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Nonsmooth Lyapunov pairs for differential inclusions governed by operators with nonempty interior domain

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Abstract The general theory of Lyapunov stability of first-order differential inclusions in Hilbert spaces has been studied by the authors in the previous paper (Adly et al. in Nonlinear Anal 75(3): 985–1008, 2012). This new contribution focuses on the case when the interior of the domain of the maximally monotone operator governing the given differential inclusion is nonempty; this includes in a natural way the finite-dimensional case. The current setting leads to simplified, more explicit criteria and permits some flexibility in the choice of the generalized subdifferentials. Some consequences of the viability of closed sets are given. Our analysis makes use of standard tools from convex and variational analysis.

Keywords Evolution differential inclusions · Maximally monotone operators · Lower semicontinuous Lyapunov pairs and functions · Invariant sets · Generalized subdifferentials

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Dedicated to Jong-Shi Pang.

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1 Introduction

In 1964, Stampacchia [32] extended the well known Lax-Milgram lemma (or variational equation) on coercive bilinear forms to convex and closed sets (representing in general some constraints). This important result applied by Fichera [18] to the Signorini problem on the elastic equilibrium of a body under unilateral constraints and by Stampacchia to the definition of the capacitary potential associated to a non symmetric bilinear form, is considered as the starting point of the theory of variational inequalities, see e.g., the excellent survey by Mazzone and Lions [21,23]. This theory was considerably extended with the work by Hartman and Stampacchia [19], in reflexive Banach spaces, for nonlinear partial differential operators arising in elasticity. Then, Lions and Stampacchia [22] extended Fichera's analysis to abstract variational inequalities associated to bilinear forms which are coercive or simply non negative in real Hilbert spaces as a tool for the study of partial differential elliptic and parabolic equations (see, also [17] and [16] for applications to the unilateral mechanics). In an abstract framework, if K is a closed and convex subset of an ambient space X and f a given element in the dual space, a variational inequality is the problem of finding $u \in K$ such that

$$\langle Au - f, v - u\rangle \geq 0 \quad \text{for each } v \in K. \tag{1}$$

The evolution analogue of (1), i.e., the problem of finding a function $t \to u(t)$, where t is the time, modelizes evolution problems such as parabolic or hyperbolic equations (see [16]). In this context, evolution variational inequalities or more generally differential inclusions have been considered as a natural generalization of ordinary differential equations (ODE's) when dealing with unilateral constraints in mechanics, for example.

On the other hand, studies were conducted in parallel in linear and nonlinear complementarity problems (a part of mathematical programming and optimization) in finite dimensional spaces with a large number of applications in economics, finance, transportation planning, Nash equilibrium and game theory. The two subjects of variational inequalities and complementarity are closely related.

In many applications, the models lead to an ordinary differential equation parametrized by a variational inequality or complementarity condition in the constraint. Such a combination appears naturally for instance in optimal control theory where the state $x(t)$ and the control $u(t)$ are related by a constraint. For illustration purpose, let us consider the following nonlinear input/output system

$$\begin{cases} \dot{x}(t) = f(t, x(t)) + B^T u(t), \quad \text{a.e. } t \in [0, T] & (2)\\ y(t) = Bx(t), & (3)\\ 0 \leq x(t) \perp y(t) \geq 0. & (4) \end{cases}$$

Here $x(t) \in \mathbb{R}^n$ is the state variable, $u(t) \in \mathbb{R}^m$ is the input variable, $y(t) \in \mathbb{R}^m$ is the output variable, $B \in \mathbb{R}^{m\times n}$ is a given matrix and $f{:}[0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a given vector field. This problem belongs to the class of piecewise smooth systems which consist of an ODE parametrized by an algebraic variable y that is required to satisfy a complementarity condition or more generally a variational inequality. This can be extended to the more general abstract framework of Differential Variational Inequalities (DVI's) studied thoroughly by Pang and Stewart [25].

Problem (2)–(4) plays an important role in nonsmooth mechanics (multibody dynamics with unilateral contact), in nonregular electrical circuits (switching systems, relay, diodes and transistors) as well as in dynamic games. Using tools from convex analysis, we show that problem (1) can be reduced to the study of a general differential inclusion governed by a maximally monotone operator which is our object problem to study in this paper. In fact, the complementarity condition (4) can be rewritten in terms of the normal cone. More precisely,

$$0 \le y(t) \perp u(t) \ge 0 \Longleftrightarrow y(t) \in -N_{\mathbb{R}^n_+}(u(t)),$$

where $N_{\mathbb{R}^n_+}$ stands for the normal cone to the positive orthant $\mathbb{R}^n_+$. Hence, relation (2) is equivalent to

$$\dot{x}(t) \in f(t, x(t)) - B^T N_{\mathbb{R}^n_+}(Bx(t)).$$

Using the chain rule formula and using the indicator function (defined in Sect. 2) $I_{R^n_+}$, we obtain

$$\dot{x}(t) \in f(t, x(t)) - \partial\varphi(x(t)), \tag{5}$$

where $\varphi(x) = (\mathrm{I}_{\mathbb{R}^n_+} \circ B)(x)$, which is a convex, lower semicontinuous (lsc for short) and proper function. Let us notice that problem (5) is of the form (6) since $A = \partial\varphi$ is a maximally monotone operator.

Equally important is the study of the stability in the sense of Lyapunov of dynamical systems due to its usefulness in system theory and engineering. This concept has been studied extensively in the literature in the smooth case. In various applications modeled by ordinary differential equations, one may be forced to work with systems that have nondifferentiable solutions. For example, Lyapunov functions (positive-definite functions, which are nonincreasing along the trajectories) used to establish a stability of a given system may be nondifferentiable. The need to extend the classical differentiable Lyapunov stability to the nonsmooth case is unavoidable when studying discontinuous systems. In practice, many systems arising in physics, engineering, biology, etc, exhibit generally nonsmooth energy functions, which are usually typical candidates for Lyapunov functions. Thus, the use of elements of nonsmooth analysis is essential [3,9,15,29].

Instead of considering inclusion (5), throughout this article we are interested in the general framework of infinite-dimensional dynamical systems, that is, systems of the form:

$$\dot{x}(t; x_0) \in f(x(\cdot; x_0)) - Ax(\cdot; x_0),\ x_0 \in \mathrm{cl}(\mathrm{Dom}\, A)\, a.e.\, t \ge 0. \tag{6}$$

In the sequel, cl(Dom A) is the closure of the domain of a maximally monotone operator $A : H \rightrightarrows H$ defined on a real Hilbert space H, possibly nonlinear and multivalued with domain Dom A, and f is a Lipschitz continuous mapping defined on cl(Dom A). A pair of proper lower semicontinuous functions $V, W : H \to \mathbb{R} \cup \{+\infty\}$ is said to form an $a-$ Lyapunov pair for (6), with some $a \geq 0$, if for all $x_0 \in$ cl(Dom A) the solution $x(\cdot; x_0)$ of (6), in the sense that will be made precise in Sect. 3, satisfies

$$e^{at} V(x(t; x_0)) + \int_s^t W(x(\tau; x_0)) d\tau \leq e^{as} V(x(s; x_0)) \quad \text{for all } t \geq s \geq 0. \tag{7}$$

Observe that when $W \equiv 0$ and $a = 0$ one recovers the classical notion of Lyapunov functions; see, e.g., [31]. The main motivation for this definition is that many stability concepts for the equilibrium sets of (6) (namely stability, asymptotic or finite-time stability, etc.) can be obtained just by choosing appropriate values for a and function W in (7). The method of Lyapunov functions has a long history that has been described in several places. We refer the reader to Clarke [13,14] for an overview of the recent developments of the theory, where he pointed out that for nonlinear systems, the Lyapunov method turns out to be essential to consider nonsmooth Lyapunov functions, even if the underlying control dynamics are themselves smooth.

Among the various contributions, Kocan and Soravia [20] characterized Lyapunov pairs in terms of viscosity solutions to a related partial differential inequality. Another well-established approach consists of characterizing Lyapunov pairs by means of the contingent derivative of the maximally monotone operator A—see e.g. Cârjă and Motreanu [10] for the case of a linear maximally monotone operator and also when A is a multivalued m-accretive operator on an arbitrary Banach space [11]. In these approaches the authors used tangency and flow-invariance arguments combined with a priori estimates and approximation. We also refer to the paper by Adly and Goeleven [1] in which smooth Lyapunov functions were used in the framework of the second order differential equations, that can be rewritten in the form of (6).

In [2], we followed a different approach that did not make use of viscosity solutions or contingent derivatives associated to the operator A. We provided general criteria for nonsmooth Lyapunov pairs associated to (6) in terms of proximal and horizon subgradients of the involved function V. Such conditions were written by considering limiting processes required by the fact that the initial condition in (6) was allowed to be any point in the closure of the domain A.

Our objective in this work is to refine the approach of [2] to the setting where the interior of the domain of the involved maximally monotone operator is nonempty. This setting subsumes the finite-dimensional case where the relative interior of the convex envelope of the domain of the operator is always nonempty. Moreover, as in [1] which deals with the smooth case, the criteria for Lyapunov pairs are checked only in the interior of the domain (or the relative interior) instead of the closure of the whole domain. In contrast to [1], this setting also ensures obtaining global Lyapunov pairs and allows us to control the whole trajectory of the solution to the given differential inclusion. This additional interiority assumption provides more explicit criteria for nonsmooth Lyapunov pairs than the ones given in [2]. Indeed, on the one hand, the conditions we present here are given at the nominal point and do not involve limiting processes. On the other hand, the current analysis is flexible regarding the choice of the subdifferentials, which can be either the proximal, the Fréchet (regular), or the limiting subdifferentials. Moreover, a notable difference with the previous results of [2] is that it is not necessary to consider the horizon subgradient, a fact that leads us to sharper conditions for Lyapunov pairs.

