Approximations for some classes of optimal control problems with state constraints and repetitive control systems

by

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Abstract

We examine the theoretical background for the development of numerical methods for solving some classes of optimal control problems. We give some results devoted to the approximation for optimal control problems with nonsmooth state constraints and extend the well-posed discrete approximation approach to the optimization problem with min-max type constraint. Also, we develop the optimal control theory for the linear dynamics in the presence of intermediate constraints and so-called linear repetitive processes. The classic approach is based on the separation theorem and a new, named constructive method is developed. These results are illustrated by solving synthesis problem for the simple dynamic system. The well-posed discrete approximation is illustrated for the linear two-dimensional cran manipulator model. The numerical aspects are discussed.

Statement of Originality

Except where explicit reference is made in the text of the thesis, this thesis contains no material published elsewhere or extracted in whole or in part from a thesis by which I have qualified for or have been awarded another degree or diploma. No other person's work has been relied upon or used without due acknowledgment in the main text and bibliography of the thesis.

Siarhei Dymkou

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Context

Preface

This report presents the investigation on the Research work "Approximations for some classes of optimal control problems with state constraints and repetitive control systems". This report mainly gives the theoretical part of this research which is based on the numerical methods that are developed. It is conjectured that the corresponding algorithms will be realized as computer programs after which they can be used in practical work.

Chapter 1 gives a new theoretical result devoted to the smooth approximation for some of the nonsmooth optimization problem in general form. A possible discrete approximation scheme is also discussed.

Chapter 2 extends the well-posed approximation approach to optimization problems with min-max type constraint as a particular case of nonsmooth constraints.

In chapter 3, we develop the optimal control theory for the dynamics that are logged by the presence of intermediate constraints. Such models can be applied to relax the optimization problems, for example those with phase restriction. These results are applied to solve an illustrative example.

In chapter 4 the well-posed discrete approximation is realized for the robot path planning problem. The numerical aspects are discussed. Two simple test optimization problems are presented.

Chapter 5 develops the optimization theory for so-called repetitive processes. These objects arise in the modelling of a lot of industrial processes and can be used for planning or learning procedures. The classic approach based on the separation theorem and a new, named constructive method are developed. These results are illustrated by solving a synthesis problem for the simple dynamic system.

In chapter 6 we investigate the links between some classes of linear repetitive processes and delay systems and apply this to analyze control theory problems arising in controllability and the optimal control of these repetitive processes.

Introduction

The past three decades have seen a continually growing interest in so-called nonsmooth theory. This is clearly related to the wide variety of applications of both practical and theoretical interest. Many physical processes have a clear nonsmooth structure (see Gilbert [34], Polak [38]). Also, the nonsmooth approach is frequently used as an analysis tool to solve a wide variety of theoretical problems (Rockafellar [55], Clarke [8]).

The difficulty with the application of the nonsmooth theory is that it is difficult. Many fundamental results of optimal control theory, which are not too difficult to use in the "smooth" case, become remarkably complex though, even in the simplest "nonsmooth" cases, and the significant efforts are required to efficiently apply them to numerical methods.

This research work has been oriented to the development of numerical methods based on this theory and their application for solving some practical problems.

We consider optimal control problems described by ordinary differential equations in the presence of trajectory nonsmooth constraints which have often arisen under the mathematical modelling of real technical processes. The examination of extremal problems with nonsmooth state restriction (max-min constraints are typical of this) have met essential difficulties in both theoretical and numerical applications (see Arutyunov and Aseev [4], Mordukhovich [45]). As far the theoretical issues then the main problem is generated by the fact that the state constraints lead to a discontinuity (jump) of the trajectories of the considered dynamical systems. This yields, in part, that the formulation of the optimality conditions uses heavy mathematical tools such as Borel measures and constrained variation functions as Lagrange multipliers, etc. In addition, the necessary optimality conditions in this case can be generated for any admissible trajectory. The bibliography on classic optimization with state constraint can be found, for example, in the survey by Hartl *et al* [36]. The nonsmooth character of state constraints gives additional difficulties for optimization of these control models. Naturally, effective numerical methods can not be constructed without proper theoretical background based on the updated nonsmooth optimization theory (see Clarke [8], Demyanov and Rubinov [10]). Note that there exists a good theory to construct a discrete approximation for optimization problems under state constraints (see Teo [64], Mordukhovich [46] and the bibliography therein). It is well known that the approximation in this case demands a careful construction of the discrete model to guarantee the needed adequacy of the model (the counterexample that demonstrates incorrect results of standard discretization can be found in [46]). The optimization problem with the nonsmooth character of the state constraint is equivalent to a collection of optimization problems with smooth inequality constraints. This collection can be very large, and hence their solving by readily available software is exorbitantly expensive. An essential and key question for both theoretical and numerical aspects of approximations are as follows : can one approximate, and in which sense, an optimal solution of the original control system by solutions of an approximate model? In order to achieve a well-posed discrete approximation ensuring the convergence of optimal solutions we need to admit, in general, the state perturbations in the discrete model. Another way to construct the correct approximation for the optimization problem under state constraint is to introduce a new (continuous-time) model without any state constraints. Surely, such simplification demands the proper modification of the model: we make worse (in some sense) the right- hand side of the control system. Some aspects of this approach for optimization problems of differential inclusions can be found in the paper by Aseev [5].

Next we consider the optimal control problem described by ordinary differential equations in presence of the trajectory constraints of the special, min-max, type. This collection can be very large, and hence solving an optimization problem with max-min constraints by solving each member of the collection can be exorbitantly expensive. We present a well-posed discrete approximation with appropriate converge results. The main complication comes from the nonsmooth phase constraints that lead to an increasing number of corresponding discrete finite-dimensional models of the constrained mathematical programming problem with nonsmooth constraints, which can be solved by using readily available software. Can one approximate, and in which sense, an optimal solution of the continuous control system by discrete solutions? In order to achieve a well-posed approximation ensuring the convergence of optimal solutions we need to admit, in general, the phase perturbations in the discrete model. It is well known that for the ordinary case of endpoints constraints the relaxation stability property connected with the so-called "hidden convexity" of differential system is necessary and sufficient for the value convergence of discrete approximations under the appropriate perturbation of endpoints constraints.

Often, the statement of the control and the path planning problem for different mechanisms may be broken into several stages. Firstly, the key role of such planning methods is to specify a geometric path in the presence of real physical obstacles. There are a number of research papers where this task is formulated as a control problem with state space restrictions [34, 38]. The development of effective numerical methods for their solution is of permanent interest [29, 37, 10]. Here, in particular, we propose to replace the given obstacles by the collection of sets that are obligatory for the pass-crossing by the considered mechanisms. These sets and their configuration choice can be pre-assigned by experts or the learning procedure, for example, and they are usually placed in the "danger" area to avoid possible collisions. The traditional continuity properties of the considered trajectories hope that the desired behaviour of the model will be satisfactory in some neighborhood of the given obstacles. The obtained trajectory can be considered then as a feasible solution to start the various improvement procedures. These procedures can also be accompanied by the formulation of suitable optimization problems. This part of the report is arranged as follows. Firstly, we establish the optimality conditions for the robot models described by linear nonstationary differential equations with nonlinear inputs and equipped by the intermediate constraints of the general form at pre-assigned moments. The cost functional is given by the convex function defined on the trajectories of the system at given moments. The next section is devoted to some applications of the developed results for the robotic motion planning in the two-dimensional case. Some simple numerical methods are proposed for the particular cases of the considered model. Finally, the robot dynamics models described by linear stationary differential equations with linear inputs is chosen to demonstrate and approve the obtained results and discretization schemes. Also this robot model is used in the course of the report's chapters to test the obtained results. The choice of this model is partially explained by the fact that there is some experimental data for it. The proposed methods can be easy extended to other dynamics which is planned to be realized in the future. The detail calculation of the nonsmooth functions needed for the computer programming is given. Also two-dimensional test optimization problems with known exact (theoretical) optimal solutions are presented.

A multipass process (termed a repetitive process in other literature) is one in which the material involved is processed by a sequence of passes, termed sweeps, of the processing tool. Such systems are characterized by two distinctive features, repetitive operation and dependence of the present pass behaviour on the behaviour of the previous passes. They arise in the modelling of a lot of industrial processes such as long-wall coal cutting, metal rolling operations and others. Metal rolling, for example, is an industrial process where deformation of the metal stock takes place between two rollers with parallel axes revolving in opposite directions through a series of passes for successive reductions. A repetitive processes of metal rolling modeling in linearized form can be presented as follows (some details can be found in [56])

$$\frac{d^2 y_k(t)}{dt^2} + \lambda_1 y_k(t) = \lambda_2 \frac{d^2 y_{k-1}(t)}{dt^2} + \lambda_1 y_{k-1}(t) + b u_k(t), \ t \in [0, t^*], \ k \in K = \{1, ..., N\},$$

where $y_k(t)$ and $y_{k-1}(t)$ denote the gauge on the current and previous passes through the rollers; λ_1, λ_2 and b are determined, in fact, by the stiffness of the metal strip and the roller mechanism properties, $u(t)$ can be interpreted as the applied force to the metal strip by the rollers.

Such dynamic systems also provide an appropriate mathematical tool for modeling chemical processes. In particular, a model of the rectification process of a many component mixture in a many-plate column can be represented by a similar model

$$\begin{aligned}
\frac{dx_s(t)}{dt} &= V_{s-1}(t)x_{s-1}(t) + V_s(t)x_s(t) - R_s(x_s(t), y_s(t)) + u_{x_s}(t), \\
\frac{dy_s(t)}{dt} &= L_{s+1}(t)y_{s+1}(t) + L_s(t)y_s(t) + R_s(x_s(t), y_s(t)) + u_{y_s}(t), \\
& \qquad t \in [0, t^*], \quad s \in K \doteq \{1, ..., N\}.
\end{aligned}$$

Here $x(s,t)$, $y(s,t)$ denote the desired material concentration on s-th plate in the gas and liquid fractions, respectively; L, V and R present the hydrodynamic characteristic of the process under consideration; u_x and u_y are the control material rows; K is a subset of integers. Some details of the model can be found in [12].

Also problem areas exist where adopting a repetitive process perspective has clear advantages over the alternatives. The development of a mature systems theory for these processes has been the subject of considerable research efforts over the past two decades which has resulted in very significant progress on systems theoretic properties. This work is devoted to the optimization theory of some classes of these objects.

The first part uses the classic approach to investigate traditional optimal control theory problems. It is well known that the separation theorem for convex sets is a quite useful approach for studying a wide class of extremal problems. Here we develop a method to establish optimality conditions in the classic form of maximum principal for multipass nonstationary continuous-discrete control systems with nonlinear inputs and nonlocal state-phase terminal constraints of the general form. The obtained results are typical for classic optimal control theory. However, their numerical realization is not a trivial task. For this reason in the next sections for the stationary case of the system model and particular case of the constraint and the cost functional we develop new optimality and sub-optimality conditions that are more suitable for the design of numerical methods and further applications. In contrast to the classic approaches of optimal control theory. In the second part of the paper we use the idea of the constructive methods reported in [29] and extend this setting to the continuous-discrete case to produce new results and constructive elements of optimization theory for the considered repetitive systems and also develop its relevant basic properties which can be of interest for others purposes. It is shown that the obtained optimality and ϵ-optimality conditions are closely related to the corresponding classic results of maximum principle and ϵ- maximum principle. The sensitivity analysis and some differential properties of the optimal controls under disturbances are discussed and their application to the optimal synthesis problem is given. It has been conjectured that such a setting could be appropriate for the development of numerical methods of optimal control problems and related studies, on which very little work has yet been reported. The obtained results yield a theoretical background for the design problem of optimal controllers for relevant basic processes.

It is already known that repetitive processes can be represented in various dynamic system forms, which can, where appropriate, be used to great effect in the control related analysis of these processes. In the Chapter 6, we investigate further the already known links between some classes of linear repetitive processes and delay systems, and apply this investigation to analyze control theory problems arising in the controllability and optimal control of these repetitive processes. In particular, so-called characteristic mappings introduced in [29] are used to establish controllability properties criteria. Next, time optimal control problems are considered, where it is well known that the separation theorem for convex sets is a useful approach for studying a wide class of extremal problems. Here we adopt this method to establish optimality conditions in the classic form.

Chapter 1

Approximation of the optimal control problem with the nonsmooth state constraints

This chapter is concerned with some aspects of approximation for an optimal control problem with nonsmooth state constraints. We establish some theoretical results devoted to the approximation of this problem by a sequence of optimization problems without the presence of the state restriction. Also we exploit the idea of the perturbations of the state constraints for a well-posed discrete approximation ensuring the convergence of optimal solutions. We then use these results for a robot model described by linear stationary differential equations with linear inputs to design the numerical methods. An essential and key question for both theoretical and numerical aspects of approximations is as follows : can one approximate, and in which sense, an optimal solution of the original control system by solutions of the approximate model? In order to achieve a well-posed discrete approximation ensuring the convergence of optimal solutions we need to admit, in general, the state perturbations in the discrete model. Another way to construct the correct approximation for the optimization problem under state constraint is to introduce a new (continuous-time) model without any state constraints.

1.1 Problem statement

In this section we use the idea of the approximation of a continuous-time model for the following optimal control problem : minimize

$$\max J(u) = \varphi(x(T,u)) \tag{1.1}$$

over absolutely continuous trajectories $x : [0,T] \to \mathbb{R}^n$ for the differential equation

$$\frac{dx(t)}{dt} = f(x,u,t),\ x(0) = x_0,\ u(t) \in U \quad \text{a.e.} \quad t \in [0,T] \tag{1.2}$$

subject to the nonsmooth state constraint of the form

$$x(t) \in G, \; t \in [0,T], \text{ where } G = \{x \in \mathbb{R}^n : g(x) \leq 0\} \tag{1.3}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, $g : \mathbb{R}^n \to \mathbb{R}$ is a continuous and directional differentiable function, x_0 is the given $n-$ vector. We denote the derivatives of g at a point x in direction $l \in \mathbb{R}^n$ by $g_x^{'}(l)$. It is assumed that the function $l \to g_x^{'}(l)$ is continuous as a function of the direction l for all x. We suppose that the function $f : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R} \to \mathbb{R}^r$ satisfies the Caratheodory conditions (i.e. f is continuous on (x,u) and measurable on t) such that the initial Cauchy problem for the differential equation (1.2) has a unique, absolutely continuous solution.

Definition 1. *We say that the function $u : T \to \mathbb{R}^r$ is admissible for (1.2) if it is measurable and satisfies the constraint $u(t) \in U$ almost everywhere $t \in T$, where U is a given compact set from $\mathbb{R}^r$. We say that the function $x : T \to \mathbb{R}^n$ is an admissible solution of (1.2) corresponding to the given admissible control $u(t)$ if it is absolutely continuous with respect to $t \in T$ and satisfies (1.2) for almost all $t \in T$.*

Next we suppose that the set of admissible controls $U(\cdot)$ is nonempty and also assume that the following condition Aseev [5] is fulfilled:

(A) There exists constants $\epsilon_0 > 0, \; \alpha < 0$ such that for all $u \in U$ and almost all $t \in [0,T]$ the following inequality

$$g_x^{'}(f(x,u,t)) \leq \alpha \tag{1.4}$$

holds for $\forall x : 0 \leq g(x) \leq \epsilon_0$.

This condition can be treated as a normality or regularity condition for the optimization problem with state constraints. In fact, the condition (A) coordinates the dynamic behaviour of the system (1.2) with the state restriction (1.3) in order to avoid a lengthy presence in the prohibited zone where the state constraints are disturbed.

We consider this condition in the form (1.4) since the given number ϵ_0 will be used in the estimates bellow.

Remark 1. *The constraint of the form (1.3) includes a wide class of the state restriction. In particular, this constraint is often given in the following min form*

$$x(t) \in G, \; t \in [0,T], \text{ where } G = \{x \in \mathbb{R}^n : \; g(x) = \min_{1 \leq i \leq m} \varphi_i(x) \leq 0\} \tag{1.5}$$

Usually it is assumed that the functions $\varphi_i(x)$ are continuously differentiable. It is known that in this case the function $g(x)$ is continuous, directional differentiable and the derivative along the direction $l \in \mathbb{R}^n$ is given by the formula

$$g_x^{'}(l) = \min_{i \in Q(x)} \left(\frac{\partial \varphi_i(x)}{\partial x}, l \right), \qquad \text{where } Q(x) = \{i : g(x) = \varphi_i(x)\} \tag{1.6}$$

1.2 Approximation by continuous time optimization problems

In order to construct the correct approximation for the optimization problem with state constraints we introduce a new model without any state constraints. Such simplification demands the corresponding modification of the right-hand side of the control system. This approach for optimization problems of differential inclusions with smooth state constraints has been proposed by Aseev [5]. In this section we use his idea for the control systems described by the ordinary differential equations in the presence of nonsmooth state constraints.

For the optimization problem (1.1) -(1.3) we introduce the following *approximation*: for each $i = 1, 2, \ldots$ consider the sequence of the optimal control problems of the form:

$$\text{maximize} J(u) = \varphi(x(T, u)) \tag{1.7}$$

over the solutions of the following equations

$$\frac{dx(t)}{dt} = (1 - ih^2(x))f(x, u, t), \;\; u(t) \in U, \quad \text{a.e. } t \in [0, T], \quad x(0) = x_0, \tag{1.8}$$

where $h(x) = \max\{0, g(x)\}$.

Thus, the original optimization problem (1.1)- (1.3) is approximated by a sequence of continuous time optimization problems without state constraints $x(t) \in G$. This relaxation is compensated by the modification of the right-hand side of the differential equation. Surely, we are interested in how the trajectories of the relaxed problem approximate the trajectories of the original system and the state constraint G ? The following results are true.

Lemma 1. *Let $i \geq \frac{1}{\epsilon_0^2}$ and the assumption (A) hold. Then each trajectory $x(t)$ of the system (1.8) with the initial condition $x(0) = x_1 : g(x_1) \leq 0$ and a fixed available control function $u(t)$ satisfies the inequality $ih^2(x(t)) < 1 \quad \forall t \in [0, T]$.*

Proof. On contrary, let there exist a moment $\hat{t} \in [0, T]$ such that $ih^2(x(\hat{t}) \geq 1$. Since $ih^2(x)$ is continuous and $ih^2(x(0)) = ih^2(x_1) = 0$ then there is a minimal $t_* \in (0, T]$ such that $ih^2(x(t_*)) = 1$ and, hence, for any $\epsilon > 0$ there is $\delta > 0$ such that $1 - \epsilon \leq ih^2(x(t)) < 1 \qquad \forall\, t \in [t_* - \delta, t_*)$. Since $i \geq 1/\epsilon_0^2$ then $0 \leq g(x(t)) \leq \epsilon_0$ for $\forall\, t \in [t_* - \delta, t_*)$. Using the properties (see details in the book by Demyanov and Rubinov [10]) of the function $h(x(t))$ we can calculate the one-sided derivative d^+h/dt for all $t \in [t_* - \delta, t_*)$

$$\frac{d^+}{dt}[ih^2(x(t)] = 2ig(x(t))g'_{x(t)}(\dot{x}(t)) \tag{1.9}$$

Estimate using (1.4) we have

$$\frac{d^+}{dt}[ih^2(x(t)] = 2ig(x(t))(1 - ih^2(x(t))g'_{x(t)}(f(x(t), u(t), t)) \leq 2ig(x(t))\alpha\epsilon \leq 0 \tag{1.10}$$

The obtained inequality contradicts to the given condition

$$1 = ih^2(x(t_*)) = \max\{ih^2(x(t)) : \; t \in [t_* - \delta, t_*]\}. \tag{1.11}$$

since we have the function $ih^2(x(t))$ that does not increase on the interval $[t_* - \delta, t_*]$, and, hence, $ih^2(x(t) \geq 1 \; \forall t \in [t_* - \delta, t_*]$ is false since t_* is the minimal time where $ih^2(x(t_*)) = 1$. Lemma is proved.♦

We can rewrite it in a more general form

$$\frac{dx(t)}{dt} = H(x, u, t), \;\; u(t) \in U, \quad \text{a.e. } t \in [0, T], \quad x(0) = x_0 \tag{1.12}$$

where the choice of H can be used to improve the required properties of the produced approximation. In general, we may vary the right-hand side of the differential equation (1.8) in the wide margins. In particular, the differential equality (1.8) can be replaced by a differential inclusion that gives the wide margin to use this choice to guarantee the required approximation properties. As an example, we present the following modification. Assume that there are constants $\alpha < 0$, $\epsilon_0 > 0$, and a continuous function $r(x)$ such that the following inequality

$$g_x^{'}(r(x)) \leq \alpha \qquad \forall x : 0 \leq g(x) \leq \epsilon_0 \tag{1.13}$$

holds. Now consider the sequence of the time optimal control problem (1.7)-(1.8) where the differential equation (1.8) is replaced by the following

$$\frac{dx(t)}{dt} = (1 - ih^2(x)) f(x, u, t) + ih^2(x) r(x). \tag{1.14}$$

For example, the choice of the function $r(x)$ is used to improve the properties of the produced differential equation. It is shown that the statement of the Lemma 1 is also true in this case. The corresponding changes of the proof after (1.9)) are given as follows :

$$\frac{d^+}{dt}[h^2(x(t)] = 2ig(x(t)) \left[g_{x(t)}^{'}(f(x(t), u(t), t))(1 - h^2(x(t)) + h^2(x) g_{x(t)}^{'} r(x(t)) \right]. \tag{1.15}$$

Since the functions $f(x, u, t)$, $x(t)$, $g_x^{'}(l)$ are continuous then there is a constant $M > 0$ such that $g_{x(t)}^{'}(f(x(t), u(t), t)) \leq M \qquad \forall t \in [0, T]$. Hence, choosing the corresponding $\epsilon > 0$ and $\delta > 0$, we have again

$$\frac{d^+}{dt}[h^2(x(t)] = 2ig(x(t)) \left[\epsilon M + \alpha(1 - \epsilon) \right] \leq 0 \quad t \in [t_* - \delta, t_*]. \tag{1.16}$$

And the required statement is obtained.

