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Canonical Finite Element Method For Solving Nonconvex Variational Problems to Post Buckling Beam Problem

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Abstract. The goal of this paper is to solve the post buckling phenomena of a large deformed elastic beam by a canonical dual mixed finite element method (CD-FEM). The total potential energy of this beam is a nonconvex functional which can be used to model both pre-and post-buckling problems. Different types of dual stress interpolations are used in order to verify the triality theory. Applications are illustrated with different boundary conditions and external loads by using semi-definite programming (SDP) algorithm. The results show that the global minimum of the total potential energy is stable buckled configuration, the local maximum solution leads to the unbuckled state, and both of these two solutions are numerically stable. While the local minimum is unstable buckled configuration and very sensitive to both stress interpolations and the external loads.

INTRODUCTION

Nonconvex variational problems have always presented serious challenges not only in numerical analysis, but also in computational mechanics and engineering sciences. The popular primal-dual interior point methods suffer from uncertain error bounds in nonconvex analysis because of the intrinsic duality gaps produced by traditional duality theories. Therefore, most nonconvex minimization problems are considered as **NP-hard** in global optimization and computer sciences [6]. Unfortunately, this fundamental difficulty is not fully recognized in computational mathematics and mechanics due to the significant gap between these fields.

Canonical duality theory is a newly developed, potentially powerful methodological theory which can transfer general multi-scale nonconvex/discrete problems in $\mathbb{R}^n$ to a unified convex dual problem in continuous space $\mathbb{R}^m$ with $m \leq n$ and without duality gap. The associated *triality theory* provides extremality criteria for both global and local optimal solutions, which can be used to develop powerful algorithms for solving general nonconvex variational problems [6]. This talk will present a CD-FEM for solving general nonconvex variational problems [3]. By using Gao-Strang's complementary-dual principle and mixed finite element discretization, the general nonconvex variational problem can be reformulated as a min-max optimization problem of a saddle function. Detailed application will be illustrated by post buckling problem of a large elastic deformations of beam, which is governed by a fourth order nonlinear differential equation. Numerical results show that our algorithm can produce very stable solutions for both global minimizer and local maximizer. However, the local minimizer is very sensitive to numerical discretization and the external loads.

NONCONVEX PROBLEM AND GAO-STRANG'S COMPLEMENTARY

Consider an elastic beam subjected to a vertical distributed lateral load $q(x)$ and compressive axial force F (see Fig.1). A nonlinear beam model was proposed by Gao [4, 5]:

$$EI.w_{,xxxx} - \alpha E w_{,x}^2 w_{,xx} + E \lambda w_{,xx} - f(x) = 0, \quad \forall x \in [0, L], \quad (1)$$

where $E > 0$, $I = 2h^3/3$, $\alpha = 3h(1 - \mu^2) > 0$ are constants with μ as the Poisson's ratio, $\lambda = (1 + \mu)(1 - \mu^2)F/E > 0$ is an integral constant, w is the transverse displacement field of the beam, $f(x) = (1 - \mu^2)q(x)$ depends mainly on

FIGURE 1. Simply supported beam model - pre-and post-buckling problems

the distributed lateral load $q(x)$; $2h$ and L represent to the height and length of the beam respectively. The axial displacement $u(x)$ is governed by the following differential equation [4]:

$$u_x = -\frac{1}{2}(1 + \mu)w_{,x}^2 - \frac{\lambda}{2h(1 + \mu)}.$$

The total potential energy of this problem is the function $\Pi(w) : \mathcal{U}_a \rightarrow \mathbb{R}$ define by

$$\Pi(w) = \int_0^L \left(\frac{1}{2}EIw_{,xx}^2 + \frac{1}{12}E\alpha w_{,x}^4 - \frac{1}{2}E\lambda w_{,x}^2 - f(x)w \right) dx = 0, \quad (2)$$

where $\mathcal{U}_a$ is the kinematically admissible space, in which certain necessary boundary conditions are given. Thus, for the given external loads $f(x)$ and λ , the primal variational problem is to find $\bar{w} \in \mathcal{U}_a$ such that

$$(\mathcal{P}) : \quad \Pi(\bar{w}) = \inf \{ \Pi(w) | w \in \mathcal{U}_a \}. \quad (3)$$