The structure of the paper is as follows. In Sect. 2 we introduce the main tools and basic results used in the paper. In Sect. 3 we give new criteria for lower semicontinuous Lyapunov pairs, achieved in Theorems 3.1, 3.2, and 3.3. Section 4 is devoted to the finite-dimensional setting.

2 Notation and main tools

Throughout the paper, H is a (real) Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$. Given a nonempty set $S \subset H$ (or $S \subset H \times \mathbb{R}$), by $\operatorname{co} S$, $\operatorname{cone} S$, and $\operatorname{aff} S$, we denote the *convex hull*, the *conic hull*, and the *affine hull* of the set S, respectively. Moreover, $\operatorname{Int} S$ is the *interior* of S, and $\operatorname{cl} S$ and $\overline{S}$ are indistinctly used for the *closure* of S (with respect to the norm topology on H).

We note $\operatorname{ri} S$ the (topological) *relative interior* of S, i.e., the interior of S in the topology relative to $\operatorname{cl}(\operatorname{aff} S)$. For $x \in H$ (or $x \in H \times \mathbb{R}$) and $\rho \geq 0$, $B_\rho(x)$ is the open ball with center x and radius ρ, and $\overline{B}_\rho(x)$ is the closure of $B_\rho(x)$ (with $B := B_1(0)$).

Our notation is the standard one used in convex and variational analysis and in monotone operator theory; see, e.g., [8,28]. The *indicator function* of $S \subset H$ is the function defined as

$$\mathrm{I}_S(x) := \begin{cases} 0 & \text{if } x \in S \\ +\infty & \text{otherwise.} \end{cases}$$

The *distance function* to S is denoted by

$$d(x, S) := \inf\{\|x - y\| \mid y \in S\},$$

and the *orthogonal projection* on S, π_S, is defined as

$$\pi_S(x) := \{y \in S \mid \|x - y\| = d(x, S)\}.$$

Given a function $\varphi : H \to \overline{\mathbb{R}}$, its *(effective) domain* and *epigraph* are defined by

$$\begin{aligned} \operatorname{Dom} \varphi &:= \{x \in H \mid \varphi(x) < +\infty\}, \\ \operatorname{epi} \varphi &:= \{(x, \alpha) \in H \times \mathbb{R} \mid \varphi(x) \leq \alpha\}. \end{aligned}$$

For $\lambda \in \mathbb{R}$, the *open upper level set* of φ at λ is

$$[\varphi > \lambda] := \{x \in H \mid \varphi(x) > \lambda\};$$

The sets $[\varphi \leq \lambda]$ and $[\varphi < \lambda]$ are defined similarly. We say that φ is proper if $\operatorname{Dom}\varphi \neq \emptyset$ and $\varphi(x) > -\infty$ for all $x \in H$. We say that φ is convex if $\operatorname{epi}\varphi$ is convex, and (weakly) lower semicontinuous if $\operatorname{epi}\varphi$ is closed with respect to the (weak topology) norm-topology on H. We denote

$$\mathscr{F}(H) := \{\varphi : H \to \overline{\mathbb{R}} \mid \varphi \text{ is proper and lsc}\},$$
$$\mathscr{F}_w(H) := \{\varphi : H \to \overline{\mathbb{R}} \mid \varphi \text{ is proper and weakly lsc}\};$$

$\mathscr{F}(H;\mathbb{R}_+)$ and $\mathscr{F}_w(H;\mathbb{R}_+)$ stand for the subsets of nonnegative functions of $\mathscr{F}(H)$ and $\mathscr{F}_w(H)$, respectively.

As maximally monotone set-valued operators play an important role in this work, it is useful to recall some of basic definitions and some of their properties. More generally, they have frequently shown themselves to be a key class of objects in both modern Optimization and Analysis; see, e.g., [4–6,8,28,30].

For an operator $A : H \rightrightarrows H$, the *domain* and the *graph* of A are given respectively by

$$\operatorname{Dom} A := \{z \in H \mid Az \neq \emptyset\} \quad \text{and} \quad \operatorname{gph} A := \{(x,y) \in H \times H \mid y \in Ax\};$$

for notational simplicity we identify the operator A to its graph. The *inverse operator* of A, denoted by A^{-1}, is defined as

$$(y,x) \in A^{-1} \Longleftrightarrow (x,y) \in A.$$

We say that an operator A is *monotone* if

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0 \quad \text{for all } (x_1,y_1),(x_2,y_2) \in A,$$

and *maximally monotone* if A is monotone and has no proper monotone extension (in the sense of graph inclusion). If A is maximally monotone, it is well-known (e.g., [30]) that $\overline{\operatorname{Dom} A}$ is convex, and Ax is convex and closed for every $x \in \operatorname{Dom} A$. Note that the domain or the range of a maximally monotone operator may fail to be convex, see, e.g., [28, page 555]. In particular, if A is the subdifferential $\partial\varphi$ of some lower semicontinuous convex and proper function $\varphi : H \to \overline{\mathbb{R}}$, then A is a classical example of a maximally monotone operator, as is a linear operator with a positive symmetric part. We know that

$$\operatorname{Dom} A \subset \operatorname{Dom}\varphi \subset \overline{\operatorname{Dom}\varphi} = \overline{\operatorname{Dom} A}.$$

For $x \in \operatorname{Dom} A$, we shall use the notation $(Ax)^\circ$ to denote the *principal section* of A, i.e., the set of points of minimal norm in Ax. For $\lambda > 0$, the resolvent and the Yoshida approximation of A are given, respectively, by

$$J_\lambda := (I + \lambda A)^{-1}, \quad A_\lambda := \frac{I - J_\lambda}{\lambda},$$

where I stands for the identity mapping on H.

We introduce next some basic concepts of nonsmooth and variational analysis (more details can be found for instance in [7,12,15,24,28]). We assume that $\varphi \in \mathscr{F}(H)$, and take $x \in \operatorname{Dom} \varphi$. We say that a vector $\xi \in H$ is a *proximal subgradient* of φ at x, and we write $\xi \in \partial_P \varphi(x)$, if there are $\rho > 0$ and $\sigma \geq 0$ such that

$$\varphi(y) \geq \varphi(x) + \langle \xi, y - x \rangle - \sigma \|y - x\|^2 \quad \text{for all } y \in B_\rho(x).$$

The set $\partial_P \varphi(x)$ is convex, possibly empty and not necessarily closed. The set $\partial_F \varphi(x)$ of *Fréchet (regular)* subgradient of φ at x is defined as the set of those $\xi \in H$ satisfying

$$\varphi(y) \geq \varphi(x) + \langle \xi, y - x \rangle + o(\|y - x\|).$$

Associated to proximal and Fréchet subdifferentials, limiting objects have been introduced:

- the *limiting Mordukhovich subdifferential* $\partial_L \varphi(x)$ of φ at x, which is the set of those $\xi \in H$ such that there exist sequences $(x_k)_{k\in\mathbb{N}}$ and $(\xi_k)_{k\in\mathbb{N}}$ satisfying $x_k \underset{\varphi}{\to} x$ (that is, $x_k \to x$ and $\varphi(x_k) \to \varphi(x)$), $\xi_k \in \partial_P \varphi(x_k)$ and $\xi_k \to \xi$;
- the *horizon (singular) subdifferential* $\partial_\infty \varphi(x)$ of φ at x, which is the set of those $\xi \in H$ such that there exist sequences $(\alpha_k)_{k\in\mathbb{N}} \subset \mathbb{R}_+$, $(x_k)_{k\in\mathbb{N}}$ and $(\xi_k)_{k\in\mathbb{N}}$ satisfying $\alpha_k \to 0^+$, $x_k \underset{\varphi}{\to} x$, $\xi_k \in \partial_P \varphi(x_k)$ and $\alpha_k \xi_k \to \xi$.

Note that the use of strong convergence in the definition of the limiting and horizontal subdifferentials above is due to the current Hilbert setting (e.g. [24, Theorem 2.34]).