Theorem 1. *Let the given assumptions in Lemma 1 and the following condition*

$$|f(x, u, t)| \leq M \quad \forall \; x \in \mathbb{R}^n, \forall u \in \mathbb{R}^r, \forall t \in [0, T]$$

hold with $M > 0$. Then for any fixed control $u(t)$ the trajectory of the system (1.8) with the initial condition $x(0) = x_1 : g(x_1) \leq 0$ satisfies the following estimate

$$\rho(x(t), G) \leq -\frac{M}{\alpha} \sqrt{\frac{1}{i}} \quad \forall t \in [0, T]. \tag{1.17}$$

starting from some $i > i_0$, where $\rho(x, G) = \min_{y \in G} \|x - y\|$ is the distance between the point x and the set G.

Proof. It follows from Lemma 1 that for any $t \in [0,T]$ the inequality $ih^2(x(t)) < 1$ holds starting from $i \geq \frac{1}{\epsilon_0^2}$, and, hence, $g(x(t)) \leq \sqrt{\frac{1}{i}}$. In which case for any ϵ_*, $0 \leq \epsilon_* \leq \epsilon_0$ the inequality $g(x(t)) \leq \sqrt{\frac{1}{i}} < \epsilon_*$ holds for all $t \in [0,T]$, starting from some $i > i_0$. If $g(x(t)) \leq 0 \quad \forall t \in [0,T]$ then $\rho(x,G) = 0$ since $x(t) \in G \quad \forall t \in [0,T]$. Let it be that the inequality $g(x(t)) \leq 0$ is not fulfilled on the interval $[0,T]$. Pick an arbitrary $\tau \in [0,T]$ where $g(x(\tau)) > 0$. Consider now the following Cauchy problem

$$\dot{y} = f(y,u(t),t), \ y(0) = x(\tau), \ t \geq 0 \tag{1.18}$$

where $u(t)$ is the control function corresponding to the given trajectory $x(t)$. This problem has a unique solution defined on the interval $[0,T]$. Since $g(y(0)) = g(x(\tau)) > 0$ and the function $g(y)$ is continuous then there is a small $\epsilon > 0$ such that $g(y(t)) \geq 0$ for almost all $t \in [0,\epsilon]$, and, due to the assumption (A)

$$g'_{y(t)}(f(y(t),u(t),t)) \leq \alpha < 0 \tag{1.19}$$

for all $t \in [0,\epsilon]$. Then calculating the directional derivative of the function $g(y(t))$ yields:

$$\frac{d^+}{dt}[g(y(t)] = g'_{\dot{y}(t)}(y(t) = g'_{y(t)}(f(y(t),u(t),t)) \leq \alpha < 0, \ t \in [0,\epsilon]. \tag{1.20}$$

Since $\frac{d^+}{dt}[g(y(t)] \leq \alpha$ for all $t \in [0,\epsilon]$ with $\alpha < 0$ then integrating the last differential inequality along some direction yields the following estimate

$$g(y(\epsilon)) \leq g(x(\tau)) + \epsilon \cdot \alpha \tag{1.21}$$

for the function $g(y(t))$. This yields that $\exists\ \hat{\tau}, \ 0 < \hat{\tau} \leq -\frac{1}{\alpha}g(x(\tau))$ such that $g(y(\hat{\tau})) = 0$. The last says that $y(\hat{\tau}) \in G$. Hence, integrating the system (1.18) leads to the following estimate

$$\rho(x(\tau),G) = \min_{y:g(y)\leq 0} \|x(\tau) - y\| \leq \|x(\tau) - y(\hat{\tau})\| \leq$$

$$\leq \|x(\tau) - x(\tau) - \int_0^{\hat{\tau}} f(y,u,t)dt\| \leq M\hat{\tau} \leq -\frac{M}{\alpha}g(x(\tau)) \leq -\frac{M}{\alpha}\sqrt{\frac{1}{i}} \tag{1.22}$$

for all $t \in [0,T]$ and $i \geq i_0$ for some integer i_0. The proof is completed. ♦

1.3 Discrete approximation

In this section we present a short description of a possible discretization scheme for the considered nonsmooth optimization problem. In the Introduction it is noted that a well-posed discrete approximation based on finite differences can be achieved if some perturbations of the state constraints are admitted in the produced discrete models. In this section, base on the the approach proposed by Mordukhovich [46] we construct a well-posed discrete model for the control model (1.1)-(1.2) with nonsmooth state constraints (1.3).

Replace the derivatives in (1.2) by the Euler finite difference

$$\dot{x}(t) \approx \frac{1}{h}[x(t+h) - x(t)] \text{ as } h \to 0 \tag{1.23}$$

Given $N = 1, 2, 3, \ldots$, let $T_N \doteq \{0, h_N, 2h_N, \ldots, T - h_N\}$ be a uniform grid on $[0, T]$ with the step size $h_N \doteq \frac{T}{N}$, and let

$$x_N(t + h_N) = x_N(t) + h_N f(x_N(t), u_N(t), t) \text{ for } t \in T_N, \ N = 1, 2, \ldots \tag{1.24}$$

be an associated sequence of discrete equations. The state constraints (1.3) are replaced by the following disturbed discrete analogous ones

$$g(x(t)) \leq \epsilon_N. \tag{1.25}$$

We say that the sequence of the problems of (1.1), (1.24), (1.25) is a discrete approximation of the problems (1.1), (1.2), (1.3) if $h_N \to 0$ and $\epsilon_N \to 0$ as $N \to \infty$.

Our purpose is to state the condition that guarantees the convergence of the optimal trajectories and the optimal criteria values for the given discrete approximation at $N \to \infty$.

First, we establish that any admissible trajectory of (1.2) can be uniformly approximated by a sequence of the discrete trajectories of (1.24). This can be done on the basis of the known results of optimal control theory. Next, using the results obtained by Mordukhovich [46] we show that the so-called relaxation stability property is sufficient for the value convergence of discrete approximation under the perturbation of the state constraints. It should be noted that the requirements of the state constraints perturbation in the discrete scheme is essential for the value convergence (some details and corresponding counterexamples can be found in the book by Mordukhovich [46]).

Let $x(t)$, $t \in T_N$ be a trajectory of the discrete equation (1.24), and for any $t \in [0, T]$ denote by t^N and t_N the points of the grid T_N nearest to t from left and right, respectively. Consider the following piecewise-linear extension of the discrete trajectories (the so-called Euler's broken line)

$$x_N(t) = x_N(t^N) + \frac{1}{h_N}[x_N(t_N) - x_N(t^N)](t - t^N) \text{ for } t \in [0, T] \tag{1.26}$$

The following result for the pointwise convergence of the extended trajectories is true.

Lemma 2. *Let $x(t)$, $t \in [0, T]$ be an admissible absolutely continuous trajectory of (1.2). Then for any partition T_N of the interval $[0, T]$ with $h_N \to 0$ as $N \to \infty$ there exists a subsequence $\{x_N(t)\}$, $t \in T_N$, of the admissible solutions of the discrete equation (1.24), the piecewise-linear extensions (1.26) of which converge uniformly to $x(t)$ on the interval $[0, T]$.*

The proof of this and next Lemmas are based on the results of [46]. Some analogies of them are given in the next Chapter for the min-max constraints.

A well-posed approximation ensuring a correct convergence of the optimal discrete trajectories of (1.1), (1.24)- (1.25) to the optimal solution of the original problem (1.1)-(1.3) exploits the following *relaxation stability property.* Along with the optimization problem (1.1)-(1.3) we consider the following relaxation (in the Gamkrelidze form): minimize the cost functional (1.1) over the set of couples of measurable functions $\{\alpha_i(t), u_i(t),\ i = 1, 2, \dots, n+1\}$ and the set of absolutely continuous trajectories $x(t),\ t \in [0,T]$ which satisfy the constraints (1.3) and the following convexified differential equations

$$\frac{dx(t)}{dt} = \sum_{i=1}^{n+1} \alpha_i(t) f(x, u_i, t), \quad \text{a.e.} \quad t \in [0,T],\ x(0) = x_0,$$

$$\alpha_i(t) \geq 0,\ \sum_{i=1}^{n+1} \alpha_i(t) = 1,\ u_i(t) \in U,\ i = 1, 2, \dots, n+1 \tag{1.27}$$

$J_C^0,\ J_R^0,\ J_N^0,\ N = 1, 2, \dots$ denote the minimal values of the cost functional (1.1) in the problems (1.2)-(1.3), (1.27) and (1.24)-(1.25), respectively.

It is said that the original optimization problem (1.1)-(1.3) is *stable with respect to relaxation* if $J_C^0 = J_R^0$.

This property is connected with the so-called *hidden convexity* of the nonconvex differential systems and it holds for the wide class of control systems such as linear systems, nonlinear systems in the absence of the state constraints and some others. Thus, the required value convergence is given by the following

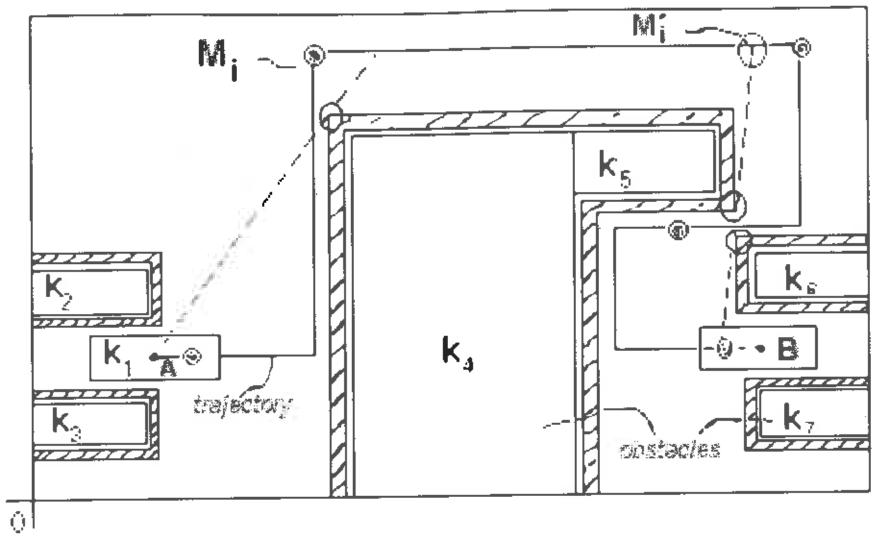
Lemma 3. *Assume the problem (1.1)- (1.3) is stable with respect to relaxation. Then there is a sequence of perturbations $\epsilon_N \to 0$ as $N \to \infty$ in (1.24), (1.25) such that one has the value convergence* $\lim\limits_{N \to \infty} J_N^0 = J_C^0$.

The proof can be obtained similarly to that given in the next Section.

1.4 Distance function in a robot path planning motion

In this section we show how the nonsmooth optimization problem of the form (1.1)-(1.3) arises in the planning motion of a two dimensional robot mechanism. We formulate the path -trek problem for the mechanism that could cross the given domain with a safety clearance along the given obstacles. The figure bellow shows a crane manipulator K_1

The task is to remove the payload from the initial position A to the final position B, avoiding the given obstacles $K_2, \dots, K_7$. Some aspects of the problem statement for this problem were discussed by Gilbert and Jonson [34], Polak and C. Neto [38] (other details of this model will also be given in Section 5). In the considered case the first stage of the path planning is to test some variants of the track by putting the sets $M_i,\ i = 1, \dots, m$ in critical places to guarantee a safe distance, and that are on the obligatory track for the path used by the crane (these sets are marked on the figure by the circles). Thus, the solution of the original path planning problem in the presence of the given obstacles is replaced by a relaxed problem with intermediate constraints. Such problem statements



can be attractive in the initial stage of the principal planning. It is conjectured that the relaxed problem admits the effective numerical solution for multiple repetition with different variants of intermediate constraints. The obtained trajectory can be used then as a starting position for the various improving procedures. These procedures can be accompanied by the formulation of suitable optimization problems.

The robot model under consideration is described as follows

$$m^i \ddot{q}^i(t) + c^i \dot{q}^i(t) = h^i u^i(t), \; i = 1, 2 \tag{1.28}$$

where m^i are the inertias of the uncoupled axes, c^i, h^i are the parameters of the dc motor drives, and u^i are the bounded motor control voltages $|u^1(t)| \leq \alpha$, $|u^2(t)| \leq \beta$. The beginning and ending points A, B give $q(0) = q_A$, $q(T) = q_B$ and $\dot{q}(0) = \dot{q}(T) = 0$. The energy required to transfer the payload in a fixed time T is

$$J(u) = \int_0^T \sum_{i=1}^{2} [(u^i(t))^2 - \gamma^i u^i(t) \dot{q}^i(t)] dt, \tag{1.29}$$

where $\gamma^i > 0$ are motor parameters. To the optimal control problem (1.28)-(1.29) we add special geometric constraints corresponding to the avoidance of the obstacle $K_i, i = 1, \ldots, m$, where K_i are some closed convex sets. These state constraints follow from the obvious requirement that the robot does not crash (that means to not touch any part of obstacles while "travelling" between them). This requirement can be presented in general form:

$$\max_{t \in [0,T]} \max_{1 \leq i \leq m} \{d_i^* - d_i(x(t))\} \leq 0 \tag{1.30}$$

where $d_i(x(t))$ denotes the distance between robot and $i - th$ obstacle at the moment t, and d_i^* denotes the safety distance between mechanism and $i - th$ obstacle. The model above we present in standard form

$$\begin{cases} \dot{x}_1 = x_2, \ \dot{x}_2 = -a_2 x_3 + b_2 u_1, \\ \dot{x}_3 = x_4, \ \dot{x}_4 = -a_4 x_4 + b_4 u_2, \ t \in [0,T] \end{cases} \tag{1.31}$$

where $q_1(t) = x_1(t), \dot{q}_1(t) = x_2(t), \ q_2(t) = x_3(t), \ \dot{q}_2(t) = x_4(t)$ and $a_2 = \frac{c_1}{m_1}, \ a_4 = \frac{c_2}{m_2}, \ b_2 = \frac{h^1}{m^1}, b_4 = \frac{h^2}{m^2}$. The geometric constraints connected with the obstacle $K_i, i = 1, \ldots, m$ can be represented by the corresponding closed and convex sets $\Omega_i \subset \mathbb{R}^4, \ i = 1, \ldots, m$ the interiors of which can be empty, in general. It is known that for each $i = 1, \ldots, m$ the distance function

$$d_i(x) = \min_{y \in \Omega_i} \|x - y\| = \|x - y_i(x)\| = \begin{cases} \sqrt{(x - y_i(x), x - y_i(x))}, & \text{if } x \notin \Omega_i \\ 0, & \text{if } x \in \Omega_i \end{cases} \tag{1.32}$$

is a convex and directional differentiable function. (Here $(\cdot, \cdot)$ means an inner product in R^4 and $y_i(x)$ denotes the element $y \in \Omega_i$ where the maximum in (1.32) is achieved). In addition, their directional derivative at the point $x = x_0$ is given as

$$(d_i)'_x(x_0) = \max_{v \in \partial d_i(x_0)} (v, l) \tag{1.33}$$

where the subdifferential $\partial d_i(x_0)$ is given by

$$\partial d_i(x_0) = \begin{cases} (y_i(x_0) - x_0) d_i^{-1}(x_0), & \text{if } x_0 \notin \Omega_i \\ (-\Gamma_i^+(x_0) \cap B_1), & \text{if } x_0 \in \Omega_i \end{cases} \tag{1.34}$$

and $\Gamma_i^+(x_0) = \{z \in \mathbb{R}^4 : (x - x_0, z) \leq 0, \ \forall x \in \Omega_i\}$, $B_1 = \{x \in \mathbb{R}^4 : \|x\| \leq 1\}$ (details see V. Demyanov and L. Vasiljev [11]).
Introduce the function

$$g(x) = \max_{1 \leq i \leq m} \{d_i^* - d_i(x)\} \tag{1.35}$$

Since the functions $d_i(x), \ i = 1, \ldots, m$ are directionally differentiable then the function $g(x)$ is also directionally differentiable and their derivative along the direction $l \in R^n$ is given by the formula

$$g'_x(l) = \max_{i \in Q(x)} \{-(d_i)'_x(l)\}, \qquad \text{where } Q(x) = \{i : \ 1 \leq i \leq m, \ g(x) = d_i^* - d_i(x)\} \tag{1.36}$$

Hence, the robot motion optimization problem can be rewritten in the required form of (2.2), (2.3):

$$\text{minimize} J(u) = \int_0^T \sum_{i=1}^{2} [(u^i(t))^2 - \gamma^i u^i(t) x^{2i}(t)] dt, \tag{1.37}$$

over the solution of the differential equations (1.31) subject to the nonsmooth state constraint

$$x(t) \in G, \ t \in [0,T], \ \text{where } G = \{x \in \mathbb{R}^4 : g(x) \leq 0\} \tag{1.38}$$

where the function $g(x)$ is given by the formula (1.35). In accordance with the proposed approach for the formulated problem we consider the following sequence of the approximating optimal control problems : for each $i = 1, 2, \ldots$ minimize the cost functional (1.37) over the solution of the following differential equations

$$\dot{x} = (1 - ih^2(x))f(x, u, t), \; x \in \mathbb{R}^4, \quad t \in [0, T] \tag{1.39}$$

where $h(x) = \max\{0, g(x)\}$, $f = (f_1, \ldots, f_4)$ with $f_1 = x_2$, $f_2 = -a_2x_3 + b_2u_1$, $f_3 = x_4$, $f_4 = -a_4x_4 + b_4u_2$. In this case the required regularity conditions of (1.4) are

$$\max_{i \in Q(x)} \{-(d_i)'_x(f(x, u, t))\} \leq \alpha < 0 \tag{1.40}$$

and they indicate on the request to avoid a lengthy presence in the prohibited zone $0 \leq g(x) \leq \epsilon_0$. This condition depends on the form of the set Ω_i. Hence, in order to guarantee the needed property it is sufficient to make a slight variation to these sets.

Thus the primary goal to leave out the state constraint in the considered robot motion model is achieved. Now the modified optimization problem can be solved by using the corresponding optimal control "nonsmooth" software.

1.5 Conclusions

This Chapter has used a continuous time approximation to develop a method to solve optimization problems with nonsmooth state constraints of a general form. The major advantage of using the proposed approximation is that it eliminates the need for solving a potentially very large collection of the constrained nonlinear programming problems which usually arise under standard approximation schemes. We present the theoretical background to construct a scheme with trajectory convergence. The modified optimization problem can be solved by using the corresponding optimal control "nonsmooth" software. It is conjectured that our approach accompanied by modern methods of nonsmooth optimization (Clarke [8], Demyanov and Rubinov [10, 61]), computational theory for optimal control (Teo [64]) and some results for optimization of special repetitive processes (see Dymkou *et al*) will be effective for the considered optimal control problems with state constraints.

In the next Chapter we consider a special case of nonsmooth constraints represented by min-max type functions for which the so-called well-posed discrete approximation is given.

Chapter 2

Discrete approximations for the optimal control problem with max-min constraints

This chapter is concerned with some aspects of approximation for a nonsmooth optimal control problem with state constraints of a special min-max kind. We discuss the idea of the perturbations of the state constraints for a well-posed discrete approximation ensuring the convergence of optimal solutions. We use these results for robot models described by linear stationary differential equations with linear inputs. A numerical method based on the discrete gradient method is proposed for the particular case of the considered model.