Clearly, before the axial load λ reaches to the Euler buckling load λ_{cr} [7], the $\Pi(w)$ is convex on $\mathcal{U}_a$ and the non-linear differential equation (1) has only one solution. When $\lambda > \lambda_{cr}$, the beam is in a post-buckling state and $\Pi(x)$ is nonconvex. Thus, eq.(1) may have at most three (strong) solutions at each material point $x \in [0, L]$ [7]: two minimizers corresponding to the two possible buckled states, one local maximizer corresponding to the possible unbuckled state. Clearly, these solutions are sensitive to both the axial load λ and the distributed lateral force field $f(x)$. Based on the canonical duality theory, the fourth-order nonconvex functional $\Pi(w)$ can be equivalently written in the Gao-Strang total complementary energy $\Xi : \mathcal{U}_a \times \mathcal{S}_a \rightarrow \mathbb{R}$ [1]:

$$\Xi(w, \sigma) = G(w, \sigma) - \int_0^L [V^*(\sigma) - f(x)w] dx, \quad (4)$$

where $G(w, \sigma) = \int_0^L (\frac{1}{2}EIw_{,xx}^2 + \frac{1}{2}\sigma w_{,x}^2) dx$ is the generalized Gao-Strang complementary gap function [1], $V^*(\sigma) = \frac{3}{4E\alpha}(\sigma + E\lambda)^2$, and $\mathcal{S}_a = \{ \sigma \in C[0, L] | \sigma(x) \geq -\lambda E \quad \forall x \in [0, L] \}$.

MIXED FINITE ELEMENT METHOD AND TRIALITY THEORY

The domain of the beam is discretized into m elements $[0, L] = \bigcup_{e=1}^m \Omega^e$ by using FEM. In each element $\Omega^e = [x_a, x_b]$, the deflection, rotating angular and dual stress for the node x_a are marked as w_a, θ_a and σ_a , respectively, and similar for x_b . Then, we have the nodal displacement vector $w_e^T = [w_a \quad \theta_a \quad w_b \quad \theta_b]$ of the e -th element and the nodal dual stress element $\sigma_e^T = [\sigma_a \quad \sigma_b]$. In each element, we use mixed finite element interpolations for both $w(x)$ and $\sigma(x)$, i.e. $w_e^h(x) = N_w^T(x)w_e$, $\sigma_e^h(x) = N_\sigma^T(x)\sigma_e$, $\forall x \in \Omega^e$. Thus, the spaces $\mathcal{U}_a$ and $\mathcal{S}_a$ can be numerically discretized to the finite-dimensional spaces $\mathcal{U}_a^h$ and $\mathcal{S}_a^h$, respectively. On the discretized feasible deformation space $\mathcal{U}_a^h$, the Gao-Strang total complementary energy can be expressed in the discretized form

$$\begin{aligned} \Xi^h(\mathbf{w}, \boldsymbol{\sigma}) &= \sum_{e=1}^m \left(\frac{1}{2} w_e^T G^e(\sigma_e) w_e - \frac{1}{2} \sigma_e^T K_e \sigma_e - \lambda_e^T \sigma_e - f_e^T w_e - c_e \right) \\ &= \frac{1}{2} \mathbf{w}^T \mathbf{G}(\boldsymbol{\sigma}) \mathbf{w} - \frac{1}{2} \boldsymbol{\sigma}^T \mathbf{K} \boldsymbol{\sigma} - \boldsymbol{\lambda}^T \boldsymbol{\sigma} - \mathbf{f}^T \mathbf{w} - c, \end{aligned} \quad (5)$$