The *Clarke subdifferential* of φ at x denoted by $\partial_C \varphi(x)$ coincides with $\overline{\text{co}}^w\{\partial_L \varphi(x) + \partial_\infty \varphi(x)\}$ (see, e.g., [24] and [28]), where the superscript w refers to the weak topology in H. Then it follows from the definition that

$$\partial_P \varphi(x) \subset \partial_F \varphi(x) \subset \partial_L \varphi(x) \subset \partial_C \varphi(x). \tag{8}$$

In particular, if φ is convex, then

$$\partial_P \varphi(x) = \partial_C \varphi(x) = \partial \varphi(x),$$

where $\partial \varphi(x)$ is the usual subdifferential of convex analysis:

$$\partial \varphi(x) := \{\xi \in H \mid \varphi(y) - \varphi(x) \geq \langle \xi, y - x \rangle \quad \text{for all } y \in H\}.$$

If φ is Gâteaux-differentiable at $x \in \operatorname{Dom} \varphi$, then we have

$$\partial_P \varphi(x) \subset \{\varphi'_G(x)\} \subset \partial_C \varphi(x).$$

If φ is C^1, then

$$\partial_P \varphi(x) \subset \{\varphi'(x)\} = \partial_C \varphi(x) \quad \text{and} \quad \partial_\infty \varphi(x) = \{\theta\}.$$

If φ is C^2, then

$$\partial_P \varphi(x) = \partial_C \varphi(x) = \{\varphi'(x)\}.$$

From a geometrical point of view, if $S \subset H$ is closed and $x \in S$, the *proximal normal cone to S at x* is

$$\mathrm{N}_S^P(x) := \partial_P \mathrm{I}_S(x).$$

or, equivalently (e.g. [12]),

$$\mathrm{N}_S^P(x) = \begin{cases} \operatorname{cone}(\pi_S^{-1}(x) - x) & \text{if } \pi_S^{-1}(x) \neq \emptyset, \\ \{\theta\} & \text{if } \pi_S^{-1}(x) = \emptyset, \end{cases}$$

where $\pi_S^{-1}(x) := \{y \in H \backslash S \mid x \in \pi_S(y)\}$. Similarly, $\mathrm{N}_S^L(x) := \partial_L \mathrm{I}_S(x) (= \partial_\infty \mathrm{I}_S(x))$ is the *limiting normal cone to S at x*, and $\mathrm{N}_S^C(x) := \overline{\mathrm{co}}^w \{\mathrm{N}_S^L(x)\}$ is the *Clarke normal cone to S at x*. In that way, the above subdifferentials of $\varphi \in \mathscr{F}(H)$ satisfy

$$\partial_P \varphi(x) = \{\xi \in H \mid (\xi, -1) \in \mathrm{N}_{\operatorname{epi}\varphi}^P(x, \varphi(x))\},$$
$$\partial_\infty \varphi(x) \subset \{\xi \in H \mid (\xi, 0) \in \mathrm{N}_{\operatorname{epi}\varphi}^P(x, \varphi(x))\}.$$

Conversely, if $\xi \in H$ is such that $(\xi, 0) \in \mathrm{N}_{\operatorname{epi}\varphi}^P(x, \varphi(x))$, then (e.g. [24, Lemma 2.37]) there exist sequences $(\alpha_k)_{k\in\mathbb{N}} \subset \mathbb{R}_+$, $(x_k)_{k\in\mathbb{N}}$ and $(\xi_k)_{k\in\mathbb{N}}$ such that $\alpha_k \to 0^+$, $x_k \underset{\varphi}{\to} x$, $\xi_k \in \alpha_k \partial_F \varphi(x_k)$ and $\xi_k \to \xi$. Note that when φ is a proper extended-real valued-convex function, we have [28, Proposition 8.12]

$$\partial_\infty \varphi(x) \subset N_{\operatorname{Dom}\varphi}(x). \tag{9}$$

We use the notation $\mathrm{T}_S(x)$ to denote the *contingent cone to S at $x \in S$* (also called *Bouligand tangent cone*) defined by

$$\mathrm{T}_S(x) := \{\xi \in H \mid x + \tau_k \xi_k \in S \text{ for some } \xi_k \to \xi \text{ and } \tau_k \to 0^+\}.$$

The Dini directional derivative of the function $\varphi\, (\in \mathscr{F}(H))$ at $x \in \operatorname{Dom}\varphi$ in the direction $v \in H$ is given by

$$\varphi'(x, v) = \liminf_{t\to 0^+, w\to v} \frac{\varphi(x + tw) - \varphi(x)}{t}.$$

The relation $\operatorname{epi}\varphi'(x, \cdot) = \mathrm{T}_{\operatorname{epi}\varphi}(x, \varphi(x))$ is verified. When $\varphi := d(\cdot, S)$ is the distance function to a closed subset S of H, then we have

$$\partial_C \varphi(x) = \mathrm{N}_S^C(x) \cap \overline{B}, \quad \text{for all x} \in S,$$

while for $x \notin S$ with $\partial_P \varphi(x) \neq \emptyset$, we have that $\pi_S(x)$ is a singleton and (see, e.g., [15])

$$\partial_P \varphi(x) = \frac{x - \pi_S(x)}{d(x, S)}.$$

Hence,

$$\partial_L \varphi(x) = \left\{ w\text{-}\lim_k \frac{x_k - \pi_S(x_k)}{\varphi(x)};\ x_k \to x \right\},$$

where w- lim stands for the weak limit. More generally, we have

$$\mathrm{N}_S^P(x) = \mathbb{R}_+ \partial_P d_S(x) \quad \text{and} \quad \mathrm{N}_S^C(x) = \overline{\mathbb{R}_+ \partial_C d_S(x)}^w$$

(with the convention that $0.\emptyset = \{\theta\}$).

Finally, we recall that $\varphi \in \mathscr{F}(\mathbb{R})$ is nonincreasing if and only if $\xi \leq 0$ for every $\xi \in \partial_P \varphi(x)$ and $x \in \mathbb{R}$, (e.g., [15]). We shall use the following Lemma:

Lemma 2.1 *Given $t_2 > t_1 \geq 0, a \neq 0$, and $b \geq 0$, we assume that an absolutely continuous function $\psi : [t_1, t_2] \to \mathbb{R}_+$ satisfies*

$$\psi'(t) \leq a\psi(t) + b \quad \textit{a.e. } t \in [t_1, t_2].$$

Then, for all $t \in [t_1, t_2]$,

$$\psi(t) \leq \left(\psi(t_1) + \frac{b}{a} \right) e^{a(t-t_1)} - \frac{b}{a}.$$

Proof Just apply Gronwall's Lemma to the function $\theta := \psi + \frac{b}{a}$. □

3 Characterization of Lyapunov pairs

In this section we provide the desired explicit criterion for lower semicontinuous (weighted-) Lyapunov pairs associated to the differential inclusion (6):

$$\dot{x}(t; x_0) \in f(x(\cdot; x_0)) - Ax(\cdot; x_0),\ \ x_0 \in \mathrm{cl}\,(\mathrm{Dom}\, A)\,,$$

where $A : H \rightrightarrows H$ is a maximally monotone operator and $f : \mathrm{cl}\,(\mathrm{Dom}\, A) \subset H \to H$ is a Lipschitz continuous mapping. Recall that for a fixed real $T > 0$ and for a given $x_0 \in \mathrm{cl}\,(\mathrm{Dom}\, A)$, we call *strong solution* of (6), the unique absolute continuous function $x(\cdot; x_0) : [0, T] \to H$, which satisfies $x(0; x_0) = x_0$ together with (see, e.g., [8])

$$\dot{x}(t; x_0) \in L^\infty_{\mathrm{loc}}((0, T], H), \tag{10}$$

$$x(t; x_0) \in \mathrm{Dom}\, A, \quad \text{for all } t > 0, \tag{11}$$

$$\dot{x}(t; x_0) \in f(x(t; x_0)) - Ax(t; x_0), \quad \text{a.e. } t \geq 0. \tag{12}$$

Existence of strong solutions is known to occur if, for instance, $x_0 \in \mathrm{Dom}\, A$, Int (co (Dom A)) $\neq \emptyset$, dim $H < \infty$, or if $A \equiv \partial\varphi$ where $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ is a lsc extended-real-valued convex proper function. Moreover, we have $\dot{x}(\cdot; x_0) \in L^\infty([0, T], H)$ if and only if $x_0 \in \mathrm{Dom}\, A$. In this later case, $x(\cdot; x_0)$ is right-differentiable at each $s \in [0, T)$ and

$$\frac{d^+ x(\cdot; x_0)}{t}(s) = f(x(s; x_0)) - \pi_{Ax(s;x_0)}(f(x(s; x_0))).$$

The strong solution also satisfies the so-called semi-group property,

$$x(s; x(t; x_0)) = x(s + t; x_0) \quad \text{for all } s, t \geq 0, \tag{13}$$

together with the relationship

$$\|x(t; x_0) - x(t; y_0)\| \leq e^{L_f t} \|x_0 - y_0\| \tag{14}$$

whenever $t \geq 0$ and $x_0, y_0 \in \mathrm{cl}(\mathrm{Dom}\, A)$; hereafter, L_f denotes the Lipschitz constant of the mapping f on cl(Dom A).

In the general case, it is well established that (6) admits a unique *weak solution* $x(\cdot; x_0) \in C(0, T; H)$ which satisfies $x(t; x_0) \in \mathrm{cl}(\mathrm{Dom}\, A)$ for all $t \geq 0$. More precisely, there exists a sequence $(z_k)_{k\in\mathbb{N}} \subset \mathrm{Dom}\, A$ converging to x_0 such that the strong solution $x_k(\cdot; z_k)$ of the equation

$$\dot{x}_k(t; z_k) \in f(x(t; z_k)) - Ax_k(t; z_k), \;\; x_k(0, z_k) = z_k, \tag{15}$$

converges uniformly to $x(\cdot; x_0)$ on $[0, T]$. Moreover, we know that (13) and (14) also hold in this case on cl(Dom A) .