2.1 Preliminary notations and definitions

We consider the following optimal control problem : minimize

$$J(u) = \varphi_0(x(0), x(T)) \rightarrow \max_{u(\cdot)} \tag{2.1}$$

over the absolutely continuous trajectories $x : [0, T] \rightarrow \mathbb{R}^n$ of the differential equation

$$\frac{dx(t)}{dt} = f(x, u, t), \ x(0) = x_0, \ u(t) \in U, \quad \text{a.e.} \quad t \in [0, T] \tag{2.2}$$

under state constraints of the form

$$\max_{t \in [0,T]} \min_{1 \leq i \leq m} \varphi_i(x(t), t) \leq 0 \tag{2.3}$$

where $\varphi_0 : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ and $\varphi_i : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ are given continuous functions, x_0 is the given $n-$vector. We suppose that

i) the function $f : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R} \rightarrow \mathbb{R}^r$ satisfies to the Caratheodory conditions (i.e. f is continuous on (x, u) and measurable on t) such that the initial Cauchy problem for the differential equation (2.2) has an unique absolutely continuous solution.

Definition 2. *We say that the function $u : T \to \mathbb{R}^r$ is admissible for (2.2) if it is measurable and satisfies the constraint $u(t) \in U$ almost everywhere $t \in T$, where U is a given compact set from $\mathbb{R}^r$. We say that the function $x : T \to \mathbb{R}^n$ is an admissible solution of (2.2) corresponding to the given admissible control $u(t)$ if it is absolutely continuous with respect to $t \in T$ and satisfies (2.2) for almost all $t \in T$.*

Next we suppose that the set of admissible controls $U(\cdot)$ is nonempty. In addition, we also suppose that

ii) the sets of admissible trajectories of (2.2) is bounded, i.e. these trajectories belong to a ball $B_r = \{x \in \mathbb{R}^n : \|x\| \leq r\}$, $0 < r < \infty$. In order to guarantee the last requirement we assume that the condition $|f(x, u, t)| \leq \mu(t) g(|x|)$ holds, where $\mu(t)$ is summable and $g(|x|) = O(|x|)$ at $|x| \to \infty$ [60].

The state constraints (2.3) demand the constructing of a *correct* discrete approximation that ensures converge results for optimal solutions.

To emphasize this we demonstrate the following counterexample.

2.1.1 Counterexample

In general, the discrete approximation schemes that are usually used to develop numerical methods, for optimization in the presence of state constraints demand careful consideration to ensure, at least, the convergence of discrete trajectories and cost values to the desired optimal values. To emphasize this point we demonstrate the following counterexample, where the typical discretization leads to incorrect results. Namely, this example shows that the cost value convergence is absent.

Consider the problem [46]

$$J(u) = -x(1) \to \min_u \tag{2.4}$$

$$\frac{dx(t)}{dt} = u, \quad x(0) = 0 \in R$$
$$u(t) \in U = \{0, \sqrt{2}\}, \quad t \in [0,1] \tag{2.5}$$

under the state constraint

$$x(1) \in \Omega = \{0, 1\} \tag{2.6}$$

Optimal control is

$$u^0(t) = \begin{cases} \sqrt{2} \text{ for } t \in T_0 \subset [0,1], \ \text{mes } T_0 = 1/\sqrt{2}, \\ 0, \ \text{for } t \in [0,1] \setminus T_0 \end{cases}$$

and

$$J^0(u^0) = -1$$

Consider a uniform grid on $[0,1]$ with step size $h_N = 1/N$ and the corresponding discrete approximation

$$x_N(t+h_N) = x_N(t) + h_N u_N(t), \ x_N(0) = 0$$
$$u_N(t) \in \{0, \sqrt{2}\}, \quad t \in \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\}$$

and

$$J_N(u) = -x_N(1) \to \min_u$$

under the same state constraint

$$x_N(1) \in \Omega = \{0, 1\} \tag{2.7}$$

It is easy to verify that for any

$$u_N(t) \in \{0, \sqrt{2}\} \longrightarrow x_N(1) = \frac{k}{N}\sqrt{2}$$

where k is some integer $k \leq N$. Thus $x_N(t) \neq 1$ for all $u_N(t)$ and, hence, an unique admissible control function for the discrete problem (2.7) is $u_N(t) \equiv 0$. Therefore,

$$J_N^0 = 0 \neq -1 = J^0$$

This means that discrete approximation (2.7) *is not a value convergence* for the original problem (2.4)-(2.6).

Thus, the key point for the value non-convergence is an incorrect approximation of the state constraint that gives *motivation* to investigate the well-posed discrete approximation for the optimization problem with nonsmooth state constraints

2.2 Discrete approximation

In this section, based on the approach proposed in [46], we construct a well-posed discrete model for the original problem (2.1)-(2.3), where the unique peculiarity is generated by the state constraints of (2.3). An essential and crucial question for both theoretical and numerical aspects of discrete approximations is as follows : can one approximate, and in which sense, an optimal solution of the continuous control system by discrete solutions? In order to achieve a well-posed approximation ensuring the convergence of optimal solutions we need to admit, in general, the state constraints perturbations in the discrete model.

Replace the derivatives in (2.2) by the Euler finite difference

$$\dot{x}(t) \approx \frac{1}{h}[x(t+h) - x(t)] \text{ as } h \to 0 \tag{2.8}$$

Given $N = 1, 2, 3, \ldots$, let $T_N \doteq \{0, h_N, 2h_N, \ldots, T - h_N\}$ be a uniform grid on $[0, T]$ with the step size $h_N \doteq \frac{T}{N}$, and let

$$x_N(t + h_N) = x_N(t) + h_N f(x_N(t), u_N(t), t) \text{ for } t \in T_N,\ N = 1, 2, \ldots \tag{2.9}$$

be an associated sequence of discrete equations. The state constraints (2.3) are replaced by the following disturbed discrete analogous ones

$$\max_{t \in T_N} \min_{1 \le i \le m} \varphi_i(x(t), t) \le \epsilon_N. \tag{2.10}$$

Definition 3. *We say that the sequence of the problems of (2.1), (2.9), (2.10) is a discrete approximation of the problem (2.1), (2.2), (2.3) if $h_N \to 0$ and $\epsilon_N \to 0$ as $N \to \infty$*

We suppose also that the condition ii) also holds for the discrete trajectories of (2.9) if $N > N_0$ is big enough.

The aim of this paper is to find the condition that guarantees the convergence of the optimal trajectories and optimal criteria values for the given discrete approximation at $N \to \infty$.

Firstly, we establish that any admissible trajectory of (2.2) can be uniformly approximated by a sequence of the discrete trajectories of (2.9). Next using the results [46] we show that the so-called relaxation stability property is sufficient for the value convergence of the discrete approximation under the correct perturbation of the state constraints. It should be noted that the requirements of the state constraints perturbation in the discrete scheme is essential for the value convergence (some details and corresponding counterexamples can be found in [46]).

Let $x(t),\ t \in T_N$ be a trajectory of the discrete equation (2.9), and for any $t \in [0, T]$ denote by t^N and t_N be the points of the grid T_N nearest to t from left and right, respectively. Consider the following piecewise-linear extension of the discrete trajectories (the so-called Euler's broken line)

$$x_N(t) = x_N(t^N) + \frac{1}{h_N}[x_N(t_N) - x_N(t^N)](t - t^N) \text{ for } t \in [0, T] \tag{2.11}$$

The following result for the pointwise convergence of the extended trajectories is true.

Lemma 4. *Let the given assumptions i)-ii) hold and $x(t),\ t \in [0, T]$ be an admissible absolutely continuous trajectory of (2.2). Then for any partition T_N of the interval $[0, T]$ with $h_N \to 0$ as $N \to \infty$ there exists a subsequence $\{x_N(t)\},\ t \in T_N,$ of the admissible solutions of the discrete equation (2.9), the piecewise-linear extensions (2.11) of which converge uniformly to $x(t)$ on the interval $[0, T]$.*

Proof. The sketch of the proof below follows, in the main [45]. Firstly, we consider the case where the control function $u(t), t \in [0, T]$ corresponding the given trajectory $x(t), t \in [0, T]$ is continuous almost everywhere on the interval $[0, T]$. For the given control function $u(t),\ t \in [0, T]$ constructs the following discrete control function $u_N(t) = u(t),\ t \in T_N,\ N = 1, 2, \ldots$, and prove that the corresponding discrete solutions $x_N(t)$ of the discrete equation (2.9) converges uniformly to $x(t)$ on the interval $[0,T]$. From (2.9) and (2.11) it follows that

$$\frac{dx_N(t)}{dt} = f(x_N(t^N), u_N(t^N), t^N), \quad t \in [0,T] \setminus T_N \tag{2.12}$$

Under the given assumption, the sequence $\{x_N(t),\ t \in [0,T],\ N = 1,2,\ldots\}$ is a uniformly bounded and equicontinuous set of functions. In accordance with the Arzela theorem, this set is precompact, and hence in this set there exists a uniformly convergent subsequence that without loss of generality is rewritten by the same letter. $x^*(t)$ denotes their limiting elements in the space $C[0,T]$ of all continuous functions on $[0,T]$ and show that $x^*(t) \equiv x(t)$.

Consider the functions of the form $\eta_N(t) = f(x_N(t^N), u_N(t^N), t^N)$ for all $t \in [0,T]$. Since the function $u(t)$ is continuous almost everywhere on $[0,T]$ then in accordance with the given discretization method for $u_N(t)$ we show that the sequence $\{\eta_N(t)\}$ converges almost everywhere to the function $f(x^*(t), u(t), t)$. Hence, taking into account condition ii), we can apply the Liebeg theorem about the limit passage under the integral sign in the following equality

$$x_N(t) = x_0 + \int_0^t \eta_N(s)ds), \quad t \in [0,T] \tag{2.13}$$

This yields that the function $x^*(t)$ is the solution of the equation (2.9) corresponding to the given control function $u = u(t)$. The required equality $x^*(t) \equiv x(t)$ follows immediately from the uniqueness assumption i) that proves the theorem for the case when the control function $u(t), t \in [0,T]$ is continuous almost everywhere on the interval $[0,T]$.

The general case of the measurable control function is reduced to the considered case on the basis of the following proposition.

For any control function $u(t)$ satisfying to the constraint $u(t) \in U$ a.e. $t \in [0,T]$ there exists a sequence of the functions $\{u_n(t)\}$, $u_n(t) \in U$ a.e. $t \in [0,T]$ which are continuous almost everywhere on $[0,T]$ such that $u_n(t) \to u(t)$ converges in some measure μ, i.e. $\forall\ \epsilon > 0 \quad \lim_{n\to\infty} \mu\{t : \|u(t) - u_n(t)\| \geq \epsilon\} = 0$. Moreover, the corresponding trajectories $x_n(t)$ converge uniformly to the function $x(t)$.

To prove this proposition we use the $C-$ property of the measurable function stated by the Luzin theorem. According to this property there exists a sequence of the positive numbers $\epsilon_k \to 0,\ k = 1,2,\ldots$ and the corresponding closed sets T_{ϵ_k} such that $\mathrm{mes}(T \setminus T_{\epsilon_k}) \leq \epsilon_k$ and the restriction of the given function $u(t)$ on each T_{ϵ_k} is continuous.

The sets $T \backslash T_{\epsilon_k}$ can be represented by a denumerable number of the disjoint intervals $(\alpha_{jk}, \beta_{jk}),\ j = 1,2,\ldots$. Consider now the functions

$$u_k(t) = \begin{cases} u(t), & t \in T_{\epsilon_k},\ k = 1,2,\ldots \\ u(\alpha_{jk}), & t \in (\alpha_{jk}, \beta_{jk}),\ j = 1,2,\ldots \end{cases} \tag{2.14}$$

From (2.14) it follows that the functions $u_k(t)$ can be only discontinued at the moments $t = \beta_{jk}$, $j = 1, 2, \ldots$. Moreover, the following inequality $\text{mes}\{t : u(t) \neq u_k(t)\} \leq \epsilon_k$ is fulfilled which gives the uniform convergence of $u_k(t) \to u(t)$ in the measure.

Thus, the desired sequence of $\{u_k(t)\}$, $k = 1, 2, \ldots$ is stated. The required uniform convergence of the corresponding trajectories follows again from the Liebig theorem [7]).

The lemma is proved. ♦

A well-posed approximation ensuring a convergence of the optimal discrete trajectories of (2.1), (2.9)- (2.10) to the optimal solution of the original problem (2.1)-(2.3) exploits the following *relaxation stability property* [33, 46].

Along with the optimization problem (2.1)-(2.3) we consider the following relaxation (in the Gamkrelidze form): minimize the cost functional (2.1) over the set of couples of measurable functions $\{\alpha_i(t), u_i(t),\ i = 1, 2, \ldots, n+1\}$ and the set of absolutely continuous trajectories $x(t)$, $t \in [0, T]$ which satisfies the constraints (2.3) and the following convexified differential equations

$$\frac{dx(t)}{dt} = \sum_{i=1}^{n+1} \alpha_i(t) f(x, u_i, t),\ x(0) = x_0,$$

$$\alpha_i(t) \geq 0,\ \sum_{i=1}^{n+1} \alpha_i(t) = 1,\ u_i(t) \in U, \quad \text{a.e.} \quad t \in [0, T],\ i = 1, 2, \ldots, n+1 \tag{2.15}$$

J_C^0, J_R^0, J_N^0, $N = 1, 2, \ldots$ denote the minimal values of the cost functional (2.1) in the problems (2.2)-(2.3), (2.15) and (2.9)-(2.10), respectively.

It is said that the original optimization problem (2.1)-(2.3) is *stable with respect to relaxation* if $J_C^0 = J_R^0$.

This property is connected with the so-called *hidden convexity* of nonconvex differential systems and it holds for the wider class of control systems such as linear systems, nonlinear systems in absence of state constraints, etc.

Lemma 5. *Assume in addition to i)-ii) that the function φ_i, $i = 0, 1, \ldots, m$ are continuous and the problem (2.1)- (2.3)is stable with respect to relaxation. Then there is a sequence of perturbations $\epsilon_N \to 0$ as $N \to \infty$ in (2.9), (2.10) such that one has a value convergence $\lim_{N\to\infty} J_N^0 = J_C^0$.*

Proof. To prove Lemma 5 it is enough to show that under the given assumption the following inequality

$$J_R^0 \leq \lim_{N\to\infty} \inf J_N^0 \leq \lim_{N\to\infty} \sup J_N^0 \leq J_C^0 \tag{2.16}$$

holds. Then the relaxation stability property leads immediately to desired equality $\lim_{N\to\infty} J_N^0 = J_C^0$. Hence, we need to state the inequality (2.16).

Let $\{x_k(t)\}$, $t \in [0, T]$, $k = 1, 2, \ldots$ be the minimizing sequence of admissible trajectories in (2.1). Under the given assumption this set is precompact in the space $C[0, T]$. Hence, there is a subsequence from $\{x_k(t)\}$ that converges uniformly to the absolutely continuous function $x^0(t)$ that realizes the infimum for the cost functional (2.1) in (2.2)-(2.3).

Let $\{T_N\}$, $N = 1, 2, \ldots$ be an arbitrary sequence of the grids on the interval $[0, T]$ with the step size $h_N \to 0$ at $N \to \infty$. Then from Lemma 4 it follows that there is a sequence of admissible discrete trajectories $x_N(t)$, $N \to \infty$ of (2.8)-(2.9), the piecewise-linear extensions (2.11) of which converge uniformly to the given limit function $x^0(t)$ on the interval $[0, T]$.

Consider now a sequence of discrete problems (2.1), (2.9)-(2.10), where the sequence $\{\epsilon_N\}$ is defined as follows

$$\epsilon_N = \max\{\max_{t \in T} \min_{1 \le i \le m} \varphi_i(x^0(t), t), \max_{t \in T} \min_{1 \le i \le m} \varphi_i(x_N(t), t)\} \tag{2.17}$$

Since the functions φ_i are continuous and $x_N(t) \to x^0(t)$ converges uniformly then $\epsilon_N \to 0$ at $N \to \infty$. Next we prove the validity of the right hand side of inequality (2.16).

Suppose now that there exists a subsequence $\{N_1\}$, $N_1 \to \infty$ such that the following inequality

$$\lim_{N_1 \to \infty} J^0_{N_1} > J^0_C = \varphi_0(x^0(0), x^0(T)) \tag{2.18}$$

holds. Since the function φ_0 is continuous and $x_{N_1}(t) \to x^0(t)$ converges uniformly then

$$J^0_n > \varphi_0(x_n(0), x_n(T)) \tag{2.19}$$

for some large numbers n from $\{N_1\}$. From (2.3) and (2.17) follows that the trajectories $x_n(t)$ are admissible for the discrete problem (2.1), (2.9)-(2.10). Due to this, the condition (2.19) is impossible which proves the validity of the right hand side of inequality (2.16).

In order to prove the left hand side of (2.16) it is enough to show that the uniform limit $x(t)$, $t \in [0, T]$ of the admissible discrete trajectories $x_N(t)$, $t \in [0, T]$ of (2.1), (2.9)-(2.10) is an admissible trajectory for the relaxed optimization problem (2.15). Since $\epsilon_N \to 0$ $N \to \infty$ and the functions φ_i are continuous then the inequality (2.10) yields that the limit function $x(t)$ that satisfies the state constraints (2.3). Next we prove that the limit trajectory $x(t)$ satisfies the following inclusion

$$\frac{dx(t)}{dt} \in \operatorname{conv} f(x(t), U, t) \tag{2.20}$$

where $f(x, U, t) = \{y \in R^n : y = f(x, u, t),\ u \in U\}$ and $\operatorname{conv} V$ denotes the convex hull of the set V. Then using the Filipov theorem [46] on measurable selectors to (2.20) yields that there exists a measurable couple $\{\alpha_i(t), u_i(t),\ i = 1, 2, \ldots, n+1\}$ that produces the limit trajectory $x(t)$ in the relaxed differential equation (2.15).

Now we need to prove (2.20). For any $\epsilon > 0$ the following inclusion

$$\frac{dx_N(t)}{dt} = f(x_N(t^N), u_N(t^N), t^N) \in [f(x(t), U, t)]_\epsilon] \tag{2.21}$$

is fulfilled for some $N \geq N_0$ and almost everywhere $t \in [0, T]$, and where $[V]_\epsilon = \{x \in R^n : \inf_{y \in V} \|x - z\| \leq \epsilon$. Further, to state the density of the set of admissible trajectories of the differential inclusion (2.20) we use the classical Mazur theorem [7]. According to this theorem the convex combination of the derivatives $\dot{x}_N(t)$ converges weakly to $\dot{x}(t)$ in the space $L_p[0,T], p \geq 1$. Due to this, we have

$$\frac{dx(t)}{dt} \in \operatorname{conv}[f(x(t),U,t)]_\epsilon \tag{2.22}$$

for all $\epsilon > 0$. In other words the following inclusion

$$\frac{dx(t)}{dt} \in \bigcap_{\epsilon>0} \operatorname{conv}[f(x(t),U,t)]_\epsilon \tag{2.23}$$

is valid. To finish the proof of (2.20) it is sufficient to establish the following equality

$$\operatorname{conv} f(x(t),U,t) = \bigcap_{\epsilon>0} \operatorname{conv}[f(x(t),U,t)]_\epsilon \tag{2.24}$$

It is obvious that $\operatorname{conv} f(x(t),U,t) \subset \bigcap\limits_{\epsilon>0} \operatorname{conv}[f(x(t),U,t)]_\epsilon$.

Now we prove the reverse inclusion

$$\bigcap_{\epsilon>0} \operatorname{conv}[f(x(t),U,t)]_\epsilon \subset \operatorname{conv} f(x(t),U,t).$$

Let $v \in \bigcap\limits_{\epsilon>0} \operatorname{conv}[f(x(t),U,t)]_\epsilon$. Then for any sequence $\epsilon_k \to 0,\ k = 1,2,\ldots$ there exists the sequence $\alpha_i^k, u_i^k,\ i = 1,2,\ldots,n+1;\ k = 1,2,\ldots$ such that

$$v = \sum_{i=1}^{n+1} \alpha_i^k v_i^k,\ v \in [f(x(t),U,t)]_{\epsilon_k},\ \alpha_i^k \geq 0,\ \sum_{i=1}^{n+1} \alpha_i^k = 1. \tag{2.25}$$

Let u_i^k be the elements from the set U that satisfies the following inequality

$$\|f(x,u_i^k,t) - v_i^k\| \leq \epsilon_k,\ i = 1,2,\ldots,n+1;\ k = 1,2,\ldots \tag{2.26}$$

Since the sets $U,\ f(x,U,t),\ P = \{\alpha_i : \alpha_i^k \geq 0,\ \sum\limits_{i=1}^{n+1} \alpha_i^k = 1\}$ are compact sets then there is such a subsequence integers $\{k\},\ k \to \infty$ that

$$u_i^k \to u_i^- \in U,\ \{\alpha_i^k\} \to \{\alpha_i^0\} \in P,\ v_i^k \to v_i^0,\ i = 1,2,\ldots,n+1. \tag{2.27}$$

It follows from (2.25)-(2.26) that

$$v_i^0 = f(x,u_i^0,t),\ i = 1,2,\ldots,n+1, \quad v = \sum_{i=1}^{n+1} \alpha_i^0 f(x,u_i^0,t) \tag{2.28}$$

This completes the proof of (2.24), (2.20) and the validity of the left hand side of (2.16), respectively. The lemma is proved. ♦

2.3 Conclusions

In this Chapter, we constructed a well -posed discrete model for the original problem (2.1)-(2.3), where the unique peculiarity is generated by the state constraints of (2.3). In order to achieve a well-posed approximation ensuring the correct convergence of optimal solutions we need to admit, in general, the state constraints perturbations in the discrete model. In Chapter 5, we use these results for robot models described by linear differential equations with constant coefficients and linear inputs where a numerical method based on the well-posed discretization is proposed for the particular case of the considered model.

