Cánovas, M., López, M., Mordukhovich, B., Parra, J. (2012)
Qualitative stability of linear infinite inequality systems under block perturbations with applications to convex systems. TOP, 20(2), 310-327.

https://doi.org/10.1007/s11750-011-0226-4

QUANTITATIVE STABILITY OF LINEAR INFINITE INEQUALITY SYSTEMS UNDER BLOCK PERTURBATIONS WITH APPLICATIONS TO CONVEX SYSTEMS[1]

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Abstract. The original motivation for this paper was to provide an efficient quantitative analysis of convex infinite (or semi-infinite) inequality systems whose decision variables run over general infinite-dimensional (resp. finite-dimensional) Banach spaces and that are indexed by an arbitrary fixed set J. Parameter perturbations on the right-hand side of the inequalities are required to be merely bounded, and thus the natural parameter space is $l_\infty(J)$. Our basic strategy consists of linearizing the parameterized convex system via splitting convex inequalities into linear ones by using the Fenchel-Legendre conjugate. This approach yields that arbitrary bounded right-hand side perturbations of the convex system turn on constant-by-blocks perturbations in the linearized system. Based on advanced variational analysis, we derive a precise formula for computing the exact Lipschitzian bound of the feasible solution map of block-perturbed linear systems, which involves only the system's data, and then show that this exact bound agrees with the coderivative norm of the aforementioned mapping. In this way we extend to the convex setting the results of [3] developed for arbitrary perturbations with no block structure in the linear framework under the boundedness assumption on the system's coefficients. The latter boundedness assumption is removed in this paper when the decision space is reflexive. The last section provides the aimed application to the convex case.

Key words. semi-infinite and infinite programming, parametric optimization, variational analysis, convex infinite inequality systems, quantitative stability, Lipschitzian bounds, generalized differentiation, coderivatives, block perturbations

AMS subject classification. 90C34, 90C25, 49J52, 49J53, 65F22

1 Introduction

This paper arose motivated by the extension to convex inequality systems of some results from [3] concerning quantitative/Lipschitz stability of feasible solutions to linear infinite and semi-infinite systems. The basic idea was to use

[1]This research was partially supported by grants MTM2008-06695-C03 (01-02) from MICINN (Spain).

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the so-called *standard linearization* by means of the *Fenchel-Legendre conjugate.* This linearization approach entails that each convex inequality is split into a generally infinite system of linear inequalities; so that a right-hand side perturbation of each convex inequality yields the same perturbation for all the linear inequalities coming from splitting the convex one. In this way, we are dealing with a linear inequality system subject to *block perturbations*. Based on this initial motivation we firstly analyze in a general framework the Lipschitz stability of linear systems under *arbitrary* block perturbations.

Indeed, the methodology of block perturbations for linear systems and their applications to convex inequalities has been previously developed in [7] to compute the distance to ill-posedness for such systems, although now the parameter spaces associated with block partitions are different from those in [7]. Going a bit further back, extreme cases of constant perturbations are implicitly present along some proofs in [1, 6]. This observation on the prominent role of constant perturbations is also pointed out in the very recent preprint [15] that provides an alternative methodology to approach directly convex systems, where the concept of perfect regularity plays a central role.

The expression obtained in the present paper for the *exact Lipschitzian bound* (also called Lipschitz modulus; see the definition below) of the feasible set mapping provides a natural extension of its linear counterpart [3, Theorem 4.6]; cf. also [1, Corollary 3.2] and [2, Theorem 1]). In this sense, the methodology and proofs themselves can be treated as major contributions of this paper. Specifically we emphasize, aside from the methodology, the usage of tools such as *coderivatives* and the *extended Ascoli formula* of Lemma 3.

Consider the linear inequality system

$$\left\{ \langle a_t^*, x\rangle \leq b_t,\ t\in T\right\} \tag{1}$$

referred to as the *nominal system*, where T is an arbitrary *index set*, $x \in X$ is a *decision variable* from a general Banach space X with its topological dual X^*, and where the function $T \ni t \mapsto (a_t^*, b_t) \in X^* \times \mathbb{R}$ providing the nominal system's data is also arbitrary. When T is infinite and X is finite-dimensional, we are dealing with *semi-infinite* systems whereas *infinite* systems allow for both infinitely many inequalities and infinite-dimensional decision spaces. Our approach involves considering a partition of the index set T denoted by

$$\mathcal{J} := \{T_j \mid j \in J\},$$

i.e., $T_j \neq \varnothing$ for all $j \in J$ and

$$T = \bigcup_{j\in J} T_j \text{ with } T_i \cap T_j = \varnothing \ \text{ if } \ i \neq j.$$

In the sequel the sets T_j, $j \in J$, in the partition are referred to as *blocks*. Then we consider the parameterized system

$$\sigma_{\mathcal{J}}(p) := \left\{ \langle a_t^*, x\rangle \leq b_t + p_j,\ t\in T_j,\ j\in J\right\}, \tag{2}$$

where the *perturbation parameter* $p = (p_j)_{j \in J}$ ranges on the Banach space $l_\infty(J)$ endowed with the norm

$$\|p\| := \sup_{j \in J} |p_j| \,.$$

The zero function $\overline{p} = 0$ is regarded as the *nominal parameter*, which corresponds to the nominal system (1), which coincides with $\sigma_{\mathcal{J}}(0)$ for every partition $\mathcal{J}$. From now on, in order to simplify the notation, the nominal system (1) is denoted just by $\sigma(0)$. The two extreme partitions are

$$\mathcal{J}_{\min} := \{T\} \text{ and } \mathcal{J}_{\max} := \left\{ \{t\} \,\middle|\; t \in T \right\} \tag{3}$$

called hereafter the *minimum partition* and the *maximum partition*, respectively.

The major goal of the paper is to analyze *quantitative stability* of the feasible set of the *linear* infinite inequality system (1) under small *block perturbations* of the right-hand side. In more detail, we focus on characterizing *Lipschitzian behavior* of the feasible solution map with computing the *exact bound* of Lipschitzian moduli by using appropriate tools of advanced variational analysis and generalized differentiation particularly based on coderivatives. The results obtained for (1) are then applied to infinite *convex* inequalities by means of their Fenchel-Legendre conjugate linearization.

If no confusion arises, we use the same notation $\|\cdot\|$ for the given norm in X and for the corresponding dual norm in X^* defined by

$$\|x^*\| := \sup_{\|x\| \le 1} \langle x^*, x \rangle \quad \text{for any} \quad x^* \in X^*,$$

where $\langle x^*, x\rangle$ stands for the standard canonical pairing. Our main attention is focused on the *feasible solution map* $\mathcal{F}_{\mathcal{J}} : l_\infty(J) \rightrightarrows X$ defined by

$$\mathcal{F}_{\mathcal{J}}(p) := \left\{ x \in X \,\middle|\, x \text{ is a solution to } \sigma_{\mathcal{J}}(p) \right\}. \tag{4}$$

The rest of the paper is organized as follows: Section 2 presents some basic definitions and key results from variational analysis and generalized differentiation needed in the sequel. In Section 3 we establish verifiable characterizations of the Lipschitz-like property of the block-perturbed feasible solution map (4) with precise computing the exact Lipschitzian bound in terms of the initial data of (1). For this computation we assume either that $\{a_t^*,\ t \in T\}$ is bounded in X^*, as in [3], or that the Banach space X of decision variables is reflexive. Section 4 presents an application of the results obtained for linear systems with block perturbations to quantitative stability analysis of feasible solutions to convex inequality systems through their conjugate linearization.