The following condition on the interior of the operator A will play a crucial role in our analysis,

$$\mathrm{Int}\, (\mathrm{co}\, (\mathrm{Dom}\, A)) \neq \emptyset. \tag{16}$$

Applying well known results from the theory of maximally monotone operators, the last assumption implies that Int (Dom A) is convex, Int (Dom A) $=$ Int (co (Dom A)) $=$ Int (cl (Dom A)), and A is locally bounded on Int (Dom A). Therefore, a (unique) strong solution of (6) always exists (see e.g. [8]).

The following technical lemma, adds more information on the qualitative behavior of this solution.

Lemma 3.1 *Let* $\bar{y} \in \mathrm{Dom}\, A$ *and* $\bar{\rho} > 0$ *be such that* $B_{\bar{\rho}}(\bar{y}) \subset \mathrm{Int}\, (\mathrm{co}\, (\mathrm{Dom}\, A))$. *Then, there exists* $\rho \in (0, \bar{\rho})$ *such that*

$$M := \sup_{z\in B_\rho(\bar{y})} \left\| (f(z) - Az)^\circ \right\| < \infty$$

and, for all $y \in B_\rho(\bar{y})$ *and* $t \geq 0$,

$$\left\| \frac{d^+ x(\cdot; y)}{dt}(t) \right\| \leq e^{L_f t} M.$$

Proof By virtue of the semi-group property (13), the following inequality holds for all $y \in \mathrm{cl}(\mathrm{Dom}\, A)$ and $0 \leq t < s$ (e.g., [8, Lemma 1.1]):

$$\|x(t+s; y) - x(t; y)\| = \|x(t; x(s; y)) - x(t; y)\| \leq e^{L_f t} \|x(s; y) - y\|. \quad (17)$$

Hence, taking limits as s goes to 0,

$$\begin{aligned} \left\| \frac{d^+ x(\cdot; y)}{dt}(t) \right\| = \lim_{s \downarrow 0} s^{-1} \|x(t+s; y) - x(t; y)\| &\leq e^{L_f t} \lim_{s \downarrow 0} s^{-1} \|x(s; y) - y\| \\ &= e^{L_f t} \left\| \frac{d^+ x(\cdot; y)}{dt}(0) \right\| \\ &= e^{L_f t} \left\| (f(y) - Ay)^\circ \right\|. \end{aligned}$$

Finally, by the monotonicity of A and the Lipschitz continuity of f, we can choose $\rho \in (0, \bar{\rho})$ such that

$$\left\| (f(y) - Ay)^\circ \right\| \leq M$$

for some constant $M \geq 0$. This concludes the proof of the lemma. □

Definition 3.1 Given functions $V \in \mathscr{F}(H)$, $W \in \mathscr{F}(H; \mathbb{R}_+)$, and a number $a \in \mathbb{R}_+$, we say that (V, W) forms an a-Lyapunov pair for (6) if for all $y \in \mathrm{cl}(\mathrm{Dom}\, A)$ we have

$$e^{at} V(x(t; y)) + \int_0^t W(x(\tau; y)) d\tau \leq V(y) \quad \text{for all } t \geq 0. \quad (18)$$

We note that if $a = 0$ and $W = 0$, then we recover the classical concept of Lyapunov functions.

The lack of regularity properties of a-Lyapunov pairs (V, W) in Definition 3.1 is mainly due to the non-smoothness of the function V. Let us remind that inequality (18) also holds if instead of W one considers its Moreau-Yosida regularization, which is Lipschitz continuous on every bounded subset of H. This follows from the following lemma (e.g [2]).

Lemma 3.2 *For every* $W \in \mathscr{F}(H; \mathbb{R}_+)$, *there exists a sequence of functions* $(W_k)_{k \in \mathbb{N}} \subset \mathscr{F}(H, \mathbb{R}_+)$ *converging to* W *(for instance,* $W_k \uparrow W$*) such that each* W_k *is Lipschitz continuous on every bounded subset of* H, *and satisfies* $W(y) > 0$ *if and only if* $W_k(y) > 0$.

The following proposition shows that, generally, Lyapunov pairs have to be checked only on the domain of the involved maximally monotone operator.

Proposition 3.1 *Let $V \in \mathscr{F}(H)$ and $W \in \mathscr{F}(H;\mathbb{R}_+)$ be given. Suppose that V satisfies*

$$\liminf_{\mathrm{Dom}\, A \ni z \to y} V(z) = V(y) \quad \text{for all } y \in \mathrm{cl}(\mathrm{Dom}\, A) \cap \mathrm{Dom}\, V. \tag{19}$$

Then, given $a \in \mathbb{R}_+$, the following statements are equivalent:

(i) (V, W) *forms an a-Lyapunov pair with respect to* Dom A*; that is, (7) holds on* Dom A*;*
(ii) (V, W) *forms an a-Lyapunov pair with respect to* cl(Dom A)*; that is, (7) holds on* cl(Dom A)*;*

Property (19) has been already used in [20], and implicitly in [26,27], among other works. It holds, if for instance, V ($\in \mathscr{F}(H)$) is convex and its effective domain has a nonempty interior such that $\mathrm{Int}(\mathrm{Dom}\, V) \subset \mathrm{Dom}\, A$.

Our starting point is the next result which characterizes a- Lyapunov pairs locally in $\mathrm{Int}(\mathrm{Dom}\, A)$. This is a specification of the analysis of [2, Theorems 3.3 and 3.4] to the current setting where $\mathrm{Int}\,(\mathrm{co}(\mathrm{Dom}\, A)) \neq \emptyset$. Here we give complete and general criteria by means of either the proximal, the Fréchet, or the limiting subdifferentials; this last one coincides with the viscosity subdifferential (see Borwein [7]). Moreover, there is no need for the horizontal subgradient.

Theorem 3.1 *Assume that* $\mathrm{Int}\,(\mathrm{co}(\mathrm{Dom}\, A)) \neq \emptyset$. *Let $V \in \mathscr{F}_w(H)$, $W \in \mathscr{F}(H;\mathbb{R}_+)$, and $a \in \mathbb{R}_+$ be given. Let $\bar{y} \in H$, $\bar{\lambda} \in [-\infty, V(\bar{y}))$, and $\bar{\rho} \in (0, +\infty]$ be such that*

$$\mathrm{Dom}\, V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}] \subset \mathrm{Int}(\mathrm{Dom}\, A).$$

Then, the following statements are equivalent provided that ∂ is either the proximal, the Fréchet, or the limiting subdifferentials:

(i) $\forall y \in \mathrm{Dom}\, V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$

$$\sup_{\xi \in \partial V(y)} \min_{v \in Ay} \langle \xi, f(y) - v \rangle + aV(y) + W(y) \leq 0;$$

(ii) $\forall y \in \mathrm{Dom}\, V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$

$$\sup_{\xi \in \partial V(y)} \left\langle \xi, f(y) - \pi_{Ay}(f(y)) \right\rangle + aV(y) + W(y) \leq 0;$$

(iii) $\forall y \in B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$ *we have*

$$e^{at} V(x(t;y)) + \int_0^t W(x(\tau;y))d\tau \leq V(y) \ \forall t \in [0, \rho(y)]\,,$$

where

$$\rho(y) := \sup\left\{ \nu > 0 \;\middle|\; \begin{array}{l} \exists \rho > 0 \; s.t. \; B_\rho(y) \subset B_{\bar\rho}(\bar y) \cap [V > \bar\lambda], \text{ and for all } t \in [0, \nu] \\ 2\,\|x(t; y) - y\| < \frac{\rho}{2} \text{ and} \\ \left| (e^{-at} - 1)V(y) - \int_0^t W(x(\tau; y))d\tau \right| < \frac{\rho}{2} \end{array} \right\}. \tag{20}$$

Remark 3.1 The constant $\rho(y)$ defined in (20) is positive whenever $y \in \mathrm{cl}(\mathrm{Dom}\, A) \cap B_{\bar\rho}(\bar y) \cap [V > \bar\lambda]$.

Proof of Theorem 3.1 For simplicity, we suppose that $W \equiv 0$ and $a = 0$ (the general case follows similarly by noting that the function W may be taken Lipschitz on bounded sets, according to Lemma 3.2).

(iii) $\Longrightarrow$ (ii). According to the sequence of inclusions (8), it is enough to give the proof for the limiting subdifferential. We will proceed into two steps.

First step Let us give the proof for the Fréchet subdifferential. Let us fix $y \in B_{\bar\rho}(\bar y) \cap [V > \bar\lambda]$ and, first, take ξ in $\partial_F V(y)$ so that $y \in B_{\bar\rho}(\bar y) \cap [V > \bar\lambda] \cap \mathrm{Dom}\, V \subset \mathrm{Dom}\, A$ and there exists $T \in (0, \rho(y))$ such that

$$\begin{aligned} \langle \xi, x(t; y) - y \rangle &\le V(x(t; y)) - V(y) + \alpha \|x(t; y) - y\|^2 \\ &\le o(\|x(t; y) - y\|) \quad \text{for all } t \in [0, T), \end{aligned}$$

where $o(.)$ is a function satisfying $o(s) \to 0$ as $s \to 0$. But $y \in \mathrm{Dom}\, A$ and so there exists a constant $l \ge 0$ such that (taking a smaller T if necessary)

$$\langle \xi, t^{-1}(x(t; y) - y) \rangle \le l\, \|x(t; y) - y\| \quad \text{for all } t \in [0, T);$$

hence, taking the limit as $t \to 0^+$ we obtain that

$$\langle \xi, f(y) - \pi_{Ay}(f(y)) \rangle \le 0;$$

that is, (ii) follows in the case when $\xi \in \partial_F V(y)$.