In the next Chapter another approach for the solving optimization problems with state constraint is developed. It is proposed to relax the state restriction by putting a collection of preassigned state constraints at the fixed moments in time.

Chapter 3

Optimization problems in the presence of intermediate constraints

An approach to the the planning of the paths of robot mechanisms in the presence of obstacles is proposed. This problem is formulated as an optimization problem with the preassigned intermediate constraints characterizing the desired path of the robot. A simple case of the robot mechanism is considered and the numerical aspects are also discussed. The chapter is arranged as follows. Firstly, we establish the optimality conditions for the control models described by linear nonstationary differential equations with nonlinear inputs and equipped with the intermediate constraints of the general form at the pre-assigned moments. The cost functional is given by the convex function defined on the trajectories of the system at given moments. The next section is devoted to an application of the developed results for the planning of the robotic motion in the two-dimensional case. A simple numerical method is proposed for the particular cases of the model under consideration.

3.1 Problem formulation

We consider the dynamics where the relationship between the control variables $u(t) \in \mathbb{R}^r$ and the object state variables $x(t) \in \mathbb{R}^n$ is described by the following system of linear differential equations with nonstationary coefficients

$$\frac{dx(t)}{dt} = A(t)x(t) + b(u(t), t), \quad t \in T \doteq [0, t_1^*] \tag{3.1}$$

with the initial conditions of the form

$$x(0) = x_0 \tag{3.2}$$

Definition 4. *We say that the function $u : T \to \mathbb{R}^r$ is available for (3.1) if it is measurable and satisfies the constraint $u(t) \in U$ almost everywhere $t \in T$, where U is a given compact set from $\mathbb{R}^r$.*

And, we say that the function $x : T \to \mathbb{R}^n$ is a solution of (3.1) corresponding to the given available control $u(t)$ if it is absolutely continuous and satisfies (3.1) for almost all $t \in T$.

Next we denote the set of available controls by $U(\cdot)$. Let $M_i,\ M_i \subset \mathbb{R}^n,\ i = 1, 2, ..., k$ be given compact convex sets.

Definition 5. *The available control $u(t)$ is said to be admissible if the corresponding solution $x(t) = x(t, x_0, u)$ of (3.1) with the initial condition (3.2) satisfies the conditions*

$$x(\tau_i) \in M_i,\ i = 1, 2, ..., k \tag{3.3}$$

where $0 < \tau_1 < \tau_2 < ... < \tau_k = t_1^*$ are given moments in T.

Next we assume that the set of admissible controls is nonempty (it can be also guaranteed by the relevant controllability assumptions). The optimal control problem is to find the admissible control $u(\cdot) \in U(\cdot)$ that minimizes the cost functional of the form

$$J(u) = \varphi(x(\tau_1), x(\tau_2), ..., x(\tau_k)) \to \min_{u \in U(\cdot)} \tag{3.4}$$

Remark 2. *The cost functional (3.4) for the planning of the robot path are often chosen in the form of the weighting distance function*

$$J(u) = \varphi(x(\tau_1), x(\tau_2), ..., x(\tau_k)) = \sum_{i=1}^{k} \alpha_i \rho(x(\tau_i, x_0, u), M_i) \to \min_{u \in U(\cdot)} \tag{3.5}$$

where $\rho(x(\tau_i, x_0, u), M_i)$ denotes the distance from the trajectory point $x(\tau_i, x_0, u)$ to the set M_i and the positive numbers α_i means the penalty coefficients for the failure requirements of (3.3).

Another effective approach for the study of robot dynamics is to use "minmax" type functions for the description of state constraint.

$$J(u) = \varphi(x(\tau_1), x(\tau_2), ..., x(\tau_k)) = \max_{1 \le i \le k} \rho(x(\tau_i, x_0, u), M_i) \to \min_{u \in U(\cdot)} \tag{3.6}$$

In the both cases the zero cost functional value means that the corresponding trajectory passes through the given sets M_i.

In the problem above we assume that the moments $\tau_i,\ i = 1, \ldots, k$ are fixed. An interesting optimization problem can be formulated in the case when this does not hold. Let Υ be the set of all collections $\tau \doteq \{\tau_1, \ldots, \tau_k\}$ satisfying the inequalities $0 < \tau_1 < \tau_2 < ... < \tau_k = t_1^*$ and Π denotes the set of all permutations $\pi \doteq \{i_1, i_2, \ldots, i_k\}$ of the indexes $1, 2, \ldots, k$. For the given control function $u(\cdot) \in U(\cdot)$ we define the cost functional as

$$J(\tau, \pi, u) = \tau_k - \tau_1 = \min_{\tau \in \Upsilon} \min_{\pi \in \Pi} \sum_{i=1}^{k} (\tau_{i+1} - \tau_i)$$

The optimization problem is now: to find the admissible control $u^0(\cdot) \in U(\cdot)$, the order $\pi^0 = \{i_1^0, i_2^0, \ldots, i_k^0\}$ of path-tracing for the pre-assigned sets M_i, $i = 1, \ldots, k$ starting at the initial moment $t = 0$ from the point $x(0) = x_0$, and the collection $\tau^0 = \{\tau_1^0, \ldots, \tau_k^0\}$ of the corresponding moments such that they satisfy the following condition

$$J(\tau^0, \pi^0, u^0) = \min_{\tau \in \Upsilon} \min_{\pi \in \Pi} \sum_{i=1}^{k} (\tau_{i+1} - \tau_i) \rightarrow \min_{u \in U(\cdot)} \tag{3.7}$$

3.2 Optimality conditions

In this section we consider the optimization problem with the cost functional of (5.4), where the function $\varphi : \mathbb{R}^{nk} \rightarrow \mathbb{R}^1$ is convex.

Next we assume that the $(n \times n)$- matrix function $A(t)$ is measurable and integrable on T, the function $b : U \times T \rightarrow \mathbb{R}^n$ is continuous with respect to $(u, t) \in U \times T$. It is easy to see that these conditions guarantee the existence and uniqueness of an absolutely continuous solution of equation (3.1) satisfying conditions (3.2) for any available control $u(t)$.

The representation of the solution of (3.1) uses the $(n \times n)$-matrix function $\Phi(\tau, t)$ defined by the following equation

$$\frac{d\Phi(\tau, t)}{d\tau} = A(\tau)\Phi(\tau, t), \quad \Phi(t, t) = E. \tag{3.8}$$

As is well known, the entries of the matrix $\Phi(\tau, t)$ are absolutely continuous functions defined on the set $T \times T$. Then the solution of (3.1)-(3.2) at the moment $t = \tau_j$ corresponding to a given available control $u \in U(\cdot)$ can be written in the form

$$x(\tau_j) = \Phi(\tau_j, 0)x_0 + \int_0^{\tau_j} \Phi(\tau_j, t)b(u(t), t)dt, \quad j = 1, 2, ...k, \tag{3.9}$$

Let $c = (c_1, c_2, ..., c_k) \in \mathbb{R}^{nk}$, where

$$c_j = \Phi(\tau_j, 0)x_0 \tag{3.10}$$

and introduce the mapping $S : U(\cdot) \rightarrow \mathbb{R}^{nk}$ as $Su = (S_1u, S_2u, ..., S_ku)$ where

$$S_ju = \int_0^{\tau_j} \Phi(\tau_j, t)b(u(t), t)dt, \; j = 1, 2, ...k. \tag{3.11}$$

Introduce now the following *Auxiliary problem* (A): find the necessary and sufficient conditions for the relation

$$z = c + Su \tag{3.12}$$

to be valid under the constraint

$$z \in M, \ \varphi(z) \le \delta, \ z \in \mathbb{R}^{nk}, \ u \in U(\cdot), \tag{3.13}$$

where $M = M_1 \times M_2 \times ... \times M_k \subset \mathbb{R}^{nk}$, and δ is a fixed number from $\mathbb{R}$.

Let

$$\mathcal{R} = \{z \in \mathbb{R}^{nk}, \ z = c + Su, \ u \in U(\cdot)\}, \ K(\delta) = \{z \in \mathbb{R}^{nk}, \ z \in M, \ \varphi(z) \le \delta\}. \tag{3.14}$$

It is easy to see that the geometric criteria for the solvability of Problem (A) is $\mathcal{R} \cap K(\delta) \neq \emptyset$. Now we establish an analytical form for this geometric criteria. To prove this we use the separation theorem for convex sets. To this end, we study the necessary properties of the sets $\mathcal{R}$ and $K(\delta)$. The main difficulties in this analysis are related to the set $\mathcal{R}$, since under the given assumptions we can easily see that $K(\delta)$ is a convex closed set.

Lemma 6. *The set $\mathcal{R}$ given by (3.14) is closed and convex.*

The proof of the Lemma is discussed in the next chapter, where a more general case of the given preposition is considered.

The next theorem gives the solvability conditions of *Auxilialary problem* (A). Denoted by (g, f) the inner product of vectors $g, \ f$ from $\mathbb{R}^{nk}$.

Theorem 2. *Problem (A) is solvable if, and only if, the following inequality*

$$\max_{\|g\|_{\mathbb{R}^{nk}}=1} \Big\{(g,c) - \max_{z\in K(\delta)} (g,z) + \min_{u\in U(\cdot)} (g, Su)\Big\} \le 0 \tag{3.15}$$

holds.

Proof. *Sufficiency.* Suppose that the condition (3.15) is valid, but the problem (A) has no solution. Then, $\mathcal{R} \cap K(\delta) = \emptyset$. The separation theorem for convex sets yields that there exists the nontrivial vector $g \in \mathbb{R}^{nk}$, $\|g\| = 1$ such that

$$\min_{z\in\mathcal{R}}(g,z) > \max_{z\in K(\delta)} (g,z). \tag{3.16}$$

Hence

$$(g,c) - \max_{z\in K(\delta)} (g,z) + \min_{u\in U(\cdot)} (g, Su) > 0 \tag{3.17}$$

that contradicts the condition (3.15).

Necessity. Suppose that the problem (A) is solvable. Then there exists $\bar{u}$ and $\bar{z}$ that satisfy the conditions (3.12)-(3.13) such that the following equality $(g,c) + (g, S\bar{u}) = (g, \bar{z})$ holds for each vector $g \in \mathbb{R}^{nk}$. Applying the maximum and minimum operations to the last yields the required inequality

$$(g,c) - \max_{z\in K(\delta)} (g,z) + \min_{u\in U(\cdot)} (g, Su) \le 0. \tag{3.18}$$

The proof is complete. ♦

Next we establish the maximum principle for the optimal control problem (3.1)-(3.4) based on the previous analysis.

Introduce the function $\Lambda : \mathbb{R} \to \mathbb{R}$ as follows

$$\Lambda(\delta) = \max_{\|g\|_{R^{nk}}=1} \{(g, c) - \max_{z \in K(\delta)} (g, z) + \max_{u \in U(\cdot)} (g, Su)\}. \tag{3.19}$$

It can be shown that the function $\Lambda(\delta) : \mathbb{R} \to \mathbb{R}$ defined by (3.19) is a non increasing continuous function. Then the optimal value of the performance index can be characterized as follows.

Theorem 3. *The control $u^0 \in U(\cdot)$ is optimal for the problem (3.1)-(3.4) if, and only if, the number $\delta^0 = J(u^0)$ is the minimal root of the equation $\Lambda(\delta) = 0$.*

Proof. *Necessety.* Let $u^0 \in U(\cdot)$ be an optimal control in the problem (3.1)-(3.4). Then u^0 is the solution of the Problem (A) with $\delta^0 = J(u^0)$. Therefore, Theorem 1 yields the inequality $\Lambda(\delta^0) \le 0$. Suppose now that $\Lambda(\delta^0) < 0$. Since $\Lambda(\delta)$ is a continuous and monotone function then there exists a number $\bar{\delta}$ such that $\bar{\delta} < \delta^0$ and $\Lambda(\bar{\delta}) \le 0$. In this case theorem 1 yields that the Problem (A) has a solution with $\delta = \bar{\delta}$. Otherwise, there is an available control $\bar{u} \in U(\cdot)$ and a vector $\bar{z} \in M$ satisfying the relations (3.11)-(3.12) for $\delta = \bar{\delta}$. Hence, $J(\bar{u}) < J(u^0)$, that contradicts the optimality of the control u^0. Therefore, $\Lambda(\delta^0) = 0$. The minimality of the root $\delta^0 = J(u^0)$ for the equation $\Lambda(\delta) = 0$ can be proved by an analogy with the above.

Sufficiency. Let $u^0 \in U(\cdot)$ be a control function such that the number $\delta^0 = J(u^0)$ is the minimal root for the equation $\Lambda(\delta) = 0$. Suppose that the control function $u^0(t)$ is not optimal for the problem (3.1)-(3.4). Then, there exists the available control function $\bar{u} \in U(\cdot)$ and a vector $\bar{z} \in M$ such that the relations $c - \bar{z} + S\bar{u} = 0$ and $J(\bar{u}) < J(u^0)$ hold. This yields that the Problem (A) has a solution for $\bar{\delta} = J(\bar{u})$, and hence, $\Lambda(\bar{\delta}) \le 0$. On the other hand, since the function $\Lambda(\delta)$ is monotone then $\Lambda(\bar{\delta}) \ge \Lambda(J(u^0)) = 0$, which contradicts the minimality of the root $\delta^0 = J(u^0)$. Consequently, it yields that u^0 is an optimal control, which completes the proof. ♦

Let $g^0 = (g_1^0, ..., g_k^0) \in \mathbb{R}^{nk}$ be a maximizing vector for $\Lambda(\delta^0)$. On the interval $T = [0, t_1^*]$ we introduce the function $\psi : \mathbb{R} \to \mathbb{R}^m$ as follows

$$\psi(t) = \sum_{i=j+1}^{k} g_i^0 \Phi(\tau_i, t), \tau_j \le t < \tau_{j+1}, j = 0, ..., k-1. \tag{3.20}$$

It is a simple task to verify that the function $\psi(t)$ satisfies the following differential equation:

$$\frac{d\psi}{dt} = -\psi'(t)A(t), \quad \psi(\tau_j - 0) - \psi(\tau_j + 0) = g_j^0, \quad j = 1, ..., k-1. \tag{3.21}$$

The optimality conditions for the optimization problem (3.1)-(3.4) are given by the following theorem.

Theorem 4. *If the number δ^0 is a minimal root of the equation $\Lambda(\delta) = 0$, then there exists an optimal control $u^0(t)$, $t \in T$ in the problem (3.1)-(3.4) such that $J(u^0) = \delta^0$ and for almost all $t \in T$ the condition*

$$\psi'(t)b(u^0(t), t) = \min_{v \in U} \psi'(t)b(v, t)), \tag{3.22}$$

holds. Here let $g^0 = (g_1^0, ..., g_k^0) \in \mathbb{R}^{nk}$ be a maximizing vector for $\Lambda(\delta^0)$ in (3.19).

Proof. Since $\Lambda(\delta^0) = 0$, then Theorem 2 yields that the Problem (A) has a solution for $\delta = \delta^0$. This implies that there exist an available control $u^0 \in U(\cdot)$ and a vector $z^0 \in M$ that satisfy the conditions (3.11)-(3.12). Hence $\varphi(z^0) = J(u^0) \leq \delta^0$. The assumption $J(u^0) < \delta^0$ leads to a contradiction with the minimality of the root δ^0. Therefore, $J(u^0) = \delta^0$, and, consequently, u^0 is the optimal control for the problem (3.1)-(3.4). Next, the function $u^0(t)$, $t \in T$ satisfies the condition

$$(g^0, Su^0) = \min_{u \in U(\cdot)} (g^0, Su). \tag{3.23}$$

Moreover, if we assume that $(g^0, Su^0) > \min\limits_{u \in U(\cdot)} (g^0, Su)$, then we have

$$\Lambda(g^0) < (g^0, c) - (g^0, z^0) + (g^0, Su) = 0, \tag{3.24}$$

which is impossible since δ^0 is a root of the equation $\Lambda(\delta) = 0$. To prove the desired optimality condition (3.22) we employ the condition (3.23). Then (we put below $\tau_0 = 0$ for brevity)

$$\min_{u \in U(\cdot)} (g^0, Su) = \min_{u \in U(\cdot)} \sum_{j=1}^{k} \int_0^{\tau_j} g_j^{0'} \Phi(\tau_j, t)b(u(t), t)dt =$$

$$= \min_{u \in U(\cdot)} \sum_{j=0}^{k+1} \int_{\tau_{j+1}}^{\tau_j} \sum_{i=j+1}^{k} g_i^{0'} \Phi(\tau_i, t)b(u(t), t)dt = \min_{u \in U(\cdot)} \sum_{j=0}^{k+1} \int_{\tau_{j+1}}^{\tau_j} \psi'(t)b(u(t), t)dt =$$

$$= \sum_{j=0}^{k+1} \int_{\tau_{j+1}}^{\tau_j} \min_{u \in U(\cdot)} \psi'(t)b(u(t), t)dt.$$

From the last equation it follows that the optimal control $u^(t)$, $t \in [0, t_1^*]$ satisfies the condition

$$\psi'(t)b(u^0(t), t) = \min_{v \in U} \psi'(t)b(v, t)),$$

which completes the proof. ♦

3.3 Robot Path Planning for the two dimensional case.

In this Section the results obtained above are modified to establish the relevant optimality conditions for the planed path of the two dimensional robot mechanism.

Consider the robotic model described by the following equations

$$\begin{cases} \dot{x_1} = x_3, \ \dot{x_2} = x_4, \\ \dot{x_3} = -a_3 x_3 + b_3 u_1, \ \dot{x_4} = -a_4 x_4 + b_4 u_2, \ t \in [t_0, t^*] \end{cases} \tag{3.25}$$

where $a_3 = \frac{c_1}{m_1}$, $a_4 = \frac{c_2}{m_2}$ (some details of this model can be found in [34]). This model can be written in the matrix form as

$$\dot{x} = Ax + Bu, \tag{3.26}$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -a_3 & 0 \\ 0 & 0 & 0 & -a_4 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ b_3 & 0 \\ 0 & b_4 \end{pmatrix}, \ u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

We suppose that the set $U(\cdot)$ of the admissible controls is the piecewise-continuous function $u(t) = (u_1(t), u_2(t))$ such that $u(t) \in P \ \ \forall t$, where $P \subset R^2$ is a nonempty compact set. The fundamental matrix $\Phi(t,\tau)(t \leq \tau)$ for (3.25) is $\Phi(t,\tau) = F(t-\tau)$ where

$$F(s) = \begin{pmatrix} 1 & 0 & \frac{1}{a_3}(1-e^{-a_3 s}) & 0 \\ 0 & 1 & 0 & \frac{1}{a_4}(1-e^{-a_4 s}) \\ 0 & 0 & e^{-a_3 s} & 0 \\ 0 & 0 & 0 & e^{-a_4 s} \end{pmatrix} \tag{3.27}$$

Then (3.9) yields that the solution of (3.25) is written as

$$x(t) = F(t-t_0)x_0 + \int_{t_0}^{t} F(t-\xi)Bu(\xi)d\xi \tag{3.28}$$

Next we consider the simplest case of the intermediate constraints, and let $M_i, M_i = 1, ..., k$ be the given points from $\mathbb{R}^2$. We also suppose that the following collection σ of the time moments

$$\sigma = \{\tau_i \in R : \ t_0 = \tau_1 \leq \tau_2 \leq ....\tau_k = t^*\} \tag{3.29}$$

is given, and the robot mechanism could pass through the given sets M_i at these moments, respectively (see, also, the Remarks 1 - 2). In general, we do not except the case when there is no admissible trajectory of (3.25), passing through the pre-assigned points M_i, $i = 1, \ldots, k$. In this case the optimization problem can be formulated as follows:

Problem 1. *For the fixed $\sigma \in T$ find the admissible control $u(\cdot) \in U(\cdot)$ that minimizes the minmax cost functional of the form*

$$\max_{1 \le i \le k} \|\{x(\tau_i) - M_i\}_{(2)}\| \to \min_{u(\cdot) \in U(\cdot)} \tag{3.30}$$

Here $\|x_{(2)}\|$ denotes a norm of the vector $x_{(2)} \in \mathbb{R}^2$ formed by first two coordinates of the vector $x \in \mathbb{R}^4$. These coordinates are chosen in accordance with the geometric character of the obstacles K_i for the robot's path (see also the figure from the previous section). The introduced cost functional

$$\varphi_\sigma(x(\tau_1), \dots, x(\tau_k)) = \max_{1 \le i \le k} \|\{x(\tau_i) - M_i\}_{(2)}\| \tag{3.31}$$

is a special case of (3.4), and hence, the results stated by the Theorems 1-3 after the proper modification can be used to solve the considered problem, in general. Most of the difficulties of this application are generated by the need to find the vector $g^0 \in \mathbb{R}^{nk}$ that is used by the optimality conditions and exploited for the solution of the conjugate system (3.21).