Our notation is basically standard in the areas of variational analysis and semi-infinite/infinite programming; see, e.g., [11, 18]. Unless otherwise stated, all the spaces under consideration are Banach. The symbol w^* signifies the weak* topology of a dual space, and thus the weak* topological limit corresponds to the weak* convergence of nets. Some particular notation will be recalled, if necessary, in the places where it is introduced.

2 Preliminaries and First Stability Results

Given a set-valued mapping $F\colon Z \rightrightarrows Y$ between Banach spaces Z and Y, we say the F is *Lipschitz-like around* $(\bar{z},\bar{y}) \in \operatorname{gph} F$, the *graph* of F, with *modulus* $\ell \geq 0$ if there are neighborhoods U of $\bar{z}$ and V of $\bar{y}$ such that

$$F(z) \cap V \subset F(u) + \ell\|z-u\|\mathbb{B}_Y \quad \text{for any} \quad z,u \in U, \tag{5}$$

where $\mathbb{B}_Y$ stands for the closed unit ball in Y. The infimum of moduli $\{\ell\}$ over all the combinations of $\{\ell,U,V\}$ satisfying (5) is called the *exact Lipschitzian bound* of F around $(\bar{z},\bar{y})$ and is labeled as $\operatorname{lip} F(\bar{z},\bar{y})$.

If $V = Y$ in (5), this relationship signifies the classical (Hausdorff) *local Lipschitzian* property of F around $\bar{z}$ with the *exact Lipschitzian bound* denoted by $\operatorname{lip} F(\bar{z})$ in this case.

It is worth mentioning that the Lipschitz-like property (also known as the Aubin or pseudo-Lipschitz property) of an arbitrary mapping $F\colon Z \rightrightarrows Y$ between Banach spaces is equivalent to other two fundamental properties in nonlinear analysis while defined for the inverse mapping $F^{-1}\colon Y \rightrightarrows Z$; namely, to the *metric regularity* of F^{-1} and to the *linear openness* of F^{-1} around $(\bar{y},\bar{z})$, with the corresponding relationships between their exact bounds (see, e.g. [13, 18, 19]). From these relationships we can easily observe the following representation for the exact Lipschitzian bound:

$$\operatorname{lip} F(\bar{z},\bar{y}) = \limsup_{(z,y)\to(\bar{z},\bar{y})} \frac{\operatorname{dist}\big(y;F(z)\big)}{\operatorname{dist}\big(z;F^{-1}(y)\big)}, \tag{6}$$

where $\inf\emptyset := \infty$ (and hence $\operatorname{dist}(x;\emptyset) = \infty$) as usual, and where $0/0 := 0$. We have accordingly that $\operatorname{lip} F(\bar{z},\bar{y}) = \infty$ if F is not Lipschitz-like around $(\bar{z},\bar{y})$.

A remarkable fact consists of the possibility to characterize pointwisely the (derivative-free) Lipschitz-like property of F around $(\bar{z},\bar{y})$—and hence its local Lipschitzian, metric regularity, and linear openness counterparts—in terms of a dual-space construction of generalized differentiation called the *coderivative* of F at $(\bar{z},\bar{y}) \in \operatorname{gph} F$. The latter is a positively homogeneous multifunction $D^*F(\bar{z},\bar{y})\colon Y^* \rightrightarrows Z^*$ defined by

$$D^*F(\bar{z},\bar{y})(y^*) := \big\{z^* \in Z^* \big|\ (z^*,-y^*) \in N\big((\bar{z},\bar{y});\operatorname{gph} F\big)\big\}, \quad y^* \in Y^*, \tag{7}$$

where $N(\cdot;\Omega)$ stands for the collection of generalized normals to a set at a given point known as the *basic*, or *limiting*, or *Mordukhovich normal cone*; see, e.g. [16, 18, 19, 20] and references therein. When both Z and Y are finite-dimensional, it is proved in [17] (cf. also [19, Theorem 9.40]) that a closed-graph mapping $F\colon Z \rightrightarrows Y$ is Lipschitz-like around $(\bar{z},\bar{y}) \in \operatorname{gph} F$ if and only if

$$D^*F(\bar{z},\bar{y})(0) = \{0\}, \tag{8}$$

and the exact Lipschitzian bound of moduli $\{\ell\}$ in (5) is computed by

$$\operatorname{lip} F(\bar{z},\bar{y}) = \|D^*F(\bar{z},\bar{y})\| := \sup\big\{\|z^*\| \ \big|\ z^* \in D^*F(\bar{z},\bar{y})(y^*),\ \|y^*\| \leq 1\big\}. \tag{9}$$

There is an extension [18, Theorem 4.10] of the coderivative criterion (8), via the so-called mixed coderivative of F at $(\bar{z}, \bar{y})$, to the case when both spaces Z and Y are Asplund (i.e., their separable subspaces have separable duals) under some additional "partial normal compactness" assumption that is automatic in finite dimensions. Also the aforementioned theorem contains an extension of the exact bound formula (9) provided that Y is Asplund while Z is finite-dimensional. Unfortunately, none of these results is applied in our setting (4) when J is infinite; the latter is our standing assumption needed, in particular, for applications to convex infinite systems developed in Section 4.

Nevertheless we show in this paper that both (8) and (9) remain valid for $\mathcal{F}_{\mathcal{J}} \colon l_\infty(J) \rightrightarrows X$ in (4) defined by the *block-perturbed* infinite system of linear inequalities (2). The graph $\operatorname{gph} \mathcal{F}_{\mathcal{J}}$ of this mapping is obviously convex, and we can easily verify that it is also closed with respect to the product topology. If the partition index set J is infinite, $l_\infty(J)$ is an infinite-dimensional Banach space, which is *never Asplund*. It is well known from functional analysis (see, e.g., [10]) that there exists an isometric isomorphism between the topological dual $l_\infty(J)^*$ and the space $ba(J)$ of additive and bounded measures on 2^J.

Given a subset S of a normed space, the notation $\operatorname{co} S$ and $\operatorname{cone} S$ stand for the convex hull and the conic convex hull of S, respectively. The symbol $\mathbb{R}_+$ signifies the interval $[0, \infty)$, and by $\mathbb{R}_+^{(J)}$ we denote the collection of all the functions $\lambda = (\lambda_j)_{j \in J} \in \mathbb{R}_+^J$ such that $\lambda_j > 0$ for only *finitely many* $j \in J$. As usual, $\operatorname{cl}^* S$ stands for the weak* (w^* in brief) topological closure of S.