Second step Let us give the proof for the limiting subdifferential. Now, we suppose that $\xi \in \partial_L V(y)$ and let the sequences $(y_k)_{k\in\mathbb{N}}$ and $(\xi_k)_{k\in\mathbb{N}}$ be such that $y_k \underset{V}{\to} y$, $\xi_k \in \partial_F V(y_k)$ and $\xi_k \to \xi$. Then, from the paragraph above, for each k we find $v_k \in Ay_k$ such that

$$\langle \xi_k, f(y_k) - v_k \rangle \le 0.$$

But since $y \in \mathrm{Int}\,(\mathrm{Dom}\, A)$ and $y_k \to y$, the maximal monotonicity of A allows us to suppose without loss of generality that the sequence (v_k) is bounded and, so, weakly convergent, up to a subsequence denoted in the same way, to some $v \in Ay$. Thus, because ξ_k strongly converges to ξ, by taking the limits as n goes to ∞ in the last inequality above we obtain that

$$\inf_{v \in Ay} \langle \xi, f(y) - v \rangle \le 0 \le \langle \xi, f(y) - v \rangle \le 0.$$

$(i) \implies (iii)$.

Again by virtue of (8), it is sufficient to give the proof for the Fréchet subdifferential. We fix $y \in \operatorname{Dom} V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$ and let $\rho > 0$ and $\nu > 0$ be such that

$$B_{\rho}(y) \subset B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}] \tag{21}$$

and

$$\sup_{t \in [0,\nu]} 2\,\|x(t; y) - y\| < \rho; \tag{22}$$

the existence of such scalars ρ and ν is a consequence of the lower semicontinuity of V and the Lipschitz continuity of $x(\cdot;\cdot)$ (see Lemma 3.1). Moreover, due to the maximal monotonicity of A, we may assume that A is bounded on $B_{\rho}(y)$. Let $T < \nu$ be fixed and define the functions $z(\cdot) : [0, T] \subset \mathbb{R}_+ \to H \times \mathbb{R}$ and $\eta(\cdot) : [0, T] \subset \mathbb{R}_+ \to \mathbb{R}_+$ as

$$z(t) := (x(t; y), V(y)),\ \eta(t) := \frac{1}{2} d^2(z(t), \operatorname{epi} V); \tag{23}$$

observe that $z(\cdot)$ and $\eta(\cdot)$ are Lipschitz continuous on $[0, T)$. Now, using a standard chain rule (e.g. [12]), for a fixed $t \in (0, T)$ it holds that

$$\partial_C \eta(t) = d(z(t), \operatorname{epi} V)\partial_C d(z(\cdot), \operatorname{epi} V)(t).$$

Hence, whenever $z(t) \in \operatorname{epi} V$ we get $\partial_C \eta(t) = \{\theta\}$. Otherwise, if $z(t) \notin \operatorname{epi} V$, then

$$\partial_C d(z(\cdot), \operatorname{epi} V)(t) \subset \overline{\operatorname{co}} \left[\bigcup_{(u,\mu) \in \Pi_{\operatorname{epi} V}(z(t)),\, u \in B_{\rho}(y)} \frac{\langle x(t; y) - u, -Ax(t; y)\rangle}{d(z(t), \operatorname{epi} V)} \right]$$

and, consequently,

$$\partial_C \eta(t) \subset \overline{\operatorname{co}} \left[\bigcup_{(u,\mu) \in \Pi_{\operatorname{epi} V}(z(t)),\, u \in B_{\rho}(y)} \langle x(t; y) - u, -Ax(t; y)\rangle \right]. \tag{24}$$

Note that the condition $u \in B_{\rho}(y)$ for $(u, \mu) \in \Pi_{\operatorname{epi} V}(z(t))$ in this last formula is a consequence of the following inequalities:

$$\begin{aligned} \|u - y\| &\leq \|x(t; y) - u\| + \|x(t; y) - y\| \\ &\leq \|(x(t; y), V(y)) - (u, \mu)\| + \|x(t; y) - y\| \\ &\leq \|(x(t; y), V(y)) - (y, V(y))\| + \|x(t; y) - y\| \\ &\leq 2\,\|x(t; y) - y\| < \rho \end{aligned}$$

(recall (22)). Take now $\xi \in Ax(t; y)$ and $(u, \mu) \in \Pi_{\operatorname{epi} V}(z(t))$ with $u \in B_{\rho}(y)$. Thus,

$$(x(t; y) - u, V(y) - \mu) \in N^P_{\operatorname{epi} V}(u, \mu),$$

and therefore $V(y)-\mu \leq 0$ and $V(u)-\mu \leq 0$. Hence, $u \in \operatorname{Dom} V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$ (recall (21)).

If $V(y) - \mu < 0$, we write $(\mu - V(y))^{-1}\,(x(t;y) - u) \in \partial V_P(u)$ and so, by the current assumption (i), we select $\upsilon \in Au$ such that

$$\left\langle (\mu - V(y))^{-1}(x(t;y) - u), -\upsilon \right\rangle \leq 0.$$

Therefore, invoking the monotonicty of A we get

$$\langle x(t;y) - u, -\xi\rangle = \langle x(t;y) - u, -\upsilon\rangle + \langle x(t;y) - u, \upsilon - \xi\rangle \leq \langle x(t;y) - u, -\upsilon\rangle \leq 0.$$

Consequently, since $\xi \in Ax(t;y)$ is arbitrary, (24) leads us to $\partial_C \eta(t) \subset \mathbb{R}_-$.

It remains to investigate the other case corresponding to $V(y) - \mu = 0$; that is, $(x(t;y) - u, 0) \in N^P_{\operatorname{epi} V}(u, V(u))$. Let us first observe that $x(t;y) - u \neq \theta$. Next, we choose an $\varepsilon > 0$ such that $B_\varepsilon(u) \subset B_\rho(y) \cap \operatorname{Int}(\operatorname{Dom} A)$ (recall that $u \in B_\rho(y) \cap \operatorname{Dom} V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}] \subset \operatorname{Int}(\operatorname{Dom} A)$) and, according to [24, Lemma 2.37 and formulas in Page 240], take $u_\varepsilon \in B_\varepsilon(u) \cap \operatorname{Dom} V$ with $|V(u) - V(u_\varepsilon)| \leq \varepsilon$, $\alpha \in (0, \varepsilon)$ and $\xi_\varepsilon \in B_\varepsilon(x(t;y) - u)$ such that $\alpha^{-1}\xi_\varepsilon \in \partial V_P(u_\varepsilon)$. Therefore, using the current assumption, select $\upsilon_\varepsilon \in Au_\varepsilon$ such that $\langle \xi_\varepsilon, -\upsilon_\varepsilon\rangle \leq 0$. Hence,

$$\langle x(t;y) - u, -\upsilon_\varepsilon\rangle \leq \varepsilon \|\upsilon_\varepsilon\| + \langle \xi_\varepsilon, -\upsilon_\varepsilon\rangle \leq \varepsilon \|\upsilon_\varepsilon\|$$

and, consequently by the monotonicity of A,

$$\begin{aligned}
\langle x(t;y) - u, -\xi\rangle &\leq \langle x(t;y) - u_\varepsilon, -\xi\rangle + \varepsilon \|\xi\| \\
&\leq \langle x(t;y) - u_\varepsilon, -\upsilon_\varepsilon\rangle + \varepsilon \|\xi\| \\
&\leq \langle x(t;y) - u, -\upsilon_\varepsilon\rangle + \|u_\varepsilon - u\| \, \|\upsilon_\varepsilon\| + \varepsilon \|\xi\| \\
&\leq 2\varepsilon \|\upsilon_\varepsilon\| + \varepsilon \|\xi\| .
\end{aligned}$$

Moreover, as $(\upsilon_\varepsilon)_{\varepsilon \leq 1} \subset B_\rho(y)$ and A is bounded on $B_\rho(y)$, by passing to the limit as ε goes to 0 we get

$$\langle x(t;y) - u, -\xi\rangle \leq 0.$$

This gives the desired inclusion $\partial_C \eta(t) \subset \mathbb{R}_-$ (recall (24)) and so establishes the proof of (iii). □

The next theorem adds more information on the uniform-like behavior of Lyapunov pairs for (6).

Theorem 3.2 *Under the condition* $\operatorname{Int}(\operatorname{co}\{\operatorname{Dom} A\}) \neq \emptyset$, *let us consider functions* $V \in \mathscr{F}_w(H)$ *and* $W \in \mathscr{F}(H; \mathbb{R}_+)$, *and a nonnegative number a. Fix* $\bar{y} \in \operatorname{Dom} V$, $\bar{\lambda} \in (-\infty, V(\bar{y}))$ *and let* $\bar{\rho} > 0$ *be such that*

$$\operatorname{Dom} V \cap [V > \bar{\lambda}] \cap B_{\bar{\rho}}(\bar{y}) \subset \operatorname{Int}(\operatorname{Dom} A).$$

Assume that for all $y \in \mathrm{Dom}\, V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$ *we have*

$$\sup_{\xi \in \partial_P V(y)} \min_{\upsilon \in Ay} \langle \xi, f(y) - \upsilon \rangle + aV(y) + W(y) \leq 0. \tag{25}$$

Then, for every $y \in \mathrm{Dom}\, V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$ *there exist* $\rho > 0$ *and* $T > 0$ *such that*

$$e^{at} V(x(t;z)) + \int_0^t W(x(\tau;z))d\tau \leq V(z), \quad \textit{for all } z \in B_\rho(y) \textit{ and } t \leq T.$$

Consequently, under condition (19), the pair (V, W) *is an a-Lyapunov pair for (6) provided that* Dom A *is open and (25) holds on* Dom A.