In order to simplify the required optimality conditions we apply the minmax theorem that establishes the conditions to transpose the operation min and max. It is well known the following fact

$$\|x\| = \max_{\|g\| \le 1} (g, x) \tag{3.32}$$

where (g, x) denotes the inner product of the vectors $g \in \mathbb{R}^n$, $x \in \mathbb{R}^n$. Therefore, the minimization problem of the norm cost function $J(u) = \|x(t_1^*\|$ defined on the reachability set $\mathcal{R}(x_0, t_1^*)$ of the linear control system can be rewritten as follows

$$J(u^0) = \|x^0(t_1^*)\| = \min_{x \in \mathcal{R}(x_0, t_1^*)} \max_{\|g\| \le 1} (g, x) = \max_{\|g\| \le 1} \min_{x \in \mathcal{R}(x_0, t_1^*)} (g, x) \tag{3.33}$$

The transposition of the min-max operation in this equality is valid if the set $\mathcal{R}(x_0, t_1^*)$ is closed and convex. The desired property of the sets involved in this optimization problem can be established, in fact, on the basis of the results from the previous section. Note that the analysis of the proofs from the previous Section show that the convexity and density properties of the required reachability set $\mathcal{R}(x_0, \tau_i)$, $i = 1, \dots, k$ are valid under the replacement of the measurable admissible functions by the piecewise- continuous functions.

Next we show that the infinite dimensional minimization problem (3.25), (3.30) is equivalent to a finite dimensional mathematical programming problem.

Theorem 5. *The optimum in the minimization problem (3.30) is equal to $\varphi_\sigma^0 = \psi(\lambda^0, \alpha^0)$, where $\psi(\lambda^0, \alpha^0)$ is the optimum of the following dual problem*

$$\psi(\lambda, \alpha) \to \max_{\lambda \in L^k,\ \alpha \in S^k}, \tag{3.34}$$

and

$$\lambda = \{\Lambda(1), \Lambda(2), ..., \Lambda(k)\} \in L^k,\ \ \Lambda(i) \in L, \quad L^k = \underbrace{L \times L \times ... \times L}_{k-times}, \quad L \doteq \{\ell \in R^2, \|\ell\| \le 1\}.$$

$$S^k \doteq \{\alpha \in \mathbb{R}^k : \alpha_i \ge 0,\ \sum_{i-1}^{k} \alpha_i = 1\}$$

$$\psi(\lambda, \alpha) = \sum_{i=1}^{k} \alpha_i[\Lambda'(i)(\{\Phi(\tau_i, t_0)x_0\}_2 - M_i)] + \sum_{q=1}^{m} \min_{u \in P} \int_{\tau_q}^{\tau_{q+1}} \sum_{i-1}^{q} \alpha_i \Lambda'(i)\{\Phi(\tau_i, \xi)Bu\}_2 d\xi$$

Proof. It is easy to see from the previous Section that for every $i = 1, \dots k$ the reachibility set $\mathcal{R}(x_0, \tau_i) \subset \mathbb{R}^4$ of the system (3.25) at the moment τ_i is a compact set. It is known that the following equality

$$\max_{1 \le i \le k} \|\{x(\tau_i) - M_i\}_{(2)}\| = \max_{\alpha \in S^k} \sum_{i=1}^{k} \alpha_i \|\{x(\tau_i) - M_i\}_{(2)}\|) \tag{3.35}$$

is valid. Then the formula (3.33) yields

$$\varphi_\sigma^0 = \min_{u \in U(\cdot)} \max_{1 \le i \le k} \|\{x(\tau_i) - M_i\}_{(2)}\| =$$

$$\min_{u \in U(\cdot)} \max_{\alpha \in S^k} \sum_{i=1}^{k} \alpha_i \|\{x(\tau_i) - M_i\}_{(2)}\| = \min_{u \in U(\cdot)} \max_{\alpha \in S^k} \sum_{i=1}^{k} \max_{\|\ell_i\| \le 1} \alpha_i(\ell_i, \{x(\tau_i) - M_i\}_{(2)}) =$$

$$= \max_{\alpha \in S^k} \min_{u \in U(\cdot)} \sum_{i=1}^{k} \max_{\|\ell_i\| \le 1} \alpha_i(\ell_i, \{x(\tau_i) - M_i\}_{(2)}) =$$

$$= \max_{\alpha \in S^k} \max_{\lambda \in L^k} \left[\sum_{i=1}^{k} \alpha_i[\Lambda'(i)(\{\Phi(\tau_i, t_0)x_0\}_2 - M_i)] + \sum_{q=1}^{k} \min_{u \in P} \int_{\tau_q}^{\tau_{q+1}} \sum_{i-1}^{q} \alpha_i \Lambda'(i)\{\Phi(\tau_i, \xi)Bu\}_2 d\xi \right]$$

This completes the proof.♦

The theorem 4 yields immediately that the optimal control for the problem (3.30) can be characterized by the following condition.

Theorem 6. *Let λ^0, α^0 be an optimal solution of the problem (3.34). Then the optimal control function $u^0(\cdot)$ on the each segment $[\tau_q, \tau_{q+1}]$, $, q = 1, ..., k$ satisfies the following condition*

$$\sum_{i=1}^{q} \alpha_i]\Lambda'^0(i)\{\Phi(\tau_i, \xi)Bu^0(\xi)\}_2 = \min_{\vartheta \in P}(\sum_{i=1}^{q} \alpha_i \Lambda'^0(i)\{\Phi(\tau_i, \xi)B\vartheta\}_2), \quad \xi \in [\tau_q, \tau_{q+1}].$$

It is obvious that the progress in the minimization problem given by the Theorem 5 is determined, in part, by the properties and the shape of the set P of the available control values. Usually, this set is presented by the simplest form— ball, polyhedron etc. Without loss of generality a we put $P = \{u \in R^2 :\ \|u\| \le \kappa\}$. In this case the optimal control $u_i^0(t)$, $i = 1, 2$ is defined

$$u_i^0(t) = -\kappa \frac{\sum_{j=1}^{i} \alpha_j \Phi(\tau_j, t)B\Lambda(j)^0}{\|\sum_{j=1}^{i} \alpha_j \Phi(\tau_j, \tau)B\Lambda(j)^0\|}, \ i = 1, 2.$$

The theorems above show that the optimal solution for the system (3.25) can be obtained via the solution of the following minimization problem:

$$\psi(\lambda,\alpha) \to \max_{\lambda\in L^k,\alpha\in S^k} \tag{3.36}$$

This is a mathematical programming problem. Let λ^0, α^0 be a solution of (3.36). Then substituting these values into the obtained above formulas for $u_i^0(t)$ we receive finally the required optimal control function of the original optimization problem. ♦

3.1. Illustrative example. Next we demonstrate the obtained results for the optimization problem (3.25) in a particular case. For sake the brevity we consider the robot mechanism that could pass only through the two points M_1, M_2 . In this case we have

$$\{\Phi(\tau_i,t_0)x_0\}_2 = \begin{pmatrix} x_{10} + x_{30}\frac{1}{a_3}(1-e^{\tau_i-t_0}) \\ x_{20} + x_{40}\frac{1}{a_4}(1-e^{\tau_i-t_0}) \end{pmatrix}$$

$$\{\Phi(\tau_i,\xi)Bu(\xi)\}_2 = \begin{pmatrix} b_3\frac{1}{a_3}(1-e^{-a_3(\tau_i-\xi)})u_1(\xi) \\ b_4\frac{1}{a_4}(1-e^{-a_4(\tau_i-\xi)})u_1(\xi) \end{pmatrix}$$

Let

$$\Lambda(i) \doteq (\Lambda_{1i},\Lambda_{2i})^T,\ M_i \doteq (\omega_{1i},\omega_{2i}),\ i=1,2$$

and

$$z_1 \doteq x_{10} + \frac{x_{30}}{a_3}(1-e^{-a_3(\tau_1-t_0)}) - \omega_{11},\ z_2 \doteq x_{20} + \frac{x_{40}}{a_4}(1-e^{-a_4(\tau_1-t_0)}) - \omega_{12},$$

$$z_3 \doteq x_{30} + \frac{x_{30}}{a_3}(1-e^{-a_3(\tau_2-t_0)}) - \omega_{21},\ z_4 \doteq x_{40} + \frac{x_{40}}{a_4}(1-e^{-a_4(\tau_2-t_0)}) - \omega_{22}$$

$$P_1(\alpha,\Lambda) \doteq \alpha_1(\Lambda_{11}z_1 + \Lambda_{12}z_2) + \alpha_2(\Lambda_{12}z_3 + \Lambda_{22}z_4)$$

Then the function $\psi(\lambda,\alpha)$ can be written in the following form

$$\begin{aligned}\psi(\lambda,\alpha) = P_1(\alpha,\lambda) &+ \int_{t_0}^{\tau_1} \min_{u\in P} \alpha_1[\Lambda_{11}\frac{b_3}{a_3}(1-e^{-a_3(\tau_1-\xi)})u_1 + \Lambda_{12}\frac{b_4}{a_4}(1-e^{-a_4(\tau_1-\xi)})u_2]d\xi + \\ &+ \int_{\tau_1}^{\tau_2=t^*} \min_{u\in P}\{\alpha_1[\Lambda_{11}\frac{b_3}{a_3}(1-e^{-a_3(\tau_1-\xi)})u_1 + \Lambda_{12}\frac{b_4}{a_4}(1-e^{-a_4(\tau_1-\xi)})u_2] + \\ &+\alpha_2[\Lambda_{21}\frac{b_3}{a_3}(1-e^{-a_3(\tau_2-\xi)})u_1 + \Lambda_{22}\frac{b_4}{a_4}(1-e^{-a_4(\tau_2-\xi)})u_2]\}\end{aligned} \tag{3.37}$$

Now we find the minimum under integrals in (3.37).

A) For the given λ and for each fixed ξ from $[t_0,\tau_1]$ we have the minimization problem

$$f_1(\xi)u_1 + f_2(\xi)u_2 \to \min_{(u_1,u_2)\in P}, \tag{3.38}$$

where

$$f_1(\xi) = \alpha_1\Lambda_{11}\frac{b_3}{a_3}(1-e^{-a_3(\tau_1-\xi)}),\ f_2(\xi) = \alpha_1\Lambda_{12}\frac{b_4}{a_4}(1-e^{-a_4(\tau_1-\xi)})$$

The function (3.38) is linear with respect to u_1, u_2 variables and hence, its minimum is achieved at the boundary of the compact P . Thus, the optimal control functions at the first time segment $[t_0, \tau_1]$ are :

$$u_1^0(\xi) = -k\frac{f_1(\xi)}{\sqrt{f_1^2(\xi) + f_2^2(\xi)}}, \quad u_2^0(\xi) = -k\frac{f_2(\xi)}{\sqrt{f_1^2(\xi) + f_2^2(\xi)}}, \; t_0 \leq \xi \leq \tau_1.$$

B) For the second segment $[\tau_1, \tau_2]$ we have the following minimization problem

$$\int\limits_{\tau_1}^{\tau_2} [g_1(\xi)u_1 + g_2(\xi)u_2]d\xi \rightarrow \min_{u_1^2+u_2^2 \leq \kappa^2}$$

where

$$g_1(\xi) = f_1(\xi) + f_3(\xi), \quad g_2(\xi) = f_2(\xi) + f_4(\xi)$$
$$f_3(\xi) = \alpha_2\Lambda_{21}\frac{b_3}{a_3}(1 - e^{-a_3(\tau_2-\xi)}), \; f_4(\xi) = \alpha_2\Lambda_{22}\frac{b_4}{a_4}(1 - e^{-a_4(\tau_2-\xi)})$$

Hence for the second interval $[\tau_1, t^*]$ we have the following optimal control functions

$$u_1^0(t) = -k\frac{g_1(\xi)}{\sqrt{g_1^2(\xi) + g_2^2(\xi)}}, \; u_2^0(t) = -k\frac{g_2(\xi)}{\sqrt{g_1^2(\xi) + g_2^2(\xi)}}$$

It should be noted that the all functions $f_i(t), g_i(t)$ under consideration are dependent on the parameters $\Lambda_{1i}, \Lambda_{2i}$ and α_i. Substituting the obtained control functions into the formula (3.37) for $\psi(\lambda, \alpha)$, we will receive a certain function of parameter λ and α. Finally, to obtain the required optimal, control the following mathematical programming problem

$$\psi(\lambda, \alpha) \rightarrow \max_{\lambda \in L^k, \alpha \in S^k} \tag{3.39}$$

should be solved. To simplify the calculation of the function $\psi(\lambda, \alpha)$ further, we can consider the case where the set of admissible control values P is given in the polyhedron form

$$P = \{(u_1, u_2) \in \mathbb{R}^2 : \; |u_1| \leq 1, \; |u_2| \leq 1\}. \tag{3.40}$$

It is easy to verify that in this case the optimal control functions are

$$u_i^0(t) = -\text{sign} f_i(t), \; t \in [t_0, \tau_1], \quad u_i^0(t) = -\text{sign} g_i(t), \; t \in [\tau_1, \tau_2], \; i = 1, 2 \tag{3.41}$$

Since the sign$f_i(t)$ is determined, in fact by the sign of the coefficient a_3, a_4 then the function $\psi(\lambda, \alpha)$ for the minimization problem (3.39) can be easy calculated.

3.4 Conclusions

It is shown above that the research of the extremal problems with state constraints have meet both essential theoretical and numerical difficulties. As it is well known, for some cases optimal control problems with state constraints can be approximated by a sequence of optimization problems without phase constraints. In particular, the principal elements of the planning of the robot's path can be stated with help of the simplest linear optimization model of the form

$$J(u) = c^{'}x(t^*) \to \max_{|u|\leq 1}, \qquad \frac{dx}{dt} = Ax + bu, \ x(0) = x_0,, \quad Hx(t^*) = g, \ t \in [0, t*] \tag{3.42}$$

where instead of the collection of the intermediate sets $M_i \subset \mathbb{R}^n$ it is satisfactory to consider the single terminal set of the form $H_i x(\tau_i) = g_i, \ i = 1, \ldots, k$. The the robot's path in the presence of the obstacles can be realized by a step-by-step optimization procedure for a simple control problem (3.42).

In general, the planning procedure can be detailed as a multistage or repetitive process. The generalization of these results for the linear repetitive processes is presented in the next Chapter, some preliminary results of which were reported in [23, 44].

Chapter 4

Discrete approximation for the robot path planning motion

In this Chapter the results obtained above are modified to develop the well-posed discrete approximation for the planning of the robot's path. We formulate the path-trek problem for the robot mechanism that could cross the given domain with the safety clearance along the given obstacles. Some aspects of the problem statement for this problem were discussed in [34].

4.1 Robot dynamics and its discrete model

The robot model under consideration is described as follows

$$m^i \ddot{q}^i(t) + c^i \dot{q}^i(t) = h^i u^i(t), \ i = 1, 2 \tag{4.1}$$

where m^i are the inertias of the uncoupled axes, c^i, h^i are parameters of the dc motor drives, and u^i are the motor control voltages. The beginning and ending robot positions at the points A, B lead to the conditions $q(0) = q_A, \ q(T) = q_B$ and $\dot{q}(0) = \dot{q}(T) = 0$. The energy required to transfer the payload in a fixed time T is

$$J(u) = \int_0^T \sum_{i=1}^{2} [(u^i(t))^2 - \gamma^i u^i(t) \dot{q}^i(t)] dt, \tag{4.2}$$

where $\gamma^i > 0$ are the motor parameters. To the optimal control problem (4.1)-(4.2) we add special geometric constraints corresponding to the avoidance of obstacles $K_i, i = 2, \ldots, m$ (see Fig. 1). These state constraints follow from the obvious requirement that the robot, denoted by $K_1(t)$ does not crash. Note that the set $K_1(t)$ describes the space occupied by the parts the of robot mechanism and their working area i.e. rotation and translation due to robotic motion, in general. In the motion course, this area can be variable, and hence, depends on the current position at the moment t, in fact. This requirement can be presented in the general form :

$$\max_{t \in [0,T]} \min_{2 \le i \le m} \{d_i^* - d_i(t)\} \le 0 \tag{4.3}$$

where $d_i(t)$ denotes the distance between the robot $K_1(t)$ and $i-th$ obstacle K_i at the moment t, and d_i^* denotes the safety distance.

We present the above in the standard form: minimize

$$J(u) = \int_0^T \sum_{i=1}^{2}[(u^i(t))^2 - \gamma^i u^i(t)x^{2i}(t)]dt, \tag{4.4}$$

over the solution of the differential equations

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -a_2x_3 + b_2u_1 \\ \dot{x_3} = x_4 \\ \dot{x_4} = -a_4x_4 + b_4u_2, \ \ t \in [0,T] \end{cases} \tag{4.5}$$

where $q_1(t) = x_1(t), \dot{q}_1(t) = x_2(t), \ q_2(t) = x_3(t), \ \dot{q}_2(t) = x_4(t)$ and $a_2 = \frac{c_1}{m_1}, \ a_4 = \frac{c_2}{m_2}, \ b_2 = \frac{h^1}{m^1}, b_4 = \frac{h^2}{m^2}$.

Now we construct a discrete model for the problem (4.4)-(4.5). Let $N > 0$ and T_N be a mesh on $[0,T]$ with N intervals and mesh points $\{\tau_0, \tau_1, ..., \tau_N\}$ with $\tau_s < \tau_{s+1}, s \in \{0,1,...,N-1\}, \ \tau_0 = 0, \ \tau_N = T$. Next we consider a uniform grid with $\tau_s = \frac{sT}{N}$.

Denote

$$P_N = \{\vartheta = (\vartheta_1, \vartheta_2) \in C[0,T], |\vartheta_1(t)| \le \alpha, \ |\vartheta_2(t)| \le \beta\}, \ t \in [0,T]$$

where $C[0,T]$ means the space of the continuous functions on the interval $[0,T]$, and $\vartheta_1, \ \vartheta_2$ are the piecewise - linear functions on the mesh T_N. It is easy to see that in order to determine the functions from P_N, $N+1$ points are required $\mu_0, \mu_1, ..., \mu_N$ (but every point $\mu_i = (\vartheta_{1i}, \vartheta_{2i})$ has two coordinates and, hence, we in fact need $2N + N$ points, in fact):

$$\vartheta_j(t) = \vartheta_{js} + \frac{N}{T}(t - \frac{sT}{N})(\vartheta_{j\ s+1} - \vartheta_{j\ s}), \ j = 1,2 \tag{4.6}$$

for any $\tau_s \le t \le \tau_{s+1}, \quad s = \overline{0, N-1}$. It is obvious that the requirement $\mu_i \in P_N$ leads to the conditions $\mid \vartheta_{1i} \mid \le \alpha, \ \mid \vartheta_{2i}) \mid \le \beta, \ i = 0, \ldots, N$. Replacing now the derivatives in (4.5) by the Euler finite difference we have the following difference system of equations

$$\begin{cases} x_N^1(s+1) - x_N^1(s) = (\tau_{s+1} - \tau_s)x_N^2(s), \\ x_N^2(s+1) - x_N^2(s) = (\tau_{s+1} - \tau_s)[a_2x_N^2(s) - b_1\vartheta_1(s)], \\ x_N^3(s+1) - x_N^3(s) = (\tau_{s+1} - \tau_s)x_N^4(s), \\ x_N^4(s+1) - x_N^4(s) = (\tau_{s+1} - \tau_s)[a_4x_N^4(s) - b_2\vartheta_2(s)], \end{cases} \tag{4.7}$$

with the initial condition

$$(x_N^1(0), x_N^2(0), x_N^3(0), x_N^4(0)) = (x_1^0, 0, 0, 0)$$

For the given piecewise-linear control function $\eta = (\vartheta_1, \vartheta_2) \in P_N$ let $x_N^\eta(s) = \{x_{1N}^\eta(s), ..., x_{4N}^\eta(s)\}$, $s = 1, 2, \ldots, N$ be the solution of the difference equations (4.7). Next, to obtain well-posed discrete counterparts for the max-min constraints, we introduce the following constraint functions

$$\max_{\tau \in T_N} \min_{i=2,\ldots,m} \Phi_{i,N}(\eta, \tau) \leq \epsilon_N, \ i = 2, \ldots, m \qquad \text{where} \quad \epsilon_N \to 0, \ N \to \infty \tag{4.8}$$

where the functions $\Phi_{i,N} : P_N \times [0, 1, ..., N] \longrightarrow R$ are defined as

$$\Phi_{i,N}(\eta, s) = d_{1i}^* - d_{1i}(x_N^\eta(s)) \tag{4.9}$$

and where

$$d_{1i}(x_N^\eta(s)) = \min_{y, z_i}\{\|y - z_i\|, \ y \in K_1(x_N^\eta(s)), \ z_i \in K_i\} \tag{4.10}$$

denotes the distance and d_{1i}^* is the safety distance between robot $K_1(x(t))$ and the obstacle K_j at the moment $t = \tau_s$.