Following the lines in [3, Theorem 3.2] and appealing to the extended Farkas Lemma (see [3, Lemma 2.1] and references therein), we have the following characterization of $D^*\mathcal{F}_{\mathcal{J}}(0, \overline{x})$, where we use the notation δ_j for the classical *Dirac measure* at $j \in J$ given by

$$\langle \delta_j, p \rangle := p_j \text{ for } p = (p_j)_{j \in J} \in l_\infty(J).$$

Proposition 1 (computing coderivatives for linear systems). *Consider any $\overline{x} \in \mathcal{F}_{\mathcal{J}}(0)$ for the mapping $\mathcal{F}_{\mathcal{J}} \colon l_\infty(J) \rightrightarrows X$ defined by* (4). *Then we have $p^* \in D^*\mathcal{F}_{\mathcal{J}}(0, \overline{x})(x^*)$ if and only if*

$$(p^*, -x^*, -\langle x^*, \overline{x} \rangle) \in \operatorname{cl}^* \operatorname{cone}\big\{ (-\delta_j, a_t^*, b_t) \;\big|\; j \in J,\; t \in T_j \big\}.$$

Let us now define the characteristic set

$$C_{\mathcal{J}}(p) := \operatorname{co}\left\{ (a_t^*, b_t + p_j),\; t \in T_j,\; j \in J \right\} \subset X^* \times \mathbb{R} \tag{10}$$

for $p \in l_\infty(J)$. Observe that $C_{\mathcal{J}}(0)$ actually does not depend on $\mathcal{J}$ but just on the nominal system (1). For this reason, we denote in what follows the $C_{\mathcal{J}}(0)$ simply by $C(0)$, i.e.,

$$C(0) := \operatorname{co}\big\{ (a_t^*, b_t),\; t \in T \big\}.$$

We say that the system $\sigma(0)$ in (1) satisfies the *strong Slater condition* (SSC) if there exists a point $\widehat{x} \in X$ such that

$$\sup_{t \in T} [\langle a_t^*, \widehat{x} \rangle - b_t] < 0.$$

In this case $\widehat{x}$ is called a *strong Slater point* (SS point in brief) for $\sigma(0)$.

Lemma 2 (equivalent descriptions of the Lipschitz-like property). *Assume that $\overline{x} \in \mathcal{F}_{\mathcal{J}}(0)$. The following statements are equivalent:*
(i) *$\mathcal{F}_{\mathcal{J}}$ is Lipschitz-like around* $(0, \overline{x})$;
(ii) $D^*\mathcal{F}_{\mathcal{J}}(0, \bar{x})(0) = \{0\}$;
(iii) $\sigma(0)$ *satisfies the SSC*;
(iv) $0 \in \operatorname{int}(\operatorname{dom} \mathcal{F}_{\mathcal{J}})$;
(v) *$\mathcal{F}_{\mathcal{J}}$ is Lipschitz-like around $(0, x)$ for all $x \in \mathcal{F}_{\mathcal{J}}(0)$*;
(vi) $(0, 0) \notin \operatorname{cl}^* C(0)$.

Proof. (i)⇒(ii) is a consequence of [18, Theorem 1.44] established for general set-valued mappings of closed graph between Banach spaces. The proof of (ii)⇒(i) follows the lines in the proof of [3, Theorem 4.1].

In the case of the maximum partition as in (3) the equivalence between (iii) and (vi) may be found in, e.g., [12, Theorem 3.1]; see also [11, Theorem 6.1]. Since (iii) and (vi) are not of parametric nature (i.e., their definitions involve just the nominal system, independently of the partition under consideration), the equivalence between them holds true. Moreover, equivalence (iii)⟺(iv) for the maximum partition trivially entails that (iii)⟹(iv) for the arbitrary partition $\mathcal{J}$, since block perturbations are a particular case of arbitrary perturbations. The reverse implication (iv)⟹(iii) holds by considering a constant perturbation $p \equiv \varepsilon$ for $\varepsilon > 0$ sufficient small to guarantee that $p \in \operatorname{int}(\operatorname{dom} \mathcal{F}_{\mathcal{J}})$ by taking into account that constant perturbations (corresponding to the minimum partition) are trivially a particular case of block perturbations. The equivalences (i)⟺(iv) and (iv)⟺(v) follows from the classical Robinson-Ursescu theorem. This completes the proof of the lemma ■

The following technical statement is of its own interest while playing an essential role in proving the main results presented in the subsequent sections. We keep the convention $0/0 := 0$. Observe that this result is not of parametric nature (i.e., no concept involving perturbation of p is used).

Lemma 3 (distance to feasible solutions). [3, Lemma 4.3] *Assume that the SSC is satisfied for the system $\sigma_{\mathcal{J}}(p)$ in (2) for $p \in l_\infty(J)$. Then for any $x \in X$ we have the representation*

$$\operatorname{dist}\big(x; \mathcal{F}_{\mathcal{J}}(p)\big) = \sup_{(x^*, \alpha) \in \operatorname{cl}^* C_{\mathcal{J}}(p)} \frac{[\langle x^*, x \rangle - \alpha]_+}{\|x^*\|}. \tag{11}$$

If furthermore the space X is reflexive, then

$$\operatorname{dist}\big(x; \mathcal{F}_{\mathcal{J}}(p)\big) = \sup_{(x^*, \alpha) \in C_{\mathcal{J}}(p)} \frac{[\langle x^*, x \rangle - \alpha]_+}{\|x^*\|}. \tag{12}$$

Remark 4 According to the extended Farkas Lemma in [3, Lemma 2.1] the feasibility of $\sigma_{\mathcal{J}}(p)$ ensures that $\alpha \leq 0$ whenever $(0,\alpha) \in \mathrm{cl}^* C_{\mathcal{J}}(p)$, and then the convention $0/0 := 0$ is applied. Moreover, [3, Example 4.4] shows that the simplified expression (12) may fail for the nonreflexive Asplund space $X = c_0$ of all sequences converging to zero endowed with the supremum norm.

3 Quantitative Stability of Linear Systems under Block Perturbations

The main result of this section is Theorem 10, where an expression for the coderivative norm and the exact Lipschitzian bound of the feasible solution set mapping of block-perturbed linear inequality systems is provided under either the coefficient boundedness $\{a_t^*,\ t \in T\}$ or the reflexivity of the decision space X. To accomplish this, we proceed the following chain of technical lemmas.

Recall that $\mathcal{F}_{\mathcal{J}} : l_\infty(J) \rightrightarrows X$ is defined by (4) with an arbitrary Banach decision space X unless otherwise stated. Moreover, the zero vector or function in all the spaces under consideration are simply denoted by 0.

Lemma 5 (relationships between exact Lipschitzian bounds of block-perturbed systems). *Let* $\overline{x} \in \mathcal{F}_{\mathcal{J}}(0)$. *Then we have*

$$\mathrm{lip}\,\mathcal{F}_{\min}(0,\overline{x}) \leq \mathrm{lip}\,\mathcal{F}_{\mathcal{J}}(0,\overline{x}) \leq \mathrm{lip}\,\mathcal{F}_{\max}(0,\overline{x})$$

in the notation of (3).