Proof We shall suppose that $W = 0$ and $a = 0$ (the general case is similar). For this aim we pick $\hat{y}$ in $\mathrm{Dom}\, V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$ ($\subset \mathrm{Int}(\mathrm{Dom}\, A)$) and we choose $\rho > 0$ such that A is bounded on $B_{2\rho}(\hat{y})$. Taking into account the lower semicontinuity of V, we have

$$B_{2\rho}(\hat{y}) \subset B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}] \cap \mathrm{Int}(\mathrm{Dom}\, A), \quad V(z) \geq V(\hat{y}) - 1 \; \forall z \in B_{2\rho}(\hat{y}); \tag{26}$$

moreover, by virtue of Lemma 3.1, we may assume that there exists a positive constant M satisfying, for all $t \geq 0$ and all $z \in B_{2\rho}(\hat{y})$,

$$\left\| \frac{d^+ x(\cdot; z)}{dt}(t) \right\| \leq e^{L_f t} M. \tag{27}$$

Hence, $\|x(t;z) - z\| \leq Mte^{L_f t}$ and therefore by (26),

$$V(x(t;z)) \geq V(\hat{y}) - 1 \geq \bar{\lambda} - 1 \tag{28}$$

for all $z \in B_\rho(\hat{y})$ and all $t \geq 0$ such that $te^{L_f t} \leq \frac{\rho}{M}$. Now writing, for all $z \in B_{2\rho}(\hat{y}) \cap \mathrm{Dom}\, V$ and $0 \leq t \leq 1$,

$$2\,\|x(t;z) - z\| \leq 2Me^{L_f} t,$$

there exists $T > 0$ such that for all $z \in B_{2\rho}(\hat{y})$ we have

$$\sup_{t \in [0,T]} 2\,\|x(t;z) - z\| < \frac{\rho}{2}.$$

Thus, since for any given $z \in B_\rho(\hat{y})$ we have $B_\rho(z) \subset B_{2\rho}(\hat{y}) \cap [V > \bar{\lambda}] \subset B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$ the main conclusion of the theorem follows from Theorem 3.1.

Finally, we assume that Dom A is open and (25) holds on Dom A. For fixed $y \in \mathrm{Int}\,\mathrm{Dom}\, A$ we introduce the nonempty set $E \subset \mathbb{R}_+$ given as

$$E := \{\lambda \in \mathbb{R}_+ \mid \; V(x(t;y)) \leq V(y) \;\; \forall t \leq \lambda\}.$$

Then, from the first part of the proof, by taking into account the continuity of $x(\cdot; y)$ and the openness of Dom A it follows that E is open and closed so that $E = \mathbb{R}_+$. Then, the fact that the pair (V, W) is an a-Lyapunov pair for (6) follows from Proposition 3.1. □

Corollary 3.1 *Under the assumption of Theorem 3.2 we also suppose that* Dom $V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$ *is compact. Then, there exists* $T > 0$ *such that*

$$e^{at} V(x(t; y)) + \int_0^t W(x(\tau; y))d\tau \leq V(y),$$

for every $y \in$ Dom $V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$ *and* $t \in [0, T]$.

Proof According to Theorem 3.2, for every $y \in$ Dom $V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$ there exist $T_y > 0$ and $\rho_y > 0$ such that

$$e^{at} V(x(t; z)) + \int_0^t W(x(\tau; z))d\tau \leq V(z), \quad \text{for all } z \in B_{\rho_y}(y) \text{ and } t \leq T_y.$$

Then, since Dom $V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$ is compact, we can find $y_1, \ldots, y_k \in$ Dom $V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$, $T_1, \ldots, T_k > 0$ and $\rho_1, \ldots, \rho_k > 0$ such that

$$\text{Dom } V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}] \subset \bigcup_{i=1,\ldots,k} B_{\rho_i}(y_i)$$

and, for each $i = 1, \ldots, k$,

$$e^{at} V(x(t; z)) + \int_0^t W(x(\tau; z))d\tau \leq V(z), \quad \text{for all } z \in B_{\rho_i}(y_i) \text{ and } t \leq T_{y_i}.$$

Consequently, the conclusion follows by taking $T = \min_{i=1,\ldots,k} T_{y_i}$. □

In the following theorem we do not assume that Int (co{Dom A}) $\neq \emptyset$.

Theorem 3.3 *We consider functions* $V \in \mathscr{F}_w(H)$ *and* $W \in \mathscr{F}(H; \mathbb{R}_+)$, *and a non-negative number* a. *Assume the existence of* $\lambda_0 > 0$ *such that for all* $y \in$ Dom V *it holds*

$$\sup_{\xi \in \partial_P V(y)} \langle \xi, f(y) - A_\lambda y \rangle + aV(y) + W(y) \leq 0 \quad \textit{for all } \lambda \leq \lambda_0.$$

Then, for every $z \in$ cl(Dom A) *we have that*

$$e^{at} V(x(t; z)) + \int_0^t W(x(\tau; z))d\tau \leq V(z), \quad \textit{for all } t \geq 0;$$

that is, (V, W) *is an* a*-Lyapunov pair for (6).*

Proof Since $\text{Dom}\, A_\lambda = H$, by Theorem 3.2 for every $\lambda \leq \lambda_0$ we have that

$$e^{at} V(x_\lambda(t; z)) + \int_0^t W(x_\lambda(\tau; z))d\tau \leq V(z), \quad \text{for all } t \geq 0 \text{ and all } z \in H,$$

where $x_\lambda(.; z)$ is the (strong) solution of the differential equation

$$\dot{x}_\lambda(t; z) = -A_\lambda x_\lambda(t; z) + f(x_\lambda(t; z)); \quad x_\lambda(0; z) = z.$$

Hence, by taking the limits as λ goes to 0 in the last inequality above, the conclusion follows due to the uniform convergence of $x_\lambda(.; z)$ to $x(.; z)$. □

4 Characterizations of finite-dimensional nonsmooth Lyapunov pairs

This section is devoted to the finite-dimensional setting. Assuming that $\dim H < \infty$, we give multiple primal and dual characterizations for nonsmooth a-Lyapunov pairs for the differential inclusion (6).

Theorem 4.1 *Assume that* $\dim H < \infty$. *Let* $V \in \mathscr{F}(H)$, $W \in \mathscr{F}(H; \mathbb{R}_+)$, *and* $a \in \mathbb{R}_+$ *be given, and let* ∂ *be either the proximal, the Fréchet, or the limiting subdifferentials. Fix* $\bar{y} \in \text{rint}(\text{cl}(\text{Dom}\, A))$ *and let* $\rho > 0$ *be such that* $B_{2\rho}(\bar{y}) \cap \text{aff}(\text{cl}(\text{Dom}\, A)) \subset \text{Dom}\, A$. *Then, the following assertions* (i)–(v) *are equivalent*:

(i) *there exists* $T > 0$ *such that for every* $y \in \text{Dom}\, A \cap \text{Dom}\, V \cap B_\rho(\bar{y})$

$$e^{at} V(x(t; y)) + \int_0^t W(x(\tau; y))d\tau \leq V(y) \quad \textit{for all } t \leq T;$$

(ii) *for every* $y \in \text{Dom}\, A \cap \text{Dom}\, V \cap B_\rho(\bar{y})$

$$\sup_{\xi \in \partial_P V(y)} \langle \xi, f(y) - \pi_{Ay}(f(y)) \rangle + aV(y) + W(y) \leq 0;$$

(iii) *for every* $y \in \text{Dom}\, A \cap \text{Dom}\, V \cap B_\rho(\bar{y})$

$$\sup_{\xi \in \partial V(y)} \inf_{\upsilon \in Ay} \langle \xi, f(y) - y^* \rangle + aV(y) + W(y) \leq 0;$$

(iv) *for every* $y \in \text{Dom}\, A \cap \text{Dom}\, V \cap B_\rho(\bar{y})$

$$V'(y; f(y) - \pi_{Ay}(f(y))) + aV(y) + W(y) \leq 0;$$

(v) *for every* $y \in \text{Dom}\, A \cap \text{Dom}\, V \cap B_\rho(\bar{y})$

$$\inf_{\upsilon \in Ay} V'(y; f(y) - \upsilon) + aV(y) + W(y) \leq 0.$$

If V is nonnegative, each one of the statements above is equivalent to

(vi) *for every* $y \in \mathrm{Dom}\, A \cap \mathrm{Dom}\, V \cap B_\rho(\bar{y})$

$$V(x(t;y)) + a\int_0^t V(x(\tau;y))d\tau + \int_0^t W(x(\tau;y))d\tau \leq V(y) \quad \textit{for all } t \geq 0.$$