where $\mathbf{w} \in \mathcal{U}_a^h \subset \mathbb{R}^{2(m+1)}$ and $\boldsymbol{\sigma} \in \mathcal{S}_a^h \subset \mathbb{R}^{\delta m+1}$ are nodal deflection and dual stress vectors, respectively, and $\mathcal{S}_a^h = \{\boldsymbol{\sigma} \in \mathbb{R}^{\delta m+1} \mid \det \mathbf{G}(\boldsymbol{\sigma}) \neq 0\}$. The scalar $\delta = 1$ or 2 corresponds to the different stress interpolation schemes; piecewise-linear stresses or piecewise-quadratic stresses, respectively. The Hessian matrix of the gap function $\mathbf{G}(\boldsymbol{\sigma}) \in \mathbb{R}^{2(m+1)} \times \mathbb{R}^{2(m+1)}$ is obtained by assembling the symmetric matrices:

$$\mathbf{G}^e(\sigma_e) = \int_{\Omega_e} \left(EI N_w'' (N_w'')^T + (N_\sigma)^T \sigma_e N_w' (N_w')^T \right) dx \quad (6)$$

The matrix $\mathbf{K} \in \mathbb{R}^{(\delta m+1)} \times \mathbb{R}^{(\delta m+1)}$ is obtained by assembling matrices $K_e = \int_{\Omega_e} \left(\frac{3}{2Ea} N_\sigma N_\sigma^T \right) dx$. The vectors, $\boldsymbol{\lambda} \in \mathbb{R}^{(\delta m+1)}$ and $\mathbf{f} \in \mathbb{R}^{2(m+1)}$ are defined by assembling the corresponding element components, and $c = \sum_{e=1}^m c_e \in \mathbb{R}$ [7]. Based on $\Xi^h(\mathbf{w}, \boldsymbol{\sigma})$, the discretized pure complementary energy $\Pi_d^h(\boldsymbol{\sigma})$ can be obtained by the canonical dual transformation

$$\Pi_d^h(\boldsymbol{\sigma}) = \min_{\mathbf{w} \in \mathcal{U}_a^h} \Xi^h(\mathbf{w}, \boldsymbol{\sigma}) = -\frac{1}{2} \mathbf{f}^T \mathbf{G}^{-1}(\boldsymbol{\sigma}) \mathbf{f} - \frac{1}{2} \boldsymbol{\sigma}^T \mathbf{K} \boldsymbol{\sigma} - \boldsymbol{\lambda}^T \boldsymbol{\sigma} - c \quad (7)$$

Theorem 1 (Triality Theory [2]) *Let $(\bar{\mathbf{w}}, \bar{\boldsymbol{\sigma}})$ be a stationary point of $\Xi^h(\mathbf{w}, \boldsymbol{\sigma})$ and $\mathcal{U}_o \times \mathcal{S}_o \subset \mathcal{U}_a^h \times \mathcal{S}_a^-$ be a neighborhood of $(\bar{\mathbf{w}}, \bar{\boldsymbol{\sigma}})$, with $\mathcal{S}_a^+ = \{\boldsymbol{\sigma} \in \mathcal{S}_a^h \mid \mathbf{G}(\boldsymbol{\sigma}) > 0\}$ and $\mathcal{S}_a^- = \{\boldsymbol{\sigma} \in \mathcal{S}_a^h \mid \mathbf{G}(\boldsymbol{\sigma}) < 0\}$.*

If $\bar{\boldsymbol{\sigma}} \in \mathcal{S}_a^+$, then $\bar{\mathbf{w}} \in \mathcal{U}_a^h$ is a global minimizer of $\Pi_p^h(\mathbf{w})$ if and only if $\bar{\boldsymbol{\sigma}} \in \mathcal{S}_a^+$ is a global maximizer of $\Pi_d^h(\boldsymbol{\sigma})$, and

$$\Pi_p^h(\bar{\mathbf{w}}) = \min_{\mathbf{w} \in \mathcal{U}_a^h} \Pi_p^h(\mathbf{w}) = \max_{\boldsymbol{\sigma} \in \mathcal{S}_a^+} \Pi_d^h(\boldsymbol{\sigma}) = \Pi_d^h(\bar{\boldsymbol{\sigma}}). \quad (8)$$