Proof. Consider the nontrivial case when SSC is satisfied at the nominal system $\sigma(0)$; otherwise all the exact Lipschitzian bounds are ∞ according to the equivalence (i)$\Longleftrightarrow$(iii) in Lemma 2). Note that the mappings $\mathcal{F}_{\min}$, $\mathcal{F}_{\mathcal{J}}$, and $\mathcal{F}_{\max}$ act in the spaces $\mathbb{R}$, $l_\infty(J)$, and $l_\infty(T)$, respectively. For each $\rho \in \mathbb{R}$ let p_ρ be the constant function $p_\rho \equiv \rho$ on J, and for each $p \in l_\infty(J)$ denote by p_T the constant by blocks function on T defined as p_j on block T_j, $j \in J$. Then the proof of the lemma relies on the observation that

$$\mathrm{dist}\left(\rho; \mathcal{F}_{\min}^{-1}(x)\right) \geq \mathrm{dist}\left(p_\rho; \mathcal{F}_{\mathcal{J}}^{-1}(x)\right) \quad \text{and} \quad \mathrm{dist}\left(p; \mathcal{F}_{\mathcal{J}}^{-1}(x)\right) \geq \mathrm{dist}\left(p_T; \mathcal{F}_{\max}^{-1}(x)\right)$$

for any $x \in X$. In more details, for the first inequality (and similarly for the second one) observe that $\mathcal{F}_{\mathcal{J}}^{-1}(x) = \varnothing$ yields $\mathcal{F}_{\min}^{-1}(x) = \varnothing$. Consider further the nontrivial case when both sets are nonempty. Thus we get for some sequence $\{\rho_r\}_{r \in \mathbb{N}} \subset \mathcal{F}_{\min}^{-1}(x)$ that

$$\mathrm{dist}\left(\rho; \mathcal{F}_{\min}^{-1}(x)\right) = \lim_{r \in \mathbb{N}} |\rho - \rho_r| = \lim_{r \in \mathbb{N}} \|p_\rho - p_{\rho_r}\| \geq \mathrm{dist}\left(p_\rho; \mathcal{F}_{\mathcal{J}}^{-1}(x)\right)$$

by taking into account that $\rho_r \in \mathcal{F}_{\min}^{-1}(x)$ if and only if $p_{\rho_r} \in \mathcal{F}_{\mathcal{J}}^{-1}(x)$.

Finally, we appeal to the Lipschitzian bound representation (6) combined with the facts that

$$\mathcal{F}_{\min}(\rho) = \mathcal{F}_{\mathcal{J}}(p_\rho) \quad \text{and} \quad \mathcal{F}_{\mathcal{J}}(p) = \mathcal{F}_{\max}(p_T),$$

which thus completes the proof of the lemma. ■

Lemma 6 (relationship between coderivative norms for block-perturbed systems). *Take any* $\overline{x} \in \mathcal{F}_{\mathcal{J}}(0)$ *and consider also the mapping* $\mathcal{F}_{\min}\colon \mathbb{R} \rightrightarrows X$. *Then we have the relationship*

$$\|D^{*}\mathcal{F}_{\min}(0,\overline{x})\| \leq \|D^{*}\mathcal{F}_{\mathcal{J}}(0,\overline{x})\|. \tag{13}$$

Proof. Observe that $\mathcal{F}_{\mathcal{J}}(0) = \mathcal{F}_{\min}(0)$ since both sets are nothing else but the nominal feasible set. Hence $\overline{x} \in \mathcal{F}_{\min}(0)$. According to the coderivative norm definition in (9), pick arbitrarily $x^{*} \in X^{*}$ with $\|x^{*}\| \leq 1$ and consider the nontrivial case when there exists $\mu \in \mathbb{R}\backslash\{0\}$ with $\mu \in D^{*}\mathcal{F}_{\min}(0,\overline{x})(x^{*})$. The coderivative calculation in Proposition 1 entails the existence of a net $\{\lambda_{\nu}\}_{\nu\in\mathcal{N}}$ with $\lambda_{\nu} = (\lambda_{t\nu})_{t\in T} \in \mathbb{R}_{+}^{(T)}$ as $\nu\in\mathcal{N}$ satisfying

$$\left(\mu, -x^{*}, -\langle x^{*},\overline{x}\rangle\right) = w^{*}\text{-}\lim_{\nu\in\mathcal{N}}\sum_{t\in T}\lambda_{t\nu}\left(-1, a_{t}^{*}, b_{t}\right). \tag{14}$$

Looking at the first coordinates in (14) and setting $\gamma_{\nu} := \sum_{t\in T}\lambda_{t\nu}$, we obtain

$$-\mu = \lim_{\nu\in\mathcal{N}}\gamma_{\nu} > 0, \tag{15}$$

and hence $\gamma_{\nu} > 0$ for ν sufficiently advanced in the directed set $\mathcal{N}$; say for all ν without loss of generality. This gives us the expression

$$\left(\mu^{-1}x^{*}, \langle\mu^{-1}x^{*},\overline{x}\rangle\right) = w^{*}\text{-}\lim_{\nu\in\mathcal{N}}\sum_{t\in T}\gamma_{\nu}^{-1}\lambda_{t\nu}\left(a_{t}^{*}, b_{t}\right) \in \operatorname{cl}{}^{*}C(0). \tag{16}$$

For each $\nu \in \mathcal{N}$ we consider the net $\eta_{\nu} = (\eta_{j\nu})_{j\in J} \in \mathbb{R}_{+}^{(J)}$ with $\eta_{j\nu} := \sum_{t\in T_j}\gamma_{\nu}^{-1}\lambda_{t\nu}$, which obviously satisfies the condition $\sum_{j\in J}\eta_{j\nu} = 1$. Since the net $\{\sum_{j\in J}\eta_{j\nu}(-\delta_j)\}_{\nu\in\mathcal{N}}$ is contained in $\mathbb{B}_{l_{\infty}(J)^{*}}$, the classical Alaoglu-Bourbaki theorem ensures that a certain subnet (indexed without relabeling by $\nu \in \mathcal{N}$) weak* converges to some $p^{*} \in l_{\infty}(J)^{*}$ with $\|p^{*}\| \leq 1$. Denoting by $e \in l_{\infty}(J)$ the function whose coordinates are identically one, we get

$$\langle p^{*}, -e\rangle = \lim_{\nu\in\mathcal{N}}\sum_{t\in j}\eta_{j\nu} = 1,$$

and hence $\|p^{*}\| = 1$. Appealing now to (16) gives us, for the subnet under consideration (recalling the definition of $\eta_{j\nu}$), the equality

$$\left(p^{*}, \mu^{-1}x^{*}, \langle\mu^{-1}x^{*},\overline{x}\rangle\right) = w^{*}\text{-}\lim_{\nu\in\mathcal{N}}\sum_{j\in J}\sum_{t\in T_j}\gamma_{\nu}^{-1}\lambda_{t\nu}\left(-\delta_j, a_{t}^{*}, b_{t}\right).$$

Employing further the coderivative description from Proposition 1 yields

$$p^{*} \in D^{*}\mathcal{F}_{\mathcal{J}}(0,\overline{x})\left(-\mu^{-1}x^{*}\right).$$

