'^*(x_0)[T(C,\delta)\cap(N(K,k_1)\times N(K,k_2))]) > m > 0. \tag{4.30}$$

Here, $f'^*(x)$ stands for the adjoint operator of $f'(x)$. Moreover, if K is sequentially normally compact, then (4.30) is equivalent to

$$d_*(0, f'^*(x_0)[T(C,0)\cap(N(K,k_0)\times N(K,k_0))]) > 0. \tag{4.31}$$

Proof. For the sufficiency part, suppose that

$$\liminf_{\substack{(k_1,k_2)\to(k_0,k_0)\\ \delta\downarrow 0^+}} d_*(0, f'^*(x_0)(T(C,\delta))\cap N(K,k_1)\times N(K,k_2)) > m > 0.$$

Since f' is continuous at x_0, for any $\varepsilon > 0$, there exist $\delta > 0$ such that

$$\|f'(x) - f'(x_0)\| < \varepsilon \quad \text{for all } \ x \in B(x_0,\delta).$$

Therefore, for all $\varepsilon > 0$,

$$\|f'(x)(y_1^*,y_2^*) - f'(x_0)(y_1^*,y_2^*)\| < \varepsilon,$$

for all $x \in B(x_0,\delta)$, $k_1,k_2 \in B(k_0,\varepsilon)$, $(y_1^*,y_2^*) \in T(C,\delta)\cap(N(K,k_1)\times N(K,k_2))$.

Consequently,

$$\begin{aligned}&\liminf_{\substack{(x,k_1,k_2)\to(x_0,k_0,k_0)\\ \delta\downarrow 0^+}} d_*(0, f'^*(x)[T(C,\delta)\cap(N(K,k_1)\times N(K,k_2))])\\ &= \liminf_{\substack{k\to k_0\\ \delta\downarrow 0^+}} d_*(0, f'^*(x_0)[T(C,\delta)\cap(N(K,k_1)\times N(K,k_2))]) > m > 0^{\cdot}\end{aligned}$$

The conclusion follows from Corollary 4.2. For the necessary part, consider the mapping $g : X \to Y$ defined by

$$g(x) := f'(x_0)(x - x_0) + f(x_0) - f(x), \ \ x \in X.$$

Since f is continuously differentiable at x_0, for any $\varepsilon > 0$ there is $\delta > 0$ such that g is Lipschitz with constant ε on $B(x_0,\delta)$. Hence in view of Theorem 3.1, the metric regularity relative to C of $F := f - K$ around (x_0,y_0) implies the one of

$$(F+g)(x) = f'(x_0)(x - x_0) + f(x_0) - K.$$

As $F+g$ is a convex multifunction, the conclusion of the necessary part follows from Proposition 4.23. The equivalence between (4.30) and (4.31) follows from Remark 4.2. □

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