Proof (iii with$\partial \equiv \partial_P$) $\Longrightarrow$ (i): We may assume that $0 \in \mathrm{Dom}\, A$ and denote $H_0 := \mathrm{lin}(\mathrm{cl}(\mathrm{Dom}\, A))$. Let $A_0 : H_0 \rightrightarrows H_0$ be the operator given by

$$A_0 y = Ay \cap H_0, \tag{29}$$

and define the Lipschitz continuous mapping $f_0 : H_0 \to H_0$ as

$$f_0(y) = \pi_{H_0}(f(y)), \tag{30}$$

where π_{H_0} denotes the orthogonal projection onto H_0. According to the Minty Theorem, it follows that A_0 is also a maximally monotone operator. Further, for every $y \in \mathrm{Dom}\, A$ we have $Ay + \mathrm{N}_{\mathrm{cl}(\mathrm{Dom}\, A)}(y) = Ay$, and therefore $Ay + H_0^{\perp} = Ay$. Hence,

$$Ay = (Ay \cap H_0) + H_0^{\perp} = A_0 y + H_0^{\perp}. \tag{31}$$

From this inequality we deduce that $\mathrm{Dom}\, A_0 = \mathrm{Dom}\, A$ and, so,

$$\mathrm{rint}(\mathrm{cl}(\mathrm{Dom}\, A)) = \mathrm{Int}(\mathrm{cl}(\mathrm{Dom}\, A_0)) = \mathrm{Int}(\mathrm{Dom}\, A_0);$$

(for the last equality see, e.g., [8, Remark 2.1- Page 33]). Further, since for $y \in \mathrm{cl}(\mathrm{Dom}\, A)$ we have

$$f_0(y) - A_0 y \subset f(y) - A_0 y + H_0^{\perp} = f(y) - Ay,$$

it follows that $x(\cdot; y)$ is the unique solution of the differential inclusion

$$\dot{x}(t;y) \in f_0(x(t;y)) - A_0 x(t;y),\ x(0,y) = y.$$

Next, we are going to show that condition (i) of Theorem 3.2 holds for the pair (A_0, f_0). Fix $y \in \mathrm{Dom}\, A \cap \mathrm{Dom}\, V \cap B_\rho(\bar{y})$ and $\xi \in \partial V(y)$ (if any). For a fixed $\varepsilon > 0$, by assumption take $\upsilon \in Ay$ in such a way that

$$\langle \xi, f(y) - \upsilon \rangle + aV(y) + W(y) \leq \varepsilon.$$

Since $f(y) \in f_0(y) + H_0^{\perp}$ and $\upsilon + H_0^{\perp} \in Ay + H_0^{\perp} = A_0 y$, we have

$$\inf_{\upsilon \in A_0 y} \langle \xi, f_0(y) - \upsilon \rangle \leq \inf_{\upsilon \in Ay} \langle \xi, f(y) - \upsilon \rangle \leq \varepsilon - aV(y) - W(y), \tag{32}$$

and condition (i) of Theorem 3.2 follows as $\varepsilon \to 0$. Consequently, by this Theorem 3.2, for every $y \in \mathrm{Dom}\, A \cap \mathrm{Dom}\, V \cap B_\rho(\bar{y})$, there exist $\hat{\rho} > 0$, small enough, and $T(y) > 0$ such that

$$e^{at}V(x(t;z)) + \int_0^t W(x(\tau;z))d\tau \le V(z) \quad \text{for all } t \le T(y) \text{ and } z \in B_{\hat{\rho}}(y).$$

Thus, as $\mathrm{Dom}\, A \cap \mathrm{Dom}\, V \cap B_\rho(\bar{y})$ is a precompact set, and $\mathrm{cl}(\mathrm{Dom}\, A \cap \mathrm{Dom}\, V \cap B_\rho(\bar{y})) \subset B_{2\rho}(\bar{y}) \cap \mathrm{aff}(\mathrm{cl}(\mathrm{Dom}\, A)) \subset \mathrm{rint}(\mathrm{cl}(\mathrm{Dom}\, A))$, instead of $T(y)$ we can choose a uniform T which gives us statement (i).

(i) $\Longrightarrow$ (iv): Fix $y \in \mathrm{Dom}\, A \cap \mathrm{Dom}\, V \cap B_\rho(\bar{y})$. Then, as shown in the paragraph above, the solution $x(t; y)$ of (6) is also the unique strong solution of the equation

$$\dot{x}(t;y) \in f_0(x(t;y)) - A_0 x(t;y),\ x(0;y) = y \in \mathrm{cl}\,(\mathrm{Dom}\, A)\,,$$

where A_0 and f_0 are defined in (29) and (30), respectively. Let $(t_n)_{n\in\mathbb{N}} \subset (0, T)$ be a sequence such that $t_n \to 0^+$ and set

$$w_n := \frac{x(t_n; y) - y}{t_n}.$$

Because $x(\cdot; y)$ is derivable from the right at 0 (recall that $y \in \mathrm{Dom}\, A$) and

$$\frac{d^+x(\cdot;y)}{dt}(0) = (f(y) - Ay)^\circ = f(y) - \pi_{Ay}(f(y)),$$

we get

$$w_n \to f(y) - \pi_{Ay}(f(y)).$$

Therefore, using the current assumption (i),

$$\begin{aligned} \frac{V(y + t_n w_n) - V(y)}{t_n} &= \frac{V(x(t_n, y)) - V(y)}{t_n} \\ &\le \frac{e^{-at_n}(1 - e^{at_n})}{t_n} V(y) - \frac{e^{-at_n}}{t_n} \int_0^{t_n} W(x(s;y))ds, \end{aligned}$$

and taking limits yields

$$\begin{aligned} V'(y; f(y) - \pi_{Ay}(f(y))) &\le \liminf_n \frac{e^{-at_n}(1 - e^{at_n})}{t_n} V(y) - \frac{e^{-at_n}}{t_n} \int_0^{t_n} W(x(s;y))ds \\ &= -aV(y) - W(y); \end{aligned}$$

this proves (iv).

(iv) $\Longrightarrow$ (v) is trivial.

(v) $\Longrightarrow$ [(iii) with $\partial \equiv \partial_L$]. Take $y \in \mathrm{Dom}\, A \cap \mathrm{Dom}\, V \cap B_\rho(\bar{y})$. For fixed $\varepsilon > 0$, by (v) we let $\upsilon \in Ay$ be such that

$$V'(y; f(y) - \upsilon) \le \varepsilon - aV(y) - W(y);$$

that is,

$$\begin{aligned}(f(y) - \upsilon, \varepsilon - aV(y) - W(y)) &\in \operatorname{epi} V'(y, \cdot) \\ &= \mathrm{T}_{\operatorname{epi} V}(y, V(y)) \subset \left[\mathrm{N}^{p}_{\operatorname{epi} V}(y, V(y))\right]^{\circ}.\end{aligned}$$

If $\xi \in \partial_P V(y)$, since $(\xi, -1) \in \mathrm{N}^{p}_{\operatorname{epi} V}(y, V(y))$, the above inequality leads to

$$\begin{aligned}\langle \xi, f(y) - \upsilon\rangle &\leq \langle (\xi, -1), (f(y) - \upsilon, \varepsilon - aV(y) - W(y))\rangle + \varepsilon - aV(y) - W(y) \\ &\leq \varepsilon - aV(y) - W(y),\end{aligned}$$

so that (ii) follows when $\varepsilon \to 0$.

If $\xi \in \partial_L V(y)$, then there are sequences (y_n) and (ξ_n) such that $y_n \to y, \xi_n \to \xi, V(\xi_n) \to V(\xi)$ and $\xi_n \in V(y_n)$ (for n sufficiently large). As just shown above, given $\varepsilon > 0$, for each n there exists $y_n^* \in Ay_n$ such that

$$\left\langle \xi_n, f(y_n) - y_n^*\right\rangle \leq \varepsilon - aV(y_n) - W(y_n).$$

Since $(y_n)_n$ converges to y, then we may suppose that $y_n^* \to \upsilon \in Ay$. Thus, passing to the limit in the above inequality, and taking into account the lower semicontinuity of V and the continuity of W, we obtain

$$\langle \xi, f(y) - \upsilon\rangle \leq \varepsilon - aV(y) - W(y).$$

This shows that (iii) holds with $\partial \equiv \partial_L$.

At this point we have proved that (i) $\Longleftrightarrow$ (iii with $\partial \equiv \partial_L$) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v). To see that (ii) is also equivalent to the other statements we observe that (ii) $\Longrightarrow$ (iii) holds obviously. On the other hand, the implication (iv) $\Longrightarrow$ (ii) follows in a similar way as in the proof of the statement (v) $\Longrightarrow$ (iii). This proves the equivalences of (i) through (v).

Finally, if V is nonnegative, (vi) is nothing else but (i) with a and W replaced by θ and $aV + W$, respectively. Thus, (vi) is equivalent to (iii). □

The following result is an immediate consequence of the previous theorem and Proposition 3.1.