Recalling (15), the positive homogeneity of the coderivative ensures

$$-\mu p^{*} \in D^{*}\mathcal{F}_{\mathcal{J}}(0,\overline{x})(x^{*}),$$

which implies by definition of the coderivative norm in (9) that

$$\|D^*\mathcal{F}_{\mathcal{J}}(0,\overline{x})\| \geq \|-\mu p^*\| = -\mu = |\mu|.$$

Since $\mu \in D^*\mathcal{F}_{\min}(0,\overline{x})(x^*)$ was chosen arbitrarily, we arrive at (13) and thus complete the proof of the lemma. ■

Remark 7 In the sequel we adopt the convention $\sup\varnothing := 0$, which makes sense while dealing with nonnegative numbers. Observe that under this convention we have for a SS point $\overline{x}$ of $\sigma(0)$ the equality

$$\sup\left\{\|u^*\|^{-1} \;\middle|\; \left(u^*, \langle u^*, \overline{x}\rangle\right) \in \mathrm{cl}^*C(0)\right\} = 0.$$

In fact, it is easy to check that for a SS point $\overline{x}$ of $\sigma(0)$ there is no element $u^* \in X^*$ satisfying $\left(u^*, \langle u^*, \overline{x}\rangle\right) \in \mathrm{cl}^*C(0)$. Note that the reciprocal is not true in general. To illustrate it, consider the system $\sigma(0) := \{tx \leq 1/t;\, t = 1, 2, \ldots\}$ in $\mathbb{R}$. On one hand, observe that $\overline{x} = 0$ is not a SS point. On the other hand, we have $\{u^* \in \mathbb{R} |\; \left(u^*, \langle u^*, \overline{x}\rangle\right) \in \mathrm{cl}^*C(0)\} = \varnothing$.

Remark 8 If SSC fails at $\sigma(0)$, then Lemma 2 ensures that $(0,0) \in \mathrm{cl}^*C(0)$. Under the convention $0^{-1} := \infty$ we have in this case that

$$\sup\left\{\|u^*\|^{-1} \;\middle|\; \left(u^*, \langle u^*, \overline{x}\rangle\right) \in \mathrm{cl}^*C(0)\right\} = \infty.$$

Lemma 9 (lower estimate of the coderivative norm for the minimum partition). *Consider the mapping* $\mathcal{F}_{\min}\colon \mathbb{R} \rightrightarrows X$ *and pick* $\overline{x} \in \mathcal{F}_{\min}(0)$*. Then we have the estimate*

$$\sup\left\{\|u^*\|^{-1} \;\middle|\; \left(u^*, \langle u^*, \overline{x}\rangle\right) \in \mathrm{cl}^*C(0)\right\} \leq \|D^*\mathcal{F}_{\min}(0,\overline{x})\|. \tag{17}$$

Proof. Let us see first that $\|D^*\mathcal{F}_{\min}(0,\overline{x})\| = \infty$ provided that the SSC fails at $\sigma(0)$. Indeed, in this case Lemma 2 yields that $(0,0) \in \mathrm{cl}^*C(0)$, which implies the existence of a net $\{\lambda_\nu\}_{\nu\in\mathcal{N}}$ with $\lambda_\nu = (\lambda_{t\nu})_{t\in T} \in \mathbb{R}_+^{(T)}$ and $\sum_{t\in T}\lambda_{t\nu} = 1$ as $\nu\in\mathcal{N}$ satisfying

$$\big(0,0\big) = w^*\text{-}\lim_{\nu\in\mathcal{N}} \sum_{t\in T} \lambda_{t\nu}\,(a_t^*, b_t).$$

The latter obviously entails that $\big(-1,0,0\big) = w^*\text{-}\lim_{\nu\in\mathcal{N}} \sum_{t\in T} \lambda_{t\nu}\,(-1, a_t^*, b_t)$, i.e., by Proposition 1 we get

$$-1 \in D^*\mathcal{F}_{\min}(0,\overline{x})(0).$$

Since $D^*\mathcal{F}_{\min}(0,\overline{x})$ is positively homogeneous, the coderivative norm definition gives us the claimed condition $\|D^*\mathcal{F}_{\min}(0,\overline{x})\| = \infty$.

Now we consider the nontrivial case when the SSC holds at $\sigma(0)$ and the set of elements $u^* \in X^*$ with $(u^*, \langle u^*, \overline{x}\rangle) \in \mathrm{cl}^*C(0)$ is nonempty. Take such an element u^*. and observe that the fulfillment of the SSC for $\sigma(0)$ ensures that

$u^* \neq 0$ according to Lemma 2. By the choice of u^*, find a net $\{\lambda_\nu\}_{\nu\in\mathcal{N}}$ with $\lambda_\nu = (\lambda_{t\nu})_{t\in T} \in \mathbb{R}_+^{(T)}$ and $\sum_{t\in T}\lambda_{t\nu} = 1$ as $\nu\in\mathcal{N}$ satisfying

$$\big(u^*, \langle u^*, \overline{x}\rangle\big) = w^*\text{-}\lim_{\nu\in\mathcal{N}} \sum_{t\in T} \lambda_{t\nu}\,(a_t^*, b_t)\,. \tag{18}$$

Then (18) can be trivially rewritten as

$$\big(-1, u^*, \langle u^*, \overline{x}\rangle\big) = w^*\text{-}\lim_{\nu\in\mathcal{N}} \sum_{t\in T} \lambda_{t\nu}\,(-1, a_t^*, b_t)\,,$$

which implies that $-1 \in D^*\mathcal{F}_{\min}(0,\overline{x})(-u^*)$. Hence hence

$$-\|u^*\|^{-1} \in D^*\mathcal{F}_{\min}(0,\overline{x})\Big(-\|u^*\|^{-1}u^*\Big)\,,$$

which ensures by the definition of the coderivative norm that

$$\|D^*\mathcal{F}_{\min}(0,\overline{x})\| \geq \|u^*\|^{-1}\,.$$

Since u^* was chosen arbitrarily from those satisfying $(u^*, \langle u^*, \overline{x}\rangle) \in \mathrm{cl}^*C(0)$, we arrive at the lower estimate (17) for the coderivative norm and thus complete the proof of this lemma. ■

Now we are ready to establish the main result of this section.

Theorem 10 (evaluation of coderivative norms for block-perturbed systems). *For any $\overline{x} \in \mathcal{F}_{\mathcal{J}}(0)$ we have the relationships*

$$\begin{aligned}\sup\Big\{\|u^*\|^{-1}\;\Big|\;\big(u^*, \langle u^*, \overline{x}\rangle\big) \in \mathrm{cl}^*C(0)\Big\} &\leq \|D^*\mathcal{F}_{\min}(0,\overline{x})\| \leq \|D^*\mathcal{F}_{\mathcal{J}}(0,\overline{x})\| \\ &\leq \mathrm{lip}\,\mathcal{F}_{\mathcal{J}}(0,\overline{x}) \leq \mathrm{lip}\,\mathcal{F}_{\max}(0,\overline{x})\,.\end{aligned}$$

Furthermore, if either the coefficient set $\{a_t^*,\ t\in T\}$ is bounded in X^* or the space X is reflexive, then all the above inequalities hold as equalities.