Remark. In fact, the theoretical results of Chapter 2 devoted to the optimization with min-max state constraint says that the correct discrete approximation uses the disturbances of all state constraints. Note that the given disturbances of the constraints are also needed to guarantee the the set P_N of the admissible *discrete* control functions is non-empty. In the course of the numerical experiments we will investigate the influence and effectiveness of different variants of the disturbances.

To calculate the cost functional value, we exploit the following facts: 1) the control $\eta(t) = (\vartheta_1(t), \vartheta_2(t))$ is a piecewise linear function on the grid T_N given in the form(4.6);

2) using the obtained collection points $x_N^\eta(0), x_N^\eta(1), ..., x_N^\eta(N)$ we construct a discrete trajectory for the robot path as the following linear functions on every segment $\tau_s \leq t \leq \tau_{s+1}$ $s = 0, \ldots, N-1$:

$$x_2(t) = x_2^\eta(s) + \frac{N}{T}(t - \frac{sT}{N})(x_2^\eta(s+1) - x_2^\eta(s)) \tag{4.11}$$

$$x_4(t) = x_4^\eta(s) + \frac{N}{T}(t - \frac{sT}{N})(x_4^\eta(s+1) - x_4^\eta(s)) \tag{4.12}$$

Thus, substituting the obtained functions (4.11)-(4.11) into (4.4) leads to the calculation of

$$\int_{\tau_s}^{\tau_{s+1}} [(u^i(t))^2 - \gamma^i u^i(t) x^{2i}(t)]dt \qquad \text{where} \quad [\tau_s, \tau_{s+1}] \doteq [\frac{sT}{N}, \frac{(s+1)T}{N}]$$

Then we have separately

$$\int_{\tau_s}^{\tau_{s+1}} (u^i(t))^2 dt = \frac{T}{N(\vartheta_{s+1}^i - \vartheta_s^i)} \int_{\vartheta_s}^{\vartheta_{s+1}} \left[\vartheta_s^i + \frac{N}{T}(t - \frac{sT}{N})(\vartheta_{s+1}^i - \vartheta_s^i)\right]^2 d[\vartheta_s^i +$$

$$\frac{N}{T}(t - \frac{sT}{N})(\vartheta_{s+1}^i - \vartheta_s^i)] = \frac{T}{N(\vartheta_{s+1}^i - \vartheta_s^i)} \int_{\vartheta_s}^{\vartheta_{s+1}} \psi^2 d\psi =$$

$$\frac{T}{N(\vartheta_{s+1}^i - \vartheta_s^i)} \left[\frac{\psi^3}{3}\Big|_{\vartheta_{s+1}^i} - \frac{\psi^3}{3}\Big|_{\vartheta_s^i}\right] = \frac{1}{3}\left((\vartheta_{s+1}^i)^2 + (\vartheta_s^i)^2 + \vartheta_{s+1}^i \vartheta_s^i\right)$$

The second integral is

$$\int_{\tau_s}^{\tau_{s+1}} (u^i(t)x^{2i}(t))dt = \int_{\frac{sT}{N})}^{\frac{(s+1)T}{N})} \Big[\vartheta_s^i + \frac{N}{T}(t - \frac{sT}{N})(\vartheta_{s+1}^i - \vartheta_s^i)\Big]\Big[x_s^i +$$

$$\frac{N}{T}(t - \frac{sT}{N})(x_{s+1}^i - x_s^i)\Big]dt == \frac{N}{T}\int_0^1 \Big[\vartheta_s^i + \frac{N}{T}(t - \frac{sT}{N})(\vartheta_{s+1}^i - \vartheta_s^i)\Big]\Big[x_s^i +$$

$$\frac{N}{T}(t - \frac{sT}{N})(x_{s+1}^i - x_s^i)\Big]d[\frac{N}{T}(t - \frac{sT}{N})] = \frac{N}{T}\int_0^1 \Big[\vartheta_s^i + \xi(\vartheta_{s+1}^i -$$

$$\vartheta_s^i)\Big]\Big[x_s^i + \xi(x_{s+1}^i - x_s^i)\Big]d\xi = \frac{N}{T}\Big[\vartheta_s^i x_s^i + \int_0^1 \xi[(\vartheta_{s+1}^i - \vartheta_s^i)x_s^i + (x_{s+1} - x_s^i)\vartheta_s^i]d\xi +$$

$$\int_0^1 \xi^2(\vartheta_{s+1}^i - \vartheta_s^i)(x_{s+1}^i - x_s^i)d\xi\Big] = \frac{N}{T}\Big[\vartheta_s^i x_s^i +$$

$$\frac{1}{2}\Big(\vartheta_{s+1}^i x_s^i - \vartheta_s^i x_s^i + \vartheta_s^i x_{s+1}^i - \vartheta_s^i x_s^i\Big) + \frac{1}{3}\Big(\vartheta_{s+1}^i x_{s+1}^i - \vartheta_{s+1}^i x_s^i - \vartheta_s^i x_{s+1}^i + \vartheta_s^i x_s^i\Big)\Big] =$$

$$= \frac{N}{T}\Big[\frac{1}{3}x_{s+1}^{2i}\vartheta_{s+1}^i + \frac{1}{3}x_s^{2i}\vartheta_s^i + \frac{1}{6}x_{s+1}^{2i}\vartheta_s^i + \frac{1}{6}x_s^{2i}\vartheta_{s+1}^i\Big]$$

Finally, we receive the following criteria value

$$J(u) = \frac{T}{N}\sum_{i=1}^{2}\sum_{s=0}^{N-1} A_{s,i} \tag{4.13}$$

with

$$A_{s,i} = \frac{1}{3}\Big((\vartheta_{s+1}^i)^2 + (\vartheta_s^i)^2 + \vartheta_{s+1}^i\vartheta_s^i\Big)$$
$$-\gamma^i\Big[\tfrac{1}{3}x_{s+1}^{2i}\vartheta_{s+1}^i + \tfrac{1}{3}x_s^{2i}\vartheta_s^i + \tfrac{1}{6}x_{s+1}^{2i}\vartheta_s^i + \tfrac{1}{6}x_s^{2i}\vartheta_{s+1}^i\Big] \tag{4.14}$$

where the required discrete variables x_s^{2i} are determined by the difference equations (4.7) and are given by the formula

$$x_s^2 = x_0^2\Big(1 + \frac{T}{N}a_2\Big)^s - \tag{4.15}$$

$$-\frac{T}{N}b_1\Big[(1 + \frac{T}{N}a_2)^{s-1}\vartheta_0^1 + (1 + \frac{T}{N}a_2)^{s-2}\vartheta_1^1 + \ldots + (1 + \frac{T}{N}a_2)\vartheta_{s-2}^1 + \vartheta_{s-1}^1\Big]$$

and

$$x_s^4 = x_0^4\Big(1 + \frac{T}{N}a_4\Big)^s - \tag{4.16}$$

$$-\frac{T}{N}b_2\Big[(1 + \frac{T}{N}a_4)^{s-1}\vartheta_0^2 + (1 + \frac{T}{N}a_4)^{s-2}\vartheta_1^2 + \ldots + (1 + \frac{T}{N}a_4)\vartheta_{s-2}^2 + \vartheta_{s-1}^2\Big]$$

Thus, we received the cost criteria as some function $J_N(\eta) = R_N(\vartheta_{01}, \vartheta_{02}, ..., \vartheta_{N1}, \vartheta_{N2})$ of the unknown control parameters $(\vartheta_{01}, \vartheta_{11}, ..., \vartheta_{N1})$ and $(\vartheta_{02}, \vartheta_{12}, ..., , \vartheta_{N2})$.

Finally, we construct the discrete approximation as the following finite-dimensional optimization problem

$$(P_{OC,N}) : \min_{\eta=\{\vartheta_{s1},\vartheta_{s2}\}\in P_N} R(\vartheta_{01}, \vartheta_{02}, ..., \vartheta_{N1}, \vartheta_{N2}) \tag{4.17}$$

subject to

$$\max_{\tau\in T_N} \min_{i=2,...,m} \Phi_{i,N}(\eta, \tau) \leq \epsilon_N, \text{ at } \epsilon_N \to 0, \ N \to \infty \tag{4.18}$$

Remark 2. The formula above shows that in order to calculate the distance function we need the functions $x_1(t), x_3(t)$. They can be defined in a similar way as the function $x_2(t), x_4(t)$ from the difference equations (4.7) and, hence, these trajectories can also be approximated by the following linear functions on each segment $\tau_s \leq t \leq \tau_{s+1}$ $s = 0, \ldots, N-1$:

$$x_1(t) = x_1(s) + \frac{N}{T}(t - \frac{sT}{N})(x_1(s+1) - x_1(s)) \tag{4.19}$$

$$x_3(t) = x_3(s) + \frac{N}{T}(t - \frac{sT}{N})(x_3(s+1) - x_3(s)) \tag{4.20}$$

where the collection of the mesh points $x_1(s), x_3(s)$ is given by the following recurrent formula

$$x_1(s+1) = x_1(s) + \frac{T}{N}x_2(s), \quad x_3(s+1) = x_3(s) + \frac{T}{N}x_4(s), \quad s = 0, \ldots, N-1 \tag{4.21}$$

and the mesh points $x_2(s), x_4(s)$ are determined by the formula (4.15)- (4.16) given above.

The problem (4.17)-(4.18) can be rewritten in the following standard nonsmooth optimization form

$$\min f^N(x, y) \tag{4.22}$$

subject to constraints

$$f^N(x, y, \epsilon) \leq 0 \tag{4.23}$$

where

$$\begin{aligned} x = (\vartheta_{01}, \vartheta_{11}, ..., \vartheta_{N1}), \quad y = (\vartheta_{02}, \vartheta_{12}, ..., \vartheta_{N2}), \\ f^N(x, y) = R_N(\vartheta_{01}, \vartheta_{11}, ..., \vartheta_{N1}, \vartheta_{02}, \vartheta_{12}, ..., \vartheta_{N2}), \\ f^N(x, y, \epsilon) = \max\{(g_s(x, y, \epsilon), f_{2s}(x), f_{3s}(y), \ s = 1, \ldots, N\}, \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} g_s(x, y, \epsilon) = \min_{i=2,...,m} \{\Phi_{i,N,\epsilon}(\vartheta_{01}, \vartheta_{11}, ..., \vartheta_{N1}, \vartheta_{02}, \vartheta_{12}, ..., \vartheta_{N2}), \tau_s\} \\ f_{2s}(x) = \max(\alpha - |x_s|), \quad s = 1, \ldots, N \\ f_{3s}(y) = \max(\beta - |y_s|), \ s = 1, \ldots, N \end{aligned} \tag{4.25}$$

(4.25) can be solved by the numerical methods [6] based on the nonsmooth optimization theory [10]. The results of numerical tests will be reported in due course.

4.2 Test optimal control examples

To verify and to check the developed numerical methods a collection of the testing examples is needed. In contrast to the well known optimization problem in the absence of state constraints, the collection of well tested optimal control problems in the presence of state constraints is very poor. This fact is explained by the serious difficulties in finding the exact solution in the functional spaces. Below we give a short description of two simple examples where the exact optimal solution is known. A detailed description and the proof of their optimality can be found in [39].

Example 1

Let the control function be partioned on the two components $u(t) = \{u_1(t), u_2(t)\}$ where the first component $|u_1(t)| \leq 1$ is bounded and the second $u_2(t)$ is free.

Consider the model of the form

$$J = \int_0^3 [2x_1(t) + \frac{1}{2}u_2^2(t)]\,dt \to \min_{u_1,u_2} \tag{4.26}$$

over the solutions

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = u_1(t) + u_2(t), \quad t \in [0,3], \end{cases} \tag{4.27}$$

subject to the constraints

$$\begin{cases} x_1(t) \leq 200, \\ -x_1(t) \leq 0, \end{cases} \tag{4.28}$$

$$|u_1(t)| \leq 1, \;\; t \in [0,T] \doteq [0,3] \tag{4.29}$$

with initial data

$$x_1(0) = 1, \;\; x_2(0) = 2/3, \;\; u_2(0) = 0. \tag{4.30}$$

The discrete model for the model above is

$$\begin{cases} x_1(s+1) = x_1(s) + hx_2(s), \\ x_2(s+1) = x_2(s) + h[u_1(s) + u_2(s)], \quad s = 0,1,2,\dots,N, \end{cases} \tag{4.31}$$

Here we replace the derivatives by Euler differences as

$$\dot{x}(t) = \frac{x(\tau_{s+1}) - x(\tau_s)}{h}$$

and denote

$$x(\tau_s) \doteq x(s), \;\; u_1(s) \doteq u_1(\tau_s), \;\; u_2(s) \doteq u_2(\tau_s)h \doteq \frac{T}{N}, \;\; 0 = \tau_0 \leq \tau_1 \leq \dots \leq \tau_{N-1} \leq \tau_N = T$$

The solution of difference equations is

$$\begin{cases} x_1(s) = x_1(0) + sh \cdot x_2(0) + h^2 \sum\limits_{i=0}^{s-2} (s-i-1)(u_1(i) + u_2(i)), \ s \geq 2 \\ x_1(1) = x_1(0) + hx_2(0) \\ x_2(s) = x_2(0) + h \sum\limits_{i=0}^{s-1} (u_1(i) + u_2(i)), \quad s = 1, 2, \ldots, N, \end{cases} \tag{4.32}$$

Finally, we need to minimize the cost function

$$J(v) = 2Tx_1(0) + T^2 x_2(0) + \frac{T^3}{N^3} \sum_{i=0}^{N-2} (N-1-i)^2 (u_1(i) + u_2(i)) +$$

$$+ \frac{T}{3N} \sum_{i=0}^{N-1} \left[\left(u_2(i) + u_2(i+1)\right)^2 - u_2(i) u_2(i+1) \right] \to \min_{u_1, u_2}$$

under the state constraints

$$\begin{cases} \max\limits_s \left\{ x_1(0) + \frac{sT}{N} x_2(0) + \frac{T^2}{N^2} \sum\limits_{i=0}^{s-2} (s-i-1) \left[u_1(i) + u_2(i)\right] - 200, \ s = 2, 3, \ldots, N \right\} \leq 0, \\ \max\limits_s \left\{ -\left[x_1(0) + \frac{sT}{N} x_2(0) + \frac{T^2}{N^2} \sum\limits_{i=0}^{s-2} (s-i-1) \left[u_1(i) + u_2(i)\right] \right] s = 2, 3, \ldots, N \right\} \leq 0 \\ \max\limits_s \left\{ 1 - |u_1(s)|, \ s = 0, 1, \ldots, N-2 \right\} \leq 0 \end{cases}$$

Optimal control for the initial problem (4.26)—(4.30) is

$$\begin{cases} u_1^0(t) = -1, \ t \in [0, 1); \\ u_1^0(t) = +1, \ t \in [1, 2); \\ u_1^0(t) = 0, \ t \in [2, 3); \end{cases} \quad u_2^0(t) = x_2^0(t), \ t \in [0, 3] \tag{4.33}$$

and the corresponding optimal trajectory is

$$\begin{cases} x_1^0(t) = -t^4/12 + t^3/2 - 3t^2/2 + 2t/3 + 1, \\ x_2^0(t) = -t^3/3 + 3t^2/2 - 3t + 2/3 \end{cases} \quad t \in [0, 1); \tag{4.34}$$

$$\begin{cases} x_1^0(t) = -t^4/12 + t^3/2 - t^2/2 - 4t/3 + 2, \\ x_2^0(t) = -t^3/3 + 3t^2/2 - t - 4/3 \end{cases} \quad t \in [1, 2); \begin{cases} x_1^0(t) = 0, \\ x_2^0(t) = 0 \end{cases}, \ t \in [2, 3); \tag{4.35}$$

The figure above illustrates the optimal trajectory for the initial problem.

Example 2

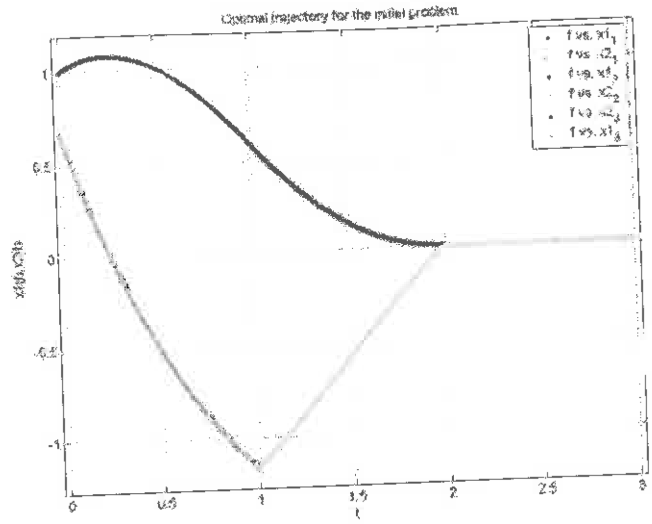


Figure 4.1: Optimal trajectory

Consider

$$J = 2\int_0^3 x_1(t)dt \to \min_u \tag{4.36}$$

over the solutions

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = u(t), \quad t \in [0,3], \end{cases} \tag{4.37}$$

subject to the constraints

$$\begin{cases} x_1(t) \leq 200, \\ -x_1(t) \leq 0, \end{cases} \tag{4.38}$$

$$|u(t)| \leq 1, \ t \in [0,T] \doteq [0,3] \tag{4.39}$$

with initial data

$$x_1(0) = 1, \ x_2(0) = 0. \tag{4.40}$$

Discrete model is as follows

$$\begin{cases} x_1(s+1) = x_1(s) + hx_2(s), \\ x_2(s+1) = x_2(s) + hu(s), \quad s = 0,1,2,\dots,N, \end{cases} \tag{4.41}$$

Here we replace the derivatives by Euler differences as

$$\dot{x}(t) = \frac{x(\tau_{s+1}) - x(\tau_s)}{h}$$

and denote

$$x(\tau_s) \doteq x(s),\ v_s \doteq u(\tau_s),\ h \doteq \frac{T}{N},\ 0 = \tau_0 \le \tau_1 \le \ldots \le \tau_{N-1} \le \tau_N = T$$

The solution of difference equations is

$$\begin{cases} x_1(s) = x_1(0) + shx_2(0) + \sum\limits_{i=0}^{s-2}(s-i-1)h^2 v_i,\ s \ge 2 \\ x_1(1) = x_1(0) + hx_2(0) \\ x_2(s) = x_2(0) + h\sum\limits_{i=0}^{s-1} v_i,\quad s = 1, 2, \ldots, N, \end{cases} \tag{4.42}$$

Finally, we are needed to minimize the function

$$J(v) = 2Tx_1(0) + T^2 x_2(0) + \frac{T^3}{N^3}\Big[(N-1)^2 v_0 + (N-2)^2 v_1 + \ldots + 2^2 v_{N-3} + v_{N-2}\Big] \to \min_{v_s} \tag{4.43}$$

under the state constraints

$$\begin{cases} \max\limits_s \big\{x_1(0) + \frac{sT}{N}x_2(0) + \frac{T^2}{N^2}\sum\limits_{i=0}^{s-2}(s-i-1)v_i - 200,\ s = 2, 3, \ldots, N\big\} \le 0, \\ \max\limits_s \big\{-[x_1(0) + \frac{sT}{N}x_2(0) + \frac{T^2}{N^2}\sum\limits_{i=0}^{s-2}(s-i-1)v_i] s = 2, 3, \ldots, N\big\} \le 0 \\ \max\limits_s \big\{1 - |v_s|,\ s = 0, 1, \ldots, N-2\big\} \le 0 \end{cases} \tag{4.44}$$

Optimal control for the initial problem (4.36)—(4.40) is

$$\begin{cases} u^0(t) = -1,\ t \in [0, 1); \\ u^0(t) = +1,\ t \in [1, 2); \\ u^0(t) = 0,\ t \in [2, 3); \end{cases} \tag{4.45}$$

and the corresponding optimal trajectory is

$$\begin{cases} x_1^0(t) = -t^2/2 + 1, \\ x_2^0(t) = -t \end{cases} t \in [0, 1); \begin{cases} x_1^0(t) = -t^2/2 - 2t + 2, \\ x_2^0(t) = -t - 2 \end{cases} t \in [1, 2);$$

$$\begin{cases} x_1^0(t) = 0, \\ x_2^0(t) = 0 \end{cases},\ t \in [2, 3); \tag{4.46}$$

4.3 Conclusions

This Chapter has used the well-posed discrete approximation to design the numerical methods for solutions to the optimization problems with max-min constraints. In common with other discrete approximations, we present the theoretical background to construct the discrete schemes with cost value and trajectory convergence. It is conjectured that our approach, accompanied by modern methods of nonsmooth optimization [10, 61] and control theory [13] will work very well for optimal control problems with special state constraints that ensure obstacle avoidance.