Corollary 4.1 *Assume that* $\dim H < \infty$. *Let* $V \in \mathscr{F}(H)$, $W \in \mathscr{F}(H; \mathbb{R}_+)$, *and* $a \in \mathbb{R}_+$ *be given, and let* ∂ *be either the proximal, the Fréchet, or the limiting subdifferentials. Assume that* $\operatorname{rint}(\operatorname{cl}(\operatorname{Dom} A)) = \operatorname{Dom} A$. *Then, under condition (19),* (V, W) *forms an a-Lyapunov pair for (6) provided that one of the following assertions holds*:

(i) *for all* $y \in \operatorname{Dom} A \cap \operatorname{Dom} V$

$$\sup_{\xi \in \partial_P V(y)} \left\langle \xi, f(y) - \pi_{Ay}(f(y))\right\rangle + aV(y) + W(y) \leq 0;$$

(ii) *for all* $y \in \mathrm{Dom}\, A \cap \mathrm{Dom}\, V$

$$\sup_{\xi \in \partial V(y)} \inf_{\upsilon \in Ay} \left\langle \xi, f(y) - y^* \right\rangle + aV(y) + W(y) \leq 0;$$

(iii) *for all* $y \in \mathrm{Dom}\, A \cap \mathrm{Dom}\, V$

$$V'(y; f(y) - \pi_{Ay}(f(y))) + aV(y) + W(y) \leq 0;$$

(iv) *for all* $y \in \mathrm{Dom}\, A \cap \mathrm{Dom}\, V$

$$\inf_{\upsilon \in Ay} V'(y; f(y) - \upsilon) + aV(y) + W(y) \leq 0.$$

In contrast to the (analytic) Definition 3.1, Lyapunov stability can also be approached from a geometrical point of view using the concept of invariance:

Definition 4.1 A non-empty closed set $S \subset H$ is said to be invariant for (6) if for all $y \in S \cap \mathrm{cl}(\mathrm{Dom}\, A)$ one has that

$$x(t; y) \in S \quad \text{for all } t \geq 0.$$

Corollary 4.2 *Assume that* $\dim H < \infty$ *and* $\mathrm{rint}(\mathrm{cl}(\mathrm{Dom}\, A)) = \mathrm{Dom}\, A$. *Then, a nonempty closed set* $S \subset H$ *is invariant for (6) if and only if one of the following assertions is satisfied*:

(i) *for all* $y \in \mathrm{Dom}\, A \cap S$

$$\sup_{\xi \in \mathrm{N}^P_{S \cap \mathrm{cl}(\mathrm{Dom}\, A)}(y)} \left\langle \xi, f(y) - \pi_{Ay}(f(y)) \right\rangle \leq 0;$$

(ii) *for all* $y \in \mathrm{Dom}\, A \cap S$

$$\sup_{\xi \in \mathrm{N}^P_{S \cap \mathrm{cl}(\mathrm{Dom}\, A)}(y)} \inf_{\upsilon \in Ay} \langle \xi, f(y) - \upsilon \rangle \leq 0;$$

(iii) *for all* $y \in \mathrm{Dom}\, A \cap S$

$$f(y) - \pi_{Ay}(f(y)) \in T_{S \cap \mathrm{cl}(\mathrm{Dom}\, A)}(y);$$

(iv) *for all* $y \in \mathrm{Dom}\, A \cap S$

$$[f(y) - Ay] \cap T_{S \cap \mathrm{cl}(\mathrm{Dom}\, A)}(y) \neq \emptyset;$$

(v) *for all* $y \in \mathrm{Dom}\, A \cap S$

$$[f(y) - Ay] \cap \overline{\mathrm{co}}\left[T_{S \cap \mathrm{cl}(\mathrm{Dom}\, A)}(y)\right] \neq \emptyset.$$

Proof It is an immediate fact that S is invariant if and only if $\mathrm{I}_{S\cap \mathrm{cl}(\mathrm{Dom}\, A)}$ is a Lyapunov function. Then, the current assertions (i) and (ii) come from statements (i) and (ii) of Theorem 4.1, respectively. Similarly, S is invariant if and only $d(\cdot, S \cap \mathrm{cl}(\mathrm{Dom}\, A))$ is a Lyapunov function. Thus, by virtue of the relationship

$$T_{S\cap \mathrm{cl}(\mathrm{Dom}\, A)}(y) = \{w \in H \mid d'(\cdot, S \cap \mathrm{cl}(\mathrm{Dom}\, A)(w) = 0\},$$

the current assertions (iii) and (iv) follow from statements (iii) and (iv) of Theorem 4.1, respectively. This shows that (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv).

It remains to show that (v) is equivalent to the other statements. We obviously have that (iv) $\Longrightarrow$ (v) and so (i) $\Longrightarrow$ (v). To prove the reverse implication it suffices to show that (v) $\Longrightarrow$ (ii). Indeed, fix $y \in S \cap \mathrm{Dom}\, A$ and $\xi \in \mathrm{N}^P_{S\cap \mathrm{cl}(\mathrm{Dom}\, A)}$. Then, by (v) there exists $\upsilon \in Ay$ such that

$$f(y) - \upsilon \in \overline{\mathrm{co}}\left[T_{S\cap \mathrm{cl}(\mathrm{Dom}\, A)}(y)\right] \subset \left[\mathrm{N}^P_{S\cap \mathrm{cl}(\mathrm{Dom}\, A)}\right]^\circ.$$

Therefore, $\langle \xi, f(y) - \upsilon\rangle \leq 0$; that is (ii) follows. □

The characterization of Gâteaux differentiable Lyapunov functions is given in the following corollary.

Corollary 4.3 *Assume that* $\dim H < \infty$ *and* $\mathrm{rint}(\mathrm{cl}(\mathrm{Dom}\, A)) = \mathrm{Dom}\, A$. *Let* $V \in \mathscr{F}(H)$, $W \in \mathscr{F}(H, \mathbb{R}_+)$, *and* $a \in \mathbb{R}_+$ *be given. If* V *is Gâteaux differentiable, then the following statements are equivalent*:

(i) (V, W) *is an* a*-Lyapunov pair for (6);*

(iii) *for every* $y \in \mathrm{Dom}\, A \cap \mathrm{Dom}\, V$

$$V'_G(y)(f(y) - \pi_{Ay}(f(y))) + aV(y) + W(y) \leq 0;$$

(iv) *for all* $y \in \mathrm{Dom}\, A \cap \mathrm{Dom}\, V$

$$\inf_{\upsilon \in Ay} V'_G(y)(f(y) - \upsilon) + aV(y) + W(y) \leq 0.$$

Finally, we treat the simple case when $A \equiv 0$ so that our inclusion (6) becomes an ordinary differential equation which reads: for every $y \in H$ there exists a unique trajectory $x(\cdot; y) \in C^1(0, \infty; H)$ such that $x(0, y) = y$ and

$$\dot{x}(t; y) = f(x(t; y)) \quad \text{for all } t \geq 0. \tag{33}$$

In this case, Theorem 3.1 gives in a simplified form the characterization of the associated a-Lyapunov pairs.

Corollary 4.4 *Assume that* $\dim H < \infty$. *Let be given* $V \in \mathscr{F}(H)$, $W \in \mathscr{F}(H; \mathbb{R}_+)$, *and* $a \in \mathbb{R}_+$. *The following statements are equivalent*:

(i) (V, W) *is an* a*-Lyapunov pair for (33);*

(ii) *for every* $y \in \text{Dom } V$

$$V'(y; f(y)) + aV(y) + W(y) \leq 0;$$

(iii) *for all* $y \in \text{Dom } V$

$$\sup_{\xi \in \partial V(y)} \langle \xi, f(y) \rangle + aV(y) + W(y) \leq 0,$$

where ∂V *stands for either the proximal, the Fréchet, the limiting, or the Clarke subdifferentials.*

Proof According to Theorem 4.1 we only need to show that (iii) is also a characterization when $\partial \equiv \partial_C$. For this aim, in view of the relationship $\partial_L \subset \partial_C$, it suffices to show that [(iii) with $\partial \equiv \partial_L$] implies [(iii) with $\partial \equiv \partial_C$]. Indeed, fix $y \in \text{Dom } V$ so that

$$\sup_{\xi \in \partial_\infty V(y)} \langle \xi, f(y) \rangle \leq 0.$$

So, according to [28], (iii with $\partial \equiv \partial_C$) follows since that

$$\begin{aligned} &\sup_{\xi \in \partial_C V(y)} \langle \xi, f(y) \rangle + aV(y) + W(y) \\ &= \sup_{\xi \in \overline{\text{co}}\{\partial_L V(y) + \partial_\infty V(y)\}} \langle \xi, f(y) \rangle + aV(y) + W(y) \leq 0. \end{aligned}$$

□

5 Concluding remarks

The main goal of this paper was to explore the existence of local nonsmooth Lyapunov pairs for a first-order evolution differential inclusion governed by a maximal monotone operator. From the mathematical point of view, our major contribution is establishing the fact that the variational criteria for the existence of Lyapunov pairs need to be verified on the interior of the domain of the operator A, while Lyapunov pairs are defined on the whole $cl(\text{Dom } A)$. The first version of these ideas appeared in [2]. The flow-invariance of a closed set has been also investigated as a particular case of the theory of Lyapunov functions. The characterization of invariance involves a "proximal aiming" condition, as well as the convex hull of the contingent cone. An important issue, left for the future work, is to go beyond maximally monotone operators, since in some applications the monotonicity assumption is not satisfied. This is the case e.g. when A coincides with the normal cone operator of a prox-regular set. It would be interesting to perform a stability analysis of differential variational inequalities involving locally-prox-regular sets. This is beyond the scope of this paper and will be a subject of a forthcoming research project. For its huge potential of applications, Lyapunov stability is used by other communities, particularly in nonlinear systems and control. As noticed by one of the referees, it should be interesting to find practical applications of the theory developed in this paper. Bridging the communities of applied mathematicians, controllers and engineers, is one of our future objectives.

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