Proof. The lower bound estimate

$$\|D^*\mathcal{F}_{\mathcal{J}}(0,\bar{x})\| \leq \mathrm{lip}\,\mathcal{F}_{\mathcal{J}}(0,\bar{x}) \tag{19}$$

is proved in [18, Theorem 1.44] for general set-valued mappings between Banach spaces. Now apply (in this order) Lemmas 9, 6, formula (19), and Lemma 5 to obtain the claimed chain of inequalities.

Consider first the case when the set $\{a_t^*,\ t\in T\}$ is bounded in X^*. Then applying [3, Theorem 4.6] adapted to the current notation gives us

$$\mathrm{lip}\,\mathcal{F}_{\max}(0,\overline{x}) \leq \sup\Big\{\|u^*\|^{-1}\;\Big|\;\big(u^*, \langle u^*, \overline{x}\rangle\big) \in \mathrm{cl}^*C(0)\Big\} \tag{20}$$

in the nontrivial case when SSC holds at $\sigma(0)$; Remark 8.

To finish the proof of this theorem, it remains to establish the same inequality (20), again in the nontrivial case when the SSC holds at $\sigma(0)$, under the assumption that X is *reflexive*, in which case the classical Mazur theorem allows us to replace the weak* closure $\mathrm{cl}^* C(0)$ of the convex set $C(0)$ by its norm closure $\mathrm{cl}\, C(0)$. Arguing by contradiction to (20), find $\beta > 0$ such that

$$\mathrm{lip}\, \mathcal{F}_{\max}(0,\overline{x}) > \beta > \sup\left\{ \|u^*\|^{-1} \;\middle|\; (u^*, \langle u^*, \overline{x}\rangle) \in \mathrm{cl}\, C(0) \right\}. \tag{21}$$

According to (6) and the first inequality in (21), there are sequences $p_r = (p_{tr})_{t\in T} \to 0$ and $x_r \to \overline{x}$ along which

$$\mathrm{dist}\big(x_r; \mathcal{F}_{\max}(p_r)\big) > \beta\, \mathrm{dist}\big(p_r; \mathcal{F}_{\max}^{-1}(x_r)\big) \text{ for all } r \in \mathbb{N}. \tag{22}$$

By the SSC at $\sigma(0)$ we have due to Lemma 2 that $\mathcal{F}_{\max}(p_r) \neq \varnothing$ for $r \in I\!N$ sufficiently large; say for all $r \in I\!N$ without loss of generality. The imposed SSC at $\sigma(0)$ is also equivalent to the inner/lower semicontinuity of $\mathcal{F}_{\max}$ around $\overline{p} = 0$ by [9, Theorem 5.1], which entails that

$$\lim_{r\to\infty} \mathrm{dist}\big(x_r; \mathcal{F}_{\max}(p_r)\big) = 0. \tag{23}$$

Moreover, it follows from (22) that the quantity

$$\begin{aligned}\mathrm{dist}\big(p_r; \mathcal{F}_{\max}^{-1}(x_r)\big) &= \sup_{t\in T} [\langle a_t^*, x_r\rangle - b_t - p_{tr}]_+ \\ &= \sup_{(x^*,\alpha)\in C_{\max}(p_r)} [\langle x^*, x_r\rangle - \alpha]_+\end{aligned} \tag{24}$$

is finite. We may assume without loss of generality that the SSC holds at $\sigma_{\max}(p_r)$ for all r. Then it follows from Lemma 3 that

$$\mathrm{dist}\big(x_r; \mathcal{F}_{\max}(p_r)\big) = \sup_{(x^*,\alpha)\in C_{\max}(p_r)} \frac{[\langle x^*, x_r\rangle - \alpha]_+}{\|x^*\|}, \quad r = 1, 2, \ldots.$$

This allows us to find $(x_r^*, \alpha_r) \in C_{\max}(p_r)$ as $r \in I\!N$ satisfying

$$0 < \mathrm{dist}\big(x_r, \mathcal{F}_{\max}(p_r)\big) - \frac{\langle x_r^*, x_r\rangle - \alpha_r}{\|x_r^*\|} < \frac{1}{r}. \tag{25}$$

Furthermore, by (22) and (24) we can choose (x_r^*, α_r) in such a way that

$$\beta\, \mathrm{dist}\big(p_r; \mathcal{F}_{\max}^{-1}(x_r)\big) < \frac{\langle x_r^*, x_r\rangle - \alpha_r}{\|x_r^*\|} \leq \frac{\mathrm{dist}\big(p_r; \mathcal{F}_{\max}^{-1}(x_r)\big)}{\|x_r^*\|}. \tag{26}$$

Since $\mathrm{dist}(p_r; \mathcal{F}_{\max}^{-1}(x_r)) > 0$ (otherwise both members of (22) would be zero), we deduce from (26) that

$$\|x_r^*\| < \frac{1}{\beta} \text{ for all } r = 1, 2, \ldots,$$

and thus, by the weak* sequential compactness of the unit ball in duals to reflexive spaces, select a subsequence $\left\{x_{r_k}^*\right\}_{k\in\mathbb{N}}$, which weak* converges to some $x^* \in X^*$ satisfying $\|x^*\| \le 1/\ \beta$. Then we get from (23) and (25) that

$$\lim_{k\in\mathbb{N}} \frac{\left\langle x_{r_k}^*, x_{r_k}\right\rangle - \alpha_{r_k}}{\left\|x_{r_k}^*\right\|} = 0,$$

which implies in turn that

$$\lim_{k\in\mathbb{N}} \left(\left\langle x_{r_k}^*, x_{r_k}\right\rangle - \alpha_{r_k}\right) = 0.$$

Since the sequence $\{x_{r_k}\}_{k\in\mathbb{N}}$ converges in norm to $\overline{x}$, the latter implies that

$$\lim_{k\in\mathbb{N}} \alpha_{r_k} = \lim_{k\in\mathbb{N}} \left\langle x_{r_k}^*, x_{r_k}\right\rangle = \langle x^*, \overline{x}\rangle\,.$$

Taking into account that for each $k \in \mathbb{N}$ we have $\left(x_{r_k}^*, \alpha_{r_k}\right) \in C_{\max}\left(p_{r_k}\right)$, there exist $\lambda_{r_k} = (\lambda_{tr_k})_{t\in T}$ such that $\lambda_{tr_k} \ge 0$, only finitely many of them are positive,

$$\sum_{t\in T} \lambda_{tr_k} = 1, \quad \text{and} \quad (x_{r_k}^*, \alpha_{r_k}) = \sum_{t\in T} \lambda_{tr_k}\left(a_t^*, b_t + p_{tr_k}\right), \quad k \in \mathbb{N}.$$