Chapter 5

Optimization problems for linear repetitive processes

The first part of this chapter uses the classic approach to investigate the traditional optimal control theory problems for the repetitive dynamics model. It is well known that the separation theorem for convex sets is quite a useful approach for studying a wide class of extreme problems. Here we develop this method to establish optimality conditions in the classic form of maximum principle for multipass nonstationary continuous-discrete control systems with nonlinear inputs and nonlocal state-phase terminal constraints of the general form. The obtained results are typical for classic optimal control theory. However, their numerical realization is not a trivial task. For this reason, in the next sections for the stationary case of the system model and the particular case of the constraint and the cost functional, we will develop new optimality and sub-optimality conditions that are more suitable for the design of numerical methods and further applications. In contrast to the classic approaches of optimal control theory,in the second part in this chapter we will use the idea of the constructive methods reported in [29] and extend this setting to the continuous-discrete case to produce new results and constructive elements of optimization theory for the considered repetitive systems and also develop its relevant basic properties which can be of interest for others purposes. It is shown that the obtained optimality and ϵ-optimality conditions are closely related to the corresponding classic results of maximum and ϵ- maximum principles. The sensitivity analysis and some differential properties of the optimal controls under disturbances are discussed and their application to the optimal synthesis problem is given. The obtained results yield a theoretical background for the design problem of optimal controllers for relevant basic processes. Some areas for short to medium term further research are also briefly discussed.

5.1 Notation and Model Definition

In practice, a repetitive process will only ever complete a finite number of passes. Hence we consider repetitive processes modeled by a system of linear differential equations with variable coefficients. Let $T = [0, t^*]$ be a given interval of values of the continuous independent variable $t \in T$ and $K = \{1, 2, ..., N\}$, $N < +\infty$ be a set of values of the discrete variable $k \in K$. Also introduce the control and state vectors as $u_k(t) \in \mathbb{R}^r$ and $x_k(t) \in \mathbb{R}^n$ respectively. Then the repetitive processes considered in this paper are described by

$$\frac{dx_k(t)}{dt} = A(t)x_k(t) + D(t)x_{k-1}(t) + b_k(u_k(t), t), \quad k \in K, \ t \in T \tag{5.1}$$

where the last nonlinear term represents the input signal actually applied to the process. To complete the description, it is necessary to specify the boundary conditions which are here taken to be of the form

$$x_k(0) = \alpha(k), k \in K, \quad x_0(t) = \beta(t), \quad t \in T \tag{5.2}$$

Note also that it is possible to augment the above model to include the fact that the pass profile can be a vector valued function of state dynamics.

Now we define the class of available and admissible input signals for the above model.

Definition 6. *We say that the function $u : K \times T \to R^r$ is available for (5.1) if it is measurable with respect to t for a fixed $k \in K$, and satisfies the constraint $u_k(t) \in U$, $k \in K$, for almost all $t \in T$, where U is a given compact set from $\mathbb{R}^r$. Also the function $x : K \times T \to R^n$ is a solution of (5.1) corresponding to the given available control $u_k(t)$ if it is absolutely continuous with respect to $t \in T$ for each fixed $k \in K$ and satisfies (5.1) for almost all $t \in T$ and each $k \in K$.*

We denote the set of available controls by $U(\cdot)$ and use M_i, $M_i \subset \mathbb{R}^n$, $i = 1, 2, ..., l$ to denote the given compact convex sets.

Definition 7. *The available control $u_k(t)$ is said to be admissible for the process (5.1) if the corresponding solution $x_k(t) = x_k(t, \alpha, \beta, u)$ of (5.1) and (5.2) satisfies*

$$x_N(\tau_i) \in M_i, \ i = 1, 2, ..., l \tag{5.3}$$

where $0 < \tau_1 < \tau_2 < ... < \tau_l = t^*$ are specified elements of T.

The optimal control problem considered in this paper can now be stated as: Minimize a cost function of the form

$$J(u) = \varphi(x_N(\tau_1), x_N(\tau_2), ..., x_N(\tau_l)) \tag{5.4}$$

for processes described by (5.1) and (5.2) in the class of admissible controls $u_k(t) \in U(\cdot)$.

We also assume that: the $n \times n$ matrix functions $A(t)$ and $D(t)$ and the $n \times 1$ function $\beta(t)$ are measurable and integrable on T, the function $b : K \times U \times T \to \mathbb{R}^n$ is continuous with respect to $(u, t) \in U \times T$ for each fixed $k \in K$ and the function $\varphi : \mathbb{R}^{nl} \to \mathbb{R}$ is convex. It is easy to see that these conditions guarantee the existence and uniqueness of an absolutely continuous solution of (5.1) and(5.2) for any available control $u_k(t)$. To guarantee the existence of optimal control, throughout this paper we assume that the set of admissible controls is non-empty.

5.1.1 Reachability set and its properties

To solve (5.1) and (5.2) we require the $n \times n$ matrix function $\Phi(\tau, t)$ be defined by the following equation

$$\frac{d\Phi(\tau,t)}{d\tau} = A(\tau)\Phi(\tau,t), \quad \Phi(t,t) = I_n \tag{5.5}$$

where I_n denotes the $n \times n$ identity matrix. Also it is well known, see, for example, [36] that the entries in the matrix $\Phi(\tau, t)$ are absolutely continuous functions defined on the set $T \times T$. Therefore, there exists a constant $0 < C < \infty$ such that $\|\Phi(\tau,t)\| \leq C$ for any $(\tau, t) \in T \times T$, where $\|\cdot\|$ denotes the standard matrix norm. We use $H^p(0, t^*)$, where $p > 0$ is an integer, to denote the set of all functions $f : (0, t^*) \to \mathbb{R}^n$, which are absolutely continuous on each closed sub-interval $[\alpha, \beta]$ from the interval $(0, t_1^*)$ and have almost everywhere integrable derivatives of order up to p on $(0, t^*)$. Also it can be shown that $H^p(0, t^*)$ is Banach space with the norm $\|f\|_H = \sum\limits_{i=0}^{p} \|f^{(i)}\|_{L_1}$ and the following inclusions $H^p(0,t^*) \subset C^p(0,t^*) \subset L_1(0,t^*)$ hold, where $C^p(0,t^*)$ denotes the space of $n \times 1$ vector functions which are continuously differentiable on $(0, t^*)$ up to order p, and $L_1(0, t^*)$ the space of $n \times 1$ vector valued functions which are integrable on $(0, t^*)$.

Now define the mapping $\mathcal{P} : L_1(0, t^*) \to H^1(0, t^*)$ by

$$(\mathcal{P}f)(\tau) = \int_0^{\tau} \Phi(\tau,t)D(t)f(t)dt, \quad \tau \in (0, t^*). \tag{5.6}$$

and its power composition $\mathcal{P}^k : H^{k-1}(0,t^*) \to H^k(0,t^*)$ as $(\mathcal{P}^k f)(\tau) = \mathcal{P}(\mathcal{P}^{k-1} f)(\tau)$, $\tau \in (0, t^*)$.

Also define the mapping $Q : L_1(0, t^*) \to H^1(0, t^*)$ by

$$(Qf)(\tau) = \int_0^{\tau} \Phi(\tau,t)f(t)dt, \quad \tau \in (0, t^*). \tag{5.7}$$

For ease of notation, the function $b_k(u_k(t), t)$, $t \in T$ is subsequently denoted by $b_u(k)$.

For the given available control $u \in U(\cdot)$ the corresponding solution of (5.1) and (5.2) at $t = \tau_j$ on pass $k = N$ can now be written in the form

$$x_N(\tau_j) = \Phi(\tau_j, 0)\alpha(N) + \sum_{i=1}^{N-1} (\mathcal{P}^i\Phi(\cdot,0))(\tau_j)\alpha(N-i) + (\mathcal{P}^N\beta)(\tau_j) +$$

$$\sum_{i=1}^{N-1} (\mathcal{P}^i Q b_u(N-i)(\tau_j)) + \int_0^{\tau_j} \Phi(\tau_j,t)b_N(u_N(t),t)dt, \; N > 1, \; j = 1, 2, \ldots, l, \tag{5.8}$$

where $\Phi(\cdot,\tau)$ denotes $\Phi(t,\tau)$ as the first variable ranges over $t \in T$ with the second variable fixed at $\tau \in T$. Also $c = (c_1, c_2, \ldots, c_l)^T \in \mathbb{R}^{nl}$, where

$$c_j = \Phi(\tau_j, 0)\alpha(N) + \sum_{i=1}^{N-1} (\mathcal{P}^i \Phi(\cdot, 0))(\tau_j)\alpha(N-i) + (\mathcal{P}^N \beta)(\tau_j), \ j = 1, 2, \ldots, k. \tag{5.9}$$

and introduce the mapping $S : U(\cdot) \to \mathbb{R}^{nl}$ as $Su = (S_1 u, S_2 u, \ldots, S_l u)^T$ where

$$S_j u = \sum_{i=1}^{N-1} (\mathcal{P}^i Q b_u(N-i)(\tau_j)) + \int_0^{\tau_j} \Phi(\tau_j, t) b_N(u_N(t), t)dt, \ j = 1, 2, \ldots, l. \tag{5.10}$$

Then we can state the following, denoted by *Problem* (A) whose solution will be used to solve the optimal control problem defined above:
Find necessary and sufficient conditions for

$$z = c + Su \tag{5.11}$$

to hold subject to

$$z \in M, \ \varphi(z) \le \delta, \ z \in \mathbb{R}^{nl}, \ u \in U(\cdot) \tag{5.12}$$

where $M = M_1 \times M_2 \times \ldots \times M_l \subset \mathbb{R}^{nl}$, and δ is a fixed number from $\mathbb{R}$.

To solve Problem (A), introduce first the following sets

$$\mathcal{R} = \{z \in \mathbb{R}^{nl}, \ z = c + Su, \ u \in U(\cdot)\}, \qquad K(\delta) = \{z \in \mathbb{R}^{nl}, \ z \in M, \ \varphi(z) \le \delta\}. \tag{5.13}$$

Then it is easy to see that the necessary and sufficient condition for Problem (A), to have a solution is $\mathcal{R} \cap K(\delta) \ne \emptyset$. In the following, we establish an analytical form of this geometric criteria which is based on the separation theorem for convex sets.

Consider first the problem of obtaining the required properties of the sets $\mathcal{R}$ and $K(\delta)$. Then the main technical difficulties here are related to the convexity and closeness of the set $\mathcal{R}$ which must be established in order to apply the separation theorem. Here we extend the results for $1D$ systems (see, for example, [7]) to a repetitive processes in order to overcome these difficulties.

Let $f : U \times \mathbb{R}^n$ be a continuous function and introduce the following set

$$Z = \left\{ z = (z_1, \ldots, z_l) \in \mathbb{R}^{nl} : z_j = \int_0^{\tau_j} f(v(t), t)dt, \quad v \in V(\cdot), \quad j = 1, 2, \ldots, l \right\}, \tag{5.14}$$

where τ_j are given points such that $0 < \tau_1 < \tau_2 < \ldots < \tau_l = t^*$, and $V(\cdot)$ is the set of all measurable functions $v : T \to \mathbb{R}^r$ such that $v(t) \in U$ for almost all $t \in T$. Then the response formulas (5.8) and (5.10) show that the required properties of the set $\mathcal{R}$ can be established by studying analogous properties for the set Z.

Now we have the following results.

Lemma 7. *Let $f : U \times T \to \mathbb{R}^n$ be a continuous function. Then for any measurable function $v(\cdot) \in V(\cdot)$ and for a given number $\varepsilon > 0$ $\exists$ a partition of the interval T by points $0 = s_0 < s_1 < \ldots < s_m = t^*$ such that*

$$\sum_{j=0}^{m-1} \int_{s_j}^{s_{j+1}} \|f(v(t), \tau_j) - f(v(t), s_j)\| dt < \varepsilon \tag{5.15}$$

holds for any τ_j satisfying $s_j \leq \tau_j \leq s_{j+1}$, $j = 0, \ldots, m$.

Proof. This is based on the so-called C-property of measurable functions [48] and, in fact, follows immediately on some routine modifications for continuous functions given in [30]. Hence the details are omitted here.

Lemma 8. *Let $f : U \times T \to \mathbb{R}^n$ be a continuous function. Then the closure $\overline{Z}$ of the set*

$$Z = \left\{ z = (z_1, \ldots, z_l) \in \mathbb{R}^{nl} : z_j = \int_0^{\tau_j} f(v(t), t) dt, \quad v \in V(\cdot), \quad j = 1, 2, \ldots, l \right\} \tag{5.16}$$

is convex.

Proof. On using Lemma 7 this is reduced to a slight modification of the results [30, 7], and hence the details are omitted here.

Remark 3. *Convexity of $\overline{Z}$ is guaranteed by the presence of the integral terms in Z. This fact, known as hidden convexity, is an important property of continuous time control systems which follows, in general, from the Lyapunov theorem on the convexity of the range of an integral operator acting on vector measures. This result is often used, see, for example, [7, 45], to prove the convexity of the reachability set for control systems which are linear in state variables.*

Formulas (5.8), (5.10) state that each integral expression in $\mathcal{R}$ contains an available control $u(s,t)$ with a fixed single value of the discrete variable s and, therefore, is independent of the others. Hence, to prove that $\mathcal{R}$ is a closed set it is sufficient to show that a set formed by controls with some fixed value of the discrete variable k, $k = 1, \ldots, N$ is closed. The simplest case is often to consider $k = N$ and then the set to be studied has the following form

$$\mathcal{R}_N = \left\{ z \in \mathbb{R}^{nl} : z_j = a_j + L_j v, \quad v(\cdot) \in V(\cdot), \quad j = 1, 2, \ldots, l \right\}. \tag{5.17}$$

Here $a_j = \Phi(\tau_j, 0)\alpha(N)$, and the mappings L_j defined on the set $V(\cdot)$ are given by

$$L_j v = \int_0^{\tau_j} \Phi(\tau_j, t) g(v(t), t)) dt$$

where $g(v(t), t)$ denotes the function $b_N(v_N(t), t)$, $t \in T$.

Lemma 9. *The set $\mathcal{R}_N$ defined by (5.17) is closed.*

Proof. Suppose that the vector sequence $\{z^n\} = \{(z_1^n, \ldots, z_l^n)^T\}$, $z_i^n \in \mathcal{R}_N$, $i = 1, \ldots, n$, converges to a point $z^* = (z_1^*, \ldots, z_l^*)^T \in \mathbb{R}^{nl}$. Then there exists a sequence $\{v^n(\cdot)\}$ of functions from $V(\cdot)$ such that $z_j^n = a_j + L_j v^n$, $j = 1, \ldots, l$ and we show that there exists a function $v^*(t)$, $t \in T$ from $V(\cdot)$ such that $z_j^* = c_j + L_j v^*$, $j = 1, \ldots, l$.

Consider the set $R(\alpha_N, 0) = \{y \in \mathbb{R}^n : y = a_1 + L_1 v,\ v \in V(\cdot)\}$. Then it is easy to see that $R(\alpha_N, 0)$ is the reachability set at $t = \tau_1$ for the following system

$$\dot{y}(t) = A(t)y(t) + g(v(t), t), \quad y(0) = \alpha(N), \quad v \in V(\cdot), \quad t \in T \tag{5.18}$$

Also, it is well known, see, for example, [45], that $R(\alpha_N, 0)$ is a closed set. Hence, for the sequence $\{z_1^n\} \to z_1^*$, $n \to \infty$, $z_1^n \in R(\alpha_N, 0)$, $n = 1, 2, \ldots$ there exists a function $v^1 \in V(\cdot)$ such that $z_1^* = a_1 + L_1 v^1$. Now introduce the sequence $\tilde{z}_2^n = \tilde{a}_2 + \tilde{L}_2 v^n$, where $\tilde{a}_2 = \Phi(\tau_2, \tau_1) z_1^*$ and $\tilde{L}_2 v^n = \int_{\tau_1}^{\tau_2} \Phi(\tau_2, t) g(v^n(t), t))dt$, i.e. $\tilde{z}_2^n$ is the solution of the system (5.18) corresponding to the function $v^n(t)$ and initial condition $y(\tau_1) = z_1^*$, where $\tilde{z}_2^n$ and $v^n(t)$ are restricted to the interval $[\tau_1, \tau_2]$. Next, we show that $\tilde{z}_2^n \to z_2^*$.

It is known [36] that the fundamental matrix $\Phi(\tau, t)$ satisifes $\Phi(\tau, s)\Phi(s, t) = \Phi(\tau, t)$, $0 \le \tau < s < t \le t^*$, and the Cauchy response formula now yields

$$\begin{aligned} z_2^n &= \Phi(\tau_2, 0)\alpha(N) + \int_0^{\tau_2} \Phi(\tau_2, t) g(v^n(t), t)dt = \Phi(\tau_2, \tau_1)\Big[\Phi(\tau_1, 0)\alpha(N) + \\ &\int_0^{\tau_1} \Phi(\tau_1, t) g(v^n(t), t)dt\Big] + \int_{\tau_1}^{\tau_2} \Phi(\tau_2, t) g(v^n(t), t)dt = \Phi(\tau_2, \tau_1) z_1^n + \int_{\tau_1}^{\tau_2} \Phi(\tau_2, t) g(v^n(t), t)dt. \end{aligned}$$

Then

$$\tilde{z}_2^n = \Phi(\tau_2, \tau_1) z_1^* + \int_{\tau_1}^{\tau_2} \Phi(\tau_2, t) g(v^n(t), t)dt.$$

Therefore

$$\|\tilde{z}_2^n - z_2^*\| \le \|\tilde{z}_2^n - z_2^n\| + \|z_2^n - z_2^*\| \le C\|z_1^n - z_1^*\| + \|z_2^n - z_2^*\|,$$

where $C = \|\Phi(\tau_2, \tau_1)\| < \infty$ is a constant. Since $z_1^n \to z_1^*, z_2^n \to z_2^*$, it follows immediately from the last inequality that also $\tilde{z}_2^n \to z_2^*$.

Introduce the set

$$R(z_1^*, \tau_1) = \{y \in \mathbb{R}^n : y = \tilde{a}_2 + \tilde{L}_2 v,\ v \in V(\cdot)\}. \tag{5.19}$$

Then it is obvious that $R(z_1^*, \tau_1)$ is the reachability set at $t = \tau_2$ for the system (5.18) and is restricted to the interval $[\tau_1, \tau_2]$ with initial condition $y(\tau_1) = z_1^*$. As shown above, $R(z_1^*, \tau_1)$ is a closed set. Therefore, for the sequence $\tilde{z}_2^* \to z_2^*$, $n \to \infty$ such that $\tilde{z}_2^n \in R(z_1^*, \tau_1)$, there exists a function $v^2(t)$, $\tau_1 \le t \le \tau_2$, $v^2 \in V(\cdot)$, such that $z_2^* = \tilde{a}_2 + \tilde{L}_2 v^2$.

In an analogous way, it can be established that on every interval $[\tau_j, \tau_{j+1}]$, there exists a function $v^{j+1} \in V(\cdot)$, $j = 1, \dots, l-1$, such that $z^*_{j+1} = \tilde{a}_{j+1} + \tilde{L}_{j+1} v^{j+1}$, where

$$\tilde{a}_{j+1} = \Phi(\tau_{j+1}, \tau_j) z^*_j, \quad \tilde{L}_{j+1} v = \int_{\tau_j}^{\tau_{j+1}} \Phi(\tau_{j+1}, t) g(v(t), t)) dt$$

Finally, we define on $T = [0, t^*]$ the function

$$v^*(t) = \begin{cases} v_1(t), & 0 \le t < \tau_1, \\ v_2(t), & \tau_1 \le t < \tau_2, \\ \dots & \dots \\ v^l(t), & \tau_{l-1} \le t \le t^* \end{cases}$$

where clearly $v^* \in V(\cdot)$. Also, it follows immediately from

$$\begin{aligned} z^*_j &= \tilde{a}_j + \tilde{L}_j v^j = \Phi(\tau_j, \tau_{j-1}) z^*_{j-1} + \int_{\tau_{j-1}}^{\tau_j} \Phi(\tau_j, t) g(v^j(t), t) dt = \Phi(\tau_j, \tau_{j-1}) \Big[\Phi(\tau_{j-1}, \tau_{j-2}) z^*_{j-2} \\ &+ \int_{\tau_{j-2}}^{\tau_{j-1}} \Phi(\tau_{j-1}, t) g(v^{j-1}(t), t) dt \Big] + \int_{\tau_{j-1}}^{\tau_j} \Phi(\tau_j, t) g(v^j(t), t) dt = \Phi(\tau_j, \tau_{j-2}) z^*_{j-2} \\ &+ \int_{\tau_{j-2}}^{\tau_{j-1}} \Phi(\tau_j, t) g(v^{j-1}(t), t) dt + \int_{\tau_{j-1}}^{\tau_j} \Phi(\tau_j, t) g(v^j(t), t)) dt = \dots = \Phi(\tau_j, 0) \alpha(N) \\ &+ \int_0^{\tau_1} \Phi(\tau_j, t) g(v^1(t), t) dt + \int_{\tau_1}^{\tau_2} \Phi(\tau_j, t) g(v^2(t), t) dt + \dots + \int_{\tau_{j-1}}^{\tau_j} \Phi(\tau_j, t) g(v^j(t), t) dt \\ &= \Phi(\tau_j, 0) \alpha(N) + \int_{\tau_0}^{\tau_j} \Phi(\tau_j, t) g(v^*(t), t) dt = a_j + L_j v^*, \quad j = 1, \dots, l, \end{aligned}$$

$v^*(t)$ is the required function. Hence $z^* \in \mathcal{R}_N$, i. e. $\mathcal{R}_N$ is a closed set and the proof is complete. ■

Note. In the cases when $k \neq N$, then the additional terms in the formulas for a_j and L_j in the set $\mathcal{R}_k$ do not change the essence of given proof.