If $\bar{\boldsymbol{\sigma}} \in \mathcal{S}_a^-$, then $\bar{\mathbf{w}} \in \mathcal{U}_o$ is a local maximizer of $\Pi_p^h(\mathbf{w})$ if and only if $\bar{\boldsymbol{\sigma}} \in \mathcal{S}_o$ is a local maximizer of $\Pi_d^h(\boldsymbol{\sigma})$, and

$$\Pi_p^h(\bar{\mathbf{w}}) = \max_{\mathbf{w} \in \mathcal{U}_o} \Pi_p^h(\mathbf{w}) \Leftrightarrow \max_{\boldsymbol{\sigma} \in \mathcal{S}_o} \Pi_d^h(\boldsymbol{\sigma}) = \Pi_d^h(\bar{\boldsymbol{\sigma}}). \quad (9)$$

Otherwise, $\bar{\mathbf{w}} \in \mathcal{U}_o$ is a local minimizer of $\Pi_p^h(\mathbf{w})$ if and only if $\bar{\boldsymbol{\sigma}} \in \mathcal{S}_o$ is a local minimizer of $\Pi_d^h(\boldsymbol{\sigma})$, and

$$\Pi_p^h(\bar{\mathbf{w}}) = \min_{\mathbf{w} \in \mathcal{U}_o} \Pi_p^h(\mathbf{w}) \Leftrightarrow \min_{\boldsymbol{\sigma} \in \mathcal{S}_o} \Pi_d^h(\boldsymbol{\sigma}) = \Pi_d^h(\bar{\boldsymbol{\sigma}}). \quad (10)$$

The canonical min-max duality (8) can be used to find global minimizer of the nonconvex problem by the canonical dual problem $\max\{\Pi_d^h(\boldsymbol{\sigma}) \mid \boldsymbol{\sigma} \in \mathcal{S}_a^+\}$, which is a concave maximization problem and can be solved easily by well-developed convex analysis and optimization techniques. The canonical double-max (9) and double-min duality (10) can be used to find the biggest local maximizer and a local minimizer of the nonconvex primal problem, respectively. It was proved in [8] that both the canonical min-max and double-max duality statements hold strongly regardless the dimensions of $\mathcal{U}_a^h$ and $\mathcal{S}_a^h$, while the canonical double-min duality statement holds weakly for $\dim \mathcal{U}_a^h \neq \dim \mathcal{S}_a^h$, but strongly if $\dim \mathcal{U}_a^h = \dim \mathcal{S}_a^h$.

SEMI-DEFINITE PROGRAMMING ALGORITHM

By Schur complement lemma [9], the canonical min-max duality (8) can be relaxed to the SDP problem:

$$\max_{\boldsymbol{\sigma}, t} t \quad \text{s.t.} \quad \mathbf{G}(\boldsymbol{\sigma}) \geq 0, \quad \begin{bmatrix} 2\mathbf{K}^{-1} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma}^T & \frac{1}{2} \mathbf{w}^T \mathbf{G}(\boldsymbol{\sigma}) \mathbf{w} - \boldsymbol{\lambda}^T \boldsymbol{\sigma} - \mathbf{f}^T \mathbf{w} - c - t \end{bmatrix} \geq 0. \quad (11)$$

Where $\mathbf{w} = \mathbf{w}(\boldsymbol{\sigma}) = \mathbf{G}^{-1}(\boldsymbol{\sigma}) \mathbf{f}$. Similar for the local maximum with the constraint $-\mathbf{G}(\boldsymbol{\sigma}) > 0$. While, for the local minimum of post-buckling problem, a new formula $\widehat{\Pi}^d(\boldsymbol{\sigma}, \mathbf{w})$ of pure complementary energy is applied:

$$\widehat{\Pi}^d(\boldsymbol{\sigma}, \mathbf{w}) = -\frac{1}{2} \mathbf{f}^T \mathbf{G}^{-1}(\boldsymbol{\sigma}) \mathbf{f} - \frac{1}{2} \mathbf{w}^T \mathbf{M}(\boldsymbol{\sigma}) \mathbf{w} - \frac{1}{2} \boldsymbol{\lambda}^T \boldsymbol{\sigma} - c, \quad (12)$$