Combining all the above gives us the relationships

$$\begin{aligned} \left(x^*, \langle x^*, \overline{x}\rangle\right) &= w^*\text{-}\lim_{k\in\mathbb{N}} (x_{r_k}^*, \alpha_{r_k}) \\ &= w^*\text{-}\lim_{k\in\mathbb{N}} \sum_{t\in T} \lambda_{tr_k}\left(a_t^*, b_t + p_{tr_k}\right) \\ &= w^*\text{-}\lim_{k\in\mathbb{N}} \sum_{t\in T} \lambda_{tr_k}\left(a_t^*, b_t\right) \in \operatorname{cl} C\left(0\right), \end{aligned}$$

where the last equality comes from $\lim_{k\to\infty} \|p_{r_k}\| = 0$. Observe finally that $x^* \neq 0$ because, by Lemma 2, the linear infinite system $\sigma\left(0\right)$ satisfies the SSC. This allows us to conclude that

$$\sup\left\{\|u^*\|^{-1} \mid \left(u^*, \langle u^*, \overline{x}\rangle\right) \in \operatorname{cl} C\left(0\right)\right\} \ge \|x^*\|^{-1} \ge \beta,$$

which contradicts (21) and thus completes the proof of the theorem. ■

We finish this section with a discussion about some consequences of the boundedness assumption on the coefficient set $\{a_t^* \mid t \in T\} \subset X^*$. First observe that this assumption yields that only ε-active indices are relevant in the computation of the supremum of the previous theorem. The following proposition provides a useful representation of the characteristic set $\left\{\left(u^*, \langle u^*, \overline{x}\rangle\right) \in \operatorname{cl}^* C\left(0\right)\right\}$, which may be rewritten as $\{(\overline{x}, -1)\}^{\perp} \cap \operatorname{cl}^* C\left(0\right)$, in terms of the sets

$$T_\varepsilon\left(\overline{x}\right) := \left\{t \in T \middle|\ \langle a_t^*, \overline{x}\rangle \ge b_t - \varepsilon\right\}, \quad \varepsilon 0.$$

Proposition 11 (limiting representation of the characteristic set). *Assume that the coefficient set $\{a_t^* \mid t \in T\}$ is bounded in X^*. Then given $\overline{x} \in \mathcal{F}_{\mathcal{J}}(0)$, we have the representation*

$$\left\{ (\overline{x}, -1) \right\}^{\perp} \cap \mathrm{cl}^{*} C(0) = \bigcap_{\varepsilon > 0} \mathrm{cl}^{*} \mathrm{co} \left\{ (a_t^*, b_t) \,\middle|\, t \in T_{\varepsilon}(\overline{x}) \right\}. \tag{27}$$

Proof. It follows the lines of justifying Step 1 in the proof of [2, Theorem 1]. Note that both sets in (27) are nonempty if and only if $\overline{x}$ is not a strong Slater point for $\sigma(0)$; see Remark 7. ■

Observe that in the continuous case considered in [1] (where T is assumed to be a compact Hausdorff space, $X = \mathbb{R}^n$, and the mapping $t \mapsto (a_t^*, b_t)$ is continuous on T) representation (27) reads as

$$\left\{ (\overline{x}, -1) \right\}^{\perp} \cap C(0) = \mathrm{co} \left\{ (a_t^*, b_t) \,\middle|\, t \in T_0(\overline{x}) \right\}.$$

The following example shows that the statement of Proposition 11 is *no longer valid* without the boundedness assumption on $\{a_t^* | \, t \in T\}$ and that in the exact bound expression of Theorem 10 via $\sup \left\{ \|u^*\|^{-1} \mid \left(u^*, \langle u^*, \overline{x} \rangle\right) \in \mathrm{cl}^{*} C(0) \right\}$ the set $\mathrm{cl}^{*} C(0)$ cannot be replaced by $\mathrm{cl}^{*}\mathrm{co}\ \{(a_t^*, b_t) \mid t \in T_{\varepsilon}(\overline{x})\}$ for some *small* $\varepsilon > 0$; i.e., it is not sufficient to consider just ε-active constraints.

Example 12 (coefficient boundedness is essential). Consider the countable linear system in $\mathbb{R}^2$*:*

$$\sigma(p) = \left\{ \begin{array}{ll} (-1)^t t x_1 \le 1 + p_t, & t = 1, 2, \ldots, \\ x_1 + x_2 \le 0 + p_0, & t = 0 \end{array} \right\}.$$

The reader can easily check that for $\overline{x} = 0 \in \mathbb{R}^2$ and $0 \le \varepsilon < 1$ we have

$$\mathrm{co}\left\{ (a_t^*, b_t) \,\middle|\, t \in T_{\varepsilon}(\overline{x}) \right\} = \left\{ (1, 1, 0) \right\} \text{ and}$$

$$\left\{ (\overline{x}, -1) \right\}^{\perp} \cap \mathrm{cl}^{*} C(0) = \left\{ (\alpha, 1, 0),\ \alpha \in \mathbb{R} \right\}$$

It follows furthermore that

$$\mathcal{F}_{\max}(p) = \{0\} \times (-\infty, p_0] \quad \text{whenever} \quad \|p\| \le 1,$$

which easily implies that $\mathrm{lip}\, \mathcal{F}_{\max}(0, \overline{x}) = 1$. Observe however that $\mathrm{lip}\, \mathcal{F}_{\max}(0, \overline{x})$ cannot be computed through $T_{\varepsilon}(\overline{x})$ for $0 < \varepsilon < 1$; in fact

$$\max \left\{ \|u^*\|^{-1} \,\middle|\, (u^*, \langle u^*, \overline{x} \rangle) \in \mathrm{cl}^{*}\mathrm{co}\left\{ (a_t^*, b_t) \,\middle|\, t \in T_{\varepsilon}(\overline{x}) \right\} \right\} = \frac{1}{\sqrt{2}}.$$

As mentioned above, it is clear that $\{(\overline{x}, -1)\}^{\perp} \cap \mathrm{cl}^{*} C(0) = \varnothing$ when $\overline{x}$ is a SS point for $\sigma(0)$. According to [3, Lemma 3.4], if $\{a_t^* \mid t \in T\}$ is bounded and $\overline{x}$ is not a SS point for $\sigma(0)$, the set $\{(\overline{x}, -1)\}^{\perp} \cap \mathrm{cl}^{*} C(0)$ is nonempty and w^*-compact in X^*. If in addition the SSC holds at $\sigma(0)$, then the latter set does not contain the origin and the supremum in Theorem 10 becomes a maximum.

4 Applications to Convex Systems

In this section we apply the results above to analyze the quantitative stability of infinite convex inequality systems by using the linearization procedure via the Fenchel-Legendre conjugate. This procedure splits each convex inequality into a block of linear ones so that a natural perturbation framework for the linearized system is a block perturbation setting. In what follows we consider the *parameterized convex inequality system* given by

$$\sigma(p) := \left\{ f_j\left(x\right) \leq p_j,\ j \in J \right\}, \tag{28}$$

where J is an arbitrary *index set*, $x \in X$ is a *decision variable* selected from a general Banach space X with its topological dual X^*, and where the functions $f_j : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, $j \in J$, are proper lower semicontinuous (lsc) and convex. As above, the functional parameter p belongs to the Banach space $l_\infty(J)$ and the zero function $\overline{p} = 0$ is regarded as the nominal parameter.