At this stage, we have established that $\mathcal{R}$ and $K(\delta)$ are closed and convex sets and the next result gives the solution of *Problem* (A), the inner product of vectors g and f from $\mathbb{R}^{nl}$ is denoted by $g^T f$.

Theorem 7. *Problem* (A) *has a solution if, and only if,*

$$\max_{\|g\|_{\mathbb{R}^{nl}}=1} \{ g^T c - \max_{z \in K(\delta)} g^T z + \min_{u \in U(\cdot)} g^T S u \} \le 0 \tag{5.20}$$

holds.

Proof. *Sufficiency.* Suppose that the condition of (5.20) is valid, but Problem (A) has no solution. Then, $\mathcal{R} \cap K(\delta) = \emptyset$ and the separation theorem for convex sets yields that there exists a nontrivial vector $g \in \mathbb{R}^{nl}$, $\|g\| = 1$ such that

$$\min_{z \in \mathcal{R}} g^T z > \max_{z \in K(\delta)} g^T z. \tag{5.21}$$

Hence

$$g^T c - \max_{z \in K(\delta)} g^T z + \min_{u \in U(\cdot)} g^T S u > 0 \tag{5.22}$$

which contradicts (5.20).

Necessity. Suppose that Problem (A) has a solution. Then there exist $\bar{u}$ and $\bar{z}$ satisfying (5.11)–(5.12) such that $g^T c + g^T S \bar{u} = g^T \bar{z}$ holds for each $g \in \mathbb{R}^{nl}$. Taking the maximum and minimum respectively of the two terms in this last expression now yield

$$g^T c - \max_{z \in K(\delta)} g^T z + \min_{u \in U(\cdot)} g^T S u \leq 0, \tag{5.23}$$

as required and the proof is complete. ∎

5.1.2 Optimality conditions and maximum principle

In this sub-section we use the results of the previous sub-section to establish the maximum principle for the optimal control problem (5.1)–(5.4).

Introduce the function $\Lambda : \mathbb{R} \to \mathbb{R}$ as

$$\Lambda(\delta) = \max_{\|g\|_{\mathbb{R}^{nl}}=1} \{g^T c - \max_{z \in K(\delta)} g^T z + \max_{u \in U(\cdot)} g^T S u\}. \tag{5.24}$$

where it can be shown that $\Lambda(\delta) : \mathbb{R} \to \mathbb{R}$ defined by (5.24) is a non increasing continuous function. Hence the optimal value of the performance index (5.4) can be characterized as follows.

Theorem 8. *The control $u^0 \in U(\cdot)$ is the optimal solution of the problem (5.1)-(5.4) if, and only if, $\delta^0 := J(u^0)$ is the smallest root of the equation $\Lambda(\delta) = 0$.*

Proof. *Necessity.* Let $u^0 \in U(\cdot)$ be an optimal control of the problem (5.1) — (5.4). Then u^0 is the solution of Problem (A) with $\delta^0 := J(u^0)$. Therefore, Theorem 7 yields that $\Lambda(\delta^0) \leq 0$.

Suppose now that $\Lambda(\delta^0) < 0$. Then since $\Lambda(\delta)$ is a continuous and monotone function, there exists a number $\bar{\delta}$ such that $\bar{\delta} < \delta^0$ and $\Lambda(\bar{\delta}) \leq 0$. Hence, Theorem 7 yields that Problem (A) has a solution where $\delta = \bar{\delta}$ since otherwise there would be an available control $\bar{u} \in U(\cdot)$ and a vector $\bar{z} \in M$ satisfying (5.11)–(5.12) for $\delta = \bar{\delta}$. Hence, $J(\bar{u}) < J(u^0)$, which contradicts the optimality of the control u^0 and therefore $\Lambda(\delta^0) = 0$. Finally, the fact that δ^0 is the smallest root of the equation $\Lambda(\delta) = 0$ can be proved as above.

Sufficiency. Let $u^0 \in U(\cdot)$ be a control function such that δ^0 is the smallest root of $\Lambda(\delta) = 0$. Suppose also that $u^0(t)$ is not an optimal solution of the problem (5.1)-(5.4). Then there exists an available control function $\bar{u} \in U(\cdot)$ and a vector $\bar{z} \in M$ such that $c - \bar{z} + S\bar{u} = 0$ and $J(\bar{u}) < J(u^0)$ holds. This establishes that Problem (A) has a solution for $\bar{\delta} = J(\bar{u})$, and hence $\Lambda(\bar{\delta}) \leq 0$.

Conversely, since the function $\Lambda(\delta)$ is monotone $\Lambda(\bar{\delta}) \geq \Lambda(J(u^0)) = 0$, which contradicts the assertion that δ^0 is the smallest root. Hence u^0 is an optimal control and the proof is complete. ■

Next, let $g^0 = (g_1^0, \dots, g_l^0)^T \in \mathbb{R}^{nl}$ be a maximizing vector for $\Lambda(\delta^0)$ and on the interval $T = [0, t^*]$ we introduce the following function $\lambda : \mathbb{R} \to \mathbb{R}^m$

$$\lambda(t) = \sum_{i=j+1}^{l} (g_i^0)^T \Phi(\tau_i, t), \quad \tau_j \leq t < \tau_{j+1}, \quad j = 0, \dots, l-1. \tag{5.25}$$

Then it is a simple task to verify that the function $\lambda(t)$ satisfies

$$\frac{d\lambda(t)}{dt} = -\lambda^T(t)A(t), \quad \lambda(\tau_j - 0) - \lambda(\tau_j + 0) = g_j^0, \quad j = 1, \dots, l-1. \tag{5.26}$$

and the optimality conditions for (5.1)–(5.4) are given by the following theorem.

Theorem 9. *If the number δ^0 is the smallest root of the equation $\Lambda(\delta) = 0$, then there exists an optimal control $u_k^0(t)$, $k \in K$, $t \in T$ for the problem (5.1)-(5.4) such that $J(u^0) = \delta^0$ and for almost all $t \in T$*

$$\psi_k^T(t) b_{N-k+1}(u_{N-k+1}^0(t), t) = \min_{v \in U} \psi_k^T(t) b_{N-k+1}(v, t), \tag{5.27}$$

holds for all $k \in K$. Here the function $\psi : K \times T \to \mathbb{R}^n$ is given by

$$\psi_k(t) = \int_0^t \psi_{k-1}^T(\tau) D(\tau) \Phi(\tau, t) d\tau, \quad \psi_1(t) = \lambda(t), \quad k \in K, \tag{5.28}$$

where the function $\lambda(t)$ is given by (5.26).

Proof. Since $\Lambda(\delta^0) = 0$, Theorem 8 yields that Problem (A) has a solution for $\delta = \delta^0$. This implies that there exists an available control $u^0 \in U(\cdot)$ and a vector $z^0 \in M$ satisfying (5.11)-(5.12). Hence $\varphi(z^0) = J(u^0) \leq \delta^0$. The assumption $J(u^0) < \delta^0$ leads to a contradiction with the assumption that δ^0 is the smallest root of the equation $\Lambda(\delta) = 0$. Therefore, $J(u^0) = \delta^0$, and, consequently, u^0 is an optimal control for (5.1)-(5.4).

The function $u_k^0(t)$, $k \in K$, $t \in T$ satisfies

$$(g^0)^T S u^0 = \min_{u \in U(\cdot)} (g^0)^T S u. \tag{5.29}$$

and if we assume that $(g^0)^T S u^0 > \min_{u \in U(\cdot)} (g^0)^T S u$, then we have that

$$\Lambda(\delta^0) < (g^0)^T c - (g^0)^T z^0 + (g^0)^T S u = 0, \tag{5.30}$$

which is impossible since δ^0 is a root of $\Lambda(\delta) = 0$. Finally, to establish the desired optimality condition (5.27) we employ (5.29). Then

$$\min_{u\in U(\cdot)} (g^0)^T Su = \min_{u\in U(\cdot)} \sum_{j=1}^{l} (g_j^0)^T \left(\sum_{i=1}^{N-1} \mathcal{P}^i Q b_u(N-i)(\tau_j) + \int_0^{\tau_j} \Phi(\tau_j, t) b_N(u_N(t), t) dt \right)$$

$$= \min_{u\in U(\cdot)} \Bigg\{ \int_0^{\tau_1} \Big[(g_1^0)^T \Phi(\tau_1, t) + \cdots + (g_l^0)^T \Phi(\tau_l, t) \Big] b_N(u_N(t), t) dt + \int_{\tau_1}^{\tau_2} \Big[(g_2^0)^T \Phi(\tau_2, t) + \cdots$$

$$+ \; (g_l^0)^T \Phi(\tau_l, t) \Big] b_N(u_N(t), t) dt + \cdots + \int_{\tau_{l-1}}^{\tau_l} (g_l^0)^T \Phi(\tau_l, t) b_N(u_N(t), t) dt + \cdots$$

$$+ \int_0^{\tau_1} \Big[(g_1^0)^T \Phi(\tau_1, t) + \cdots + (g_l^0)^T \Phi(\tau_l, t) \Big] D(t) \int_0^t \Phi(t, s) b_{N-1}(u_{N-1}(t), t) ds dt$$

$$+ \int_{\tau_1}^{\tau_2} \Big[(g_2^0)^T \Phi(\tau_2, t) + \cdots + (g_l^0)^T \Phi(\tau_l, t) \Big] D(t) \int_0^t \Phi(t, s) b_{N-1}(u_{N-1}(t), t) ds dt + \cdots$$

$$+ \int_{\tau_{l-1}}^{\tau_l} (g_l^0)^T \Phi(\tau_l, t) D(t) \int_0^t \Phi(t, s) b_{N-1}(u_{N-1}(t), t) ds dt + \cdots + \int_0^{\tau_1} \Big[(g_1^0)^T \Phi(\tau_1, t) + \cdots$$

$$+ \; (g_l^0)^T \Phi(\tau_l, t) \Big] D(t) \mathcal{P}^{N-1} Q b_u(1)(t) dt + \cdots + \int_{\tau_{l-1}}^{\tau_l} (g_l^0)^T \Phi(\tau_l, t) D(t) \mathcal{P}^{N-1} Q b_u(1)(t) dt \Bigg\}$$

$$= \min_{u\in U(\cdot)} \left\{ \psi_1^T(t) b_N(u_N(t), t) + \cdots + \psi_N^T(t) b_1(u_1(t), t) \right\} = \sum_{k\in K} \min_{v\in U} \psi_k^T(t) b_{N-k+1}(v, t).$$

which yields (5.27) and the proof is complete. ■

Remark 4. *The analysis just presented gives the optimal control solution in the standard maximum principle which is not necessarily compatible with the numerical computations as required in applications. Hence we proceed here to develop new optimality and sub-optimality conditions which are more suitable for such purposes. Also we only consider the stationary case as this is the most relevant in terms of applications.*

5.2 Stationary Differential Linear Repetitive Processes

In this Section, the process (5.1) is assumed to be stationary, and the pass constraints (5.3) and the cost function (5.4) have a special form as detailed below. Also the solutions here are, in effect, developed by extending the constructive methods approach developed in [29] to a repetitive process setting.

The processes considered in this section are described in $\mathbb{R}^n$ by the following linear matrix differential equation

$$\frac{dx_k(t)}{dt} = Ax_k(t) + Dx_{k-1}(t) + bu_k(t), \quad k \in K = \{1, \cdots, N\}, \; t \in T = [0, t^*] \tag{5.31}$$

with boundary conditions

$$x_k(0) = \alpha_k, \quad k \in K, \quad x_0(t) = f(t), \quad t \in T, \tag{5.32}$$

and a pass end, or terminal, constraint of the form

$$H_s x_k(t^*) = g_k, \qquad k \in K, \tag{5.33}$$

Here $b \in \mathbb{R}^n$ and $g_k \in \mathbb{R}^m$ are $n \times 1$ and $m \times 1$ specified vectors, $A, D, H_k,\; k \in K$ are constant matrices of $n \times n$, $n \times n$ and $m \times n$ dimensions, respectively. In addition, we assume that the matrix A has simple eigenvalues $\lambda_j,\; 1 \leq i \leq n$, and that it is a stable matrix in the sense that $Re\lambda_i < 0,\; 1 \leq i \leq n$.

Definition 8. *For every $k \in K$ the piecewise continuous function $u_k : T \to \mathbb{R}$ is termed an admissible control for pass k if it satisfies*

$$|u_k(t)| \leq 1, \; t \in T. \tag{5.34}$$

The optimization problem is to find the admissible controls $u_1(t), ..., u_N(t)$ such that the corresponding solution of the system (5.31)–(5.33) maximizes the following cost function

$$\max_{u_k} J(u), \; J(u) = \sum_{k \in K} p_k^T x_k(t^*) \tag{5.35}$$

where $p_k,\; k = 1, \ldots, n$ are given $n \times 1$ vectors.

5.2.1 Optimality conditions for supporting control functions.

In the first step, note that the solution of (5.31)–(5.32) (with no terminal conditions of (5.33)) can be written as follows

$$x_k(t) = \sum_{j=1}^{k} K_j(t)\alpha_{k+1-j} + \int_0^t K_k(t-\tau) D f(\tau) d\tau +$$

$$\sum_{j=1}^{k} \int_0^t K_j(t-\tau) b u_{k+1-j}(\tau) d\tau, \; k = 1, \ldots, N \tag{5.36}$$

where the $K_i(t)$ are the solutions of the following $n \times n$ matrix differential equations

$$\dot{K}_1(t) = AK_1(t), \quad \dot{K}_i(t) = AK_i(t) + DK_{i-1}(t), \quad i = 2, \ldots, N. \tag{5.37}$$

with initial conditions

$$K_1(0) = E, \quad K_i = 0, \quad i = 2, \ldots, N. \tag{5.38}$$

Also it is easily shown that these solutions have the following properties

$$\begin{aligned}
K_j(t-\sigma) &= \int_\sigma^t K_{j-k}(t-\tau) D K_k(\tau-\sigma) d\tau, \quad 0 \le \sigma < t \le t^*, \quad \forall k = 1, \ldots, j-1; \\
K_j(t-\sigma) &= \sum_{s=1}^{j} K_s(t-\tau) K_{j+1-s}(\tau-\sigma) \qquad j = 2, \ldots, N-1,
\end{aligned} \tag{5.39}$$

which will be used below.

Now using (5.36)we can rewrite the optimization problem in the following integral form

$$\max_{u_1,\ldots,u_N} J(u), \; J(u) = \sum_{j=1}^{N} \int_0^{t^*} c_j(\tau) u_j(\tau) d\tau + \gamma \tag{5.40}$$

subject to the terminal conditions (5.32) and the control constraints (5.34), which can also be rewritten as

$$\begin{cases}
\int_0^{t^*} g_{11}(\tau) u_1(\tau) d\tau & = h_1, \\
\int_0^{t^*} \Big[g_{21}(\tau) u_1(\tau) + g_{22}(\tau) u_2(\tau) \Big] d\tau & = h_2, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\int_0^{t^*} \Big[g_{N1}(\tau) u_1(\tau) + \ldots + g_{NN}(\tau) u_N(\tau) \Big] d\tau & = h_N,
\end{cases} \tag{5.41}$$

and

$$|u_k(\tau)| \le 1, \; \tau \in [0, t^*], \quad k = 1, \ldots, N,$$

where

$$\begin{aligned}
\gamma &= \sum_{k=1}^{N} \sum_{j=1}^{k} p_k^T K_j(t^*) \alpha_{k+1-j} + \sum_{k=1}^{N} \int_0^{t^*} p_k^T K_k(t^*-\tau) D f(\tau) d\tau, \\
c_j(\tau) &= \sum_{k=j}^{N} p_k^T K_{k+1-j}(t^*-\tau) b, \quad j = 1, \ldots, N, \quad g_{kj}(\tau) = H_k K_{k+1-j}(t^*-\tau) b, \quad j \le k, \\
h_k &= g_k - \sum_{j=1}^{k} H_k K_j(t^*) \alpha_{k+1-j} - \int_0^{t^*} H_k K_k(t^*-\tau) D f(\tau) d\tau \quad k = 1, \ldots, N.
\end{aligned}$$

Also we require the following concepts.

Definition 9. *For each fixed k,* $1 \le k \le N$, *the instance* $0 < \tau_{k1} < \tau_{k2} < \cdots < \tau_{km} < t^*$ *are termed supporting and their collection* $\tau^k_{sup} \triangleq \{\tau_{k1}, \ldots, \tau_{km}\}$ *is termed as the support of pass k for the problem (5.31)–(5.35) if the matrix* $G^k_{sup} := \{g_{kk}(\tau_{k1}), \ldots, g_{kk}(\tau_{km})\}$ *is non-singular.*

Note also that from (5.37) we have that $g_{kk}(\tau) = H_k e^{A(t^*-\tau)}b$. Therefore the existence of the support τ^k_{sup} is guaranteed by the controllability of the pair $\{A, b\}$.

Definition 10. *A pair* $\{\tau^k_{sup}, u_k(t), k = 1, ..., N\}$ *consisting of a support* τ^k_{sup} *and admissible control functions* $u_k(t), t \in T$ *is termed as a supporting control function for (5.31)–(5.35).*

Remark 5. *These last two definitions are motivated as follows.*

Often an optimal control problem solution has the so-called bang-bang form, i.e. the control function takes only boundary values in the admissible set U. *If* $U \doteq \{-1 \le u \le +1\}$ *then* $u^0(t) = \pm 1$ *(the 'switch on/switch off' regime). Also the switching times are constructive elements in the design of the optimal controller. Hence, our goal is to apply these key elements directly to the optimality conditions and hence we use the supporting time instances and control.*

Let $\{\tau^k_{sup}, u_k(t), k = 1, \ldots, N\}$ be a support control function and construct a sequence of $m \times 1$ vectors $\{\nu^{(k)},\ k = 1, \ldots, N\}$ by solving following set of linear algebraic equations

$$\begin{cases} (\nu^{(N)})^T G^N_{sup} - c^{(N)}_{sup} & = 0, \\ (\nu^{(N-1)})^T G^{N-1}_{sup} + (\nu^{(N)})^T F^N_{(N-1)sup} - c^{(N-1)}_{sup} & = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ (\nu^{(1)})^T G^1_{sup} + (\nu^{(2)})^T F^2_{1sup} + \cdots + (\nu^{(N)})^T F^N_{1sup} - c^{(1)}_{sup} & = 0 \end{cases} \tag{5.42}$$

where the vectors $c^{(k)}_{sup}$ and matrices F^k_{jsup} are given by

$$\begin{aligned} c^{(k)}_{sup} &\triangleq \left(c_k(\tau_{k1}), \ldots, c_k(\tau_{km})\right)^T, \qquad k = 1, \ldots, N, \\ F^k_{jsup} &\triangleq \Big(g_{kj}(\tau_{j1}), \ldots, g_{kj}(\tau_{jm})\Big), \qquad k > j, \quad j = 1, \ldots, N-1. \end{aligned}$$

Now define the so-called co-control function as

$$\Delta(t) = (\Delta_1(t), \ldots, \Delta_N(t))^T := \hat{\nu}' \hat{G}(t) - c(t),$$

where $\hat{\nu} = (\nu^{(1)}, \ldots, \nu^{(N)})^T$, $c(t) = (c_1(t), \ldots, c_N(t))^T$, and $\hat{G}(t)$ is an $mN \times mN$ matrix function of triangular form, whose rows are the $m \times 1$ vector functions $g_{ij}(t)$ of (5.41). Also introduce the nonsingular $mN \times mN$ matrix

$$\tilde{G}_{sup} := \begin{pmatrix} g_{kj}(t), & t \in \tau^k_{sup} \\ j \le k, & k = 1, ..., N \end{pmatrix} =: \begin{pmatrix} G^1_