Where $\mathbf{M}(\boldsymbol{\sigma})$ is obtained by assembling matrices $M^e(\sigma_e) = \int_{\Omega_e} \frac{1}{2} \left((N_\sigma)^T \sigma_e N_w' (N_w')^T \right) dx$. Thus, the SDP relaxation for the canonical double-min duality (10) should be equivalent to

$$\min_{\boldsymbol{\sigma}, t} t \quad \text{s.t.} \quad -\mathbf{G}(\boldsymbol{\sigma}) > 0, \quad \begin{bmatrix} -2\mathbf{G}(\boldsymbol{\sigma}) & \mathbf{f} \\ \mathbf{f}^T & \frac{1}{2} \mathbf{w}^T \mathbf{M}(\boldsymbol{\sigma}) \mathbf{w} + \frac{1}{2} \boldsymbol{\lambda}^T \boldsymbol{\sigma} + c + t \end{bmatrix} \geq 0. \quad (13)$$

With an initial point of $\mathbf{w}$, the $\boldsymbol{\sigma}$ is computed for all the post buckling problems by applying SDP algorithm to find the optimal solution $\mathbf{w}^*$. A software package named SeDuMi [10] is used to apply SDP.

FIGURE 2. Mixed mesh of different dual stress interpolations (in 5 elements)

FIGURE 3. Post-buckled solutions for simply supported beam (left) and cantilever beam (right) under uniformly distributed load

NUMERICAL SOLUTIONS

According to the triality theory, the local minimizer of the total potential energy can be obtained only if $\dim \mathcal{U}_a^h = \dim \mathcal{S}_a^h$. Thus, we relied on the different stress interpolation schemes, for $\delta = 1$ ($\dim \mathcal{U}_a^h \neq \dim \mathcal{S}_a^h$), $\delta = 2$ ($\dim \mathcal{U}_a^h \approx \dim \mathcal{S}_a^h$) and mixed mesh scheme for different stress interpolation ($\dim \mathcal{U}_a^h = \dim \mathcal{S}_a^h$), as shown in see Fig.2. After applying the boundary conditions, the dimension of displacement vector $\mathbf{w} \in \mathcal{U}_a^h$ is “ $2m$ ” for both simply supported beam and cantilever beam. Different boundary conditions and different external loads with $\lambda > \lambda_{cr}$ are investigated. The approximate deflections are illustrated in Fig.3, where $E = 1000Pa$, $\mu = 0.3$, $L = 1m$, $h = 0.05m$ and $m = 51$. The violet and orange curves represent to the global minimum and local maximum of the total potential energy, respectively. While the local minimum is represented by the red, blue and green curves which are designed by three different mesh schemes, that are linear, quadratic and mixed dual stress interpolations, respectively.

CONCLUSIONS

This paper presents a CD-FEM for post-buckling analysis of a large deformations elastic beam which is governed by a fourth order non-linear differential equation. The generalized total complementary energy $\Xi(\mathbf{w}, \boldsymbol{\sigma})$ associated with this model is a non-convex functional. Combining the generalized total complementary energy and the proposed formula of pure complementary energy (12), with the triality theory, a canonical duality algorithm is investigated for solving post-buckling problems by using SDP algorithm. The triality theory is verified by using different types of dual stress interpolations. The numerical results show that, the global minimum of the total potential energy “i.e. global maximum of $\Xi(\mathbf{w}, \boldsymbol{\sigma})$, see (7) and canonical min-max duality (8)” is a stable buckled configuration, while the local extrema present unstable deformation states and the solutions of unstable buckled state is very sensitive for each of the stress interpolations, axial load λ and distributed lateral force $f(x)$.

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