Hereafter we denote by $\mathcal{F}$ the *feasible solution map* of (28); i.e., $\mathcal{F} : l_\infty(J) \rightrightarrows X$ is defined by

$$\mathcal{F}(p) := \left\{ x \in X \,\middle|\, x \text{ is a solution to } \sigma(p) \right\}. \tag{29}$$

The convex system $\sigma(p)$ with $p \in l_\infty(J)$ can be *linearized* by using the *Fenchel-Legendre conjugate* $f_j^* : X^* \to \overline{\mathbb{R}}$ for each function f_j given by

$$f_j^*\left(u^*\right) := \sup\left\{ \langle u^*, x\rangle - f_j\left(x\right) \,\middle|\, x \in X \right\} = \sup\left\{ \langle u^*, x\rangle - f_j\left(x\right) \,\middle|\, x \in \mathrm{dom} f_j \right\},$$

where $\mathrm{dom} f_j := \{x \in X \mid f_j\left(x\right) < \infty\}$ is the effective domain of f_j. Specifically, under the current assumptions on each f_j its conjugate f_j^* is also a proper lsc convex function such that

$$f_j^{**} = f_j \text{ on } X \text{ with } f_j^{**} := \left(f_j^*\right)^*.$$

In this way, for each $j \in J$, the inequality $f_j\left(x\right) \leq p_j$ turns out to be equivalent to the linear system

$$\left\{ \langle u^*, x\rangle - f_j^*\left(u^*\right) \leq p_j,\ u^* \in \mathrm{dom} f_j^* \right\}$$

in the sense that they have the same solution sets.

In order to link to the notation of the previous sections, put

$$T := \left\{ (j, u^*) \in J \times X^* \mid u^* \in \mathrm{dom} f_j^* \right\}$$

and note that T is partitioned as

$$T = \bigcup_{j \in J} T_j, \qquad \text{where } T_j := \{j\} \times \mathrm{dom} f_j^*. \tag{30}$$

In this way the right-hand side perturbations on the nominal convex system $\sigma(0)$ correspond to block perturbations of the linearized nominal system $\sigma_{\mathcal{J}}(0)$

with the partition $\mathcal{J} := \{T_j \mid j \in J\}$. It is important to realize to this end that $\mathcal{F}$ and $\mathcal{F}_{\mathcal{J}}$ are *exactly the same mapping*.

Recall that the *epigraph* of a function $h\colon X \to \overline{\mathbb{R}}$ is defined by

$$\operatorname{epi} h := \big\{(x,\gamma) \in X \times \mathbb{R} \big|\ x \in \operatorname{dom} h,\ h(x) \le \gamma\big\}.$$

It is easy to see that the convex counterpart of the set $C_{\mathcal{J}}(p)$ in (10) is

$$\begin{aligned} C(p) &:= \operatorname{co}\ \big\{\big(u^*, f_j^*(u^*) + p_j\big) \mid j \in J,\ u^* \in \operatorname{dom} f_j^*\big\} \\ &= \operatorname{co}\big(\bigcup_{j\in J} \operatorname{gph}(f_j - p_j)^*\big) \subset X^* \times \mathbb{R}. \end{aligned} \tag{31}$$

For more details the reader is addressed to [8] and particularly to the extended Farkas' Lemma, which may be found in [8, Theorem 4.1].

In this convex setting the SSC at $\sigma(0)$ reads as $\sup_{t\in T} f_t(\widehat{x}) < 0$ for some $\widehat{x} \in X$. Note that $\widehat{x}$ is a strong Slater point for $\sigma(0)$ if and only if the same happens for the linearized system $\sigma_{\mathcal{J}}(0)$, i.e., $\sup_{(j,u^*)\in T}\{\langle u^*, \widehat{x}\rangle - f_j^*(u^*)\} < 0$.

The next result, which follows from its linear counterpart in Proposition 1, computes the coderivative of the solution map (29) to the original infinite convex system (28) in terms of its initial data.

Proposition 13 (computing coderivatives for convex systems). *Consider $\overline{x} \in \mathcal{F}(0)$ for the solution map* (29) *to the convex system* (28)*. Then we have $p^* \in D^*\mathcal{F}(0,\overline{x})(x^*)$ if and only if*

$$\big(p^*, -x^*, -\langle x^*, \overline{x}\rangle\big) \in \operatorname{cl}^*\operatorname{cone}\big(\bigcup_{j\in J} \big[\{-\delta_j\} \times \operatorname{gph} f_j^*\big]\big). \tag{32}$$

The next major result of the paper provides a precise computation of the exact Lipschitzian bound of the solution map (29) in the case when either the set $\bigcup_{j\in J} \operatorname{dom} f_j^*$ is bounded in X^* (this is the convex counterpart of the boundedness of $\{a_t^* \mid t \in T\}$) or the decision Banach space X is reflexive. Before this we show that the boundedness assumption, which looks quite natural in the linear setting, may fail in very simple convex examples.

Example 14 (failure of the bounded ness assumption for convex systems). Consider the following single inequality involving one-dimensional decision and parameter variables:

$$x^2 \le p \ \text{ for } \ x, p \in \mathbb{R}. \tag{33}$$

Note that the linearized system associated with (33) reads as follows:

$$\left\{ux \le \frac{u^2}{4} + p, \quad u \in \mathbb{R}\right\},$$

and thus the coefficient boundedness assumption fails.

Theorem 15 (evaluation of the coderivative norm for convex systems). *For any $\overline{x} \in \mathcal{F}(0)$ we have the relationships*

$$\sup\Big\{\,\|u^*\|^{-1}\,\Big|\,\big(u^*, \langle u^*, \overline{x}\rangle\big) \in \mathrm{cl}^*\mathrm{co}\,\Big(\bigcup_{j\in J}\mathrm{gph}\, f_j{}^*\Big)\Big\}$$
$$\leq \|D^*\mathcal{F}(0,\overline{x})\| \leq \mathrm{lip}\,\mathcal{F}(0,\overline{x})\,.$$

If furthermore either the set $\bigcup_{j\in J}\mathrm{dom}\, f_j^*$ is bounded in X^* or the space X is reflexive, then the above inequalities hold as equalities.

Proof. It follows from Theorem 10 applied to the linearized system with block perturbations by the linearization procedure and discussions above. ■

Remark 16 After the publication of [3], Alex Ioffe drew our attention to the possible connections of some of the results therein with those obtained in [14] for general set-valued mappings of convex graph. Examining this approach, we were able to check, in particular, that the result of [3, Corollary 4.7] on the computing the exact Lipschitzian bound of linear infinite systems via the coderivative norm under the coefficient boundedness can be obtained by applying Theorem 3 and Proposition 5 from [14] by involving some technicalities.

Remark 17 The main results of this paper were basically obtained at the end of 2008 during the visit of the third author to the University of Alicante and the Miguel Hernández University of Elche and then were presented at several meetings in 2009-10 and also written in [5]. During the final revision of the manuscript we have become familiar with the very recent preprint [15] where, under a certain uniform boundedness condition held by replacing our functions f_j with $\max\{-1, f_j\}$, the equality in Theorem 15 is obtained with no coefficient boundedness or reflexivity assumptions by a completely different approach.

Remark 18 Following our approach in [4], the coderivative calculations presented above allow us to develop necessary optimality conditions of both lower subdifferential and upper subdifferential types for nonsmooth problems of semi-infinite and infinite programming with feasible sets given by infinite systems of convex inequalities; see [5, Section 6] for more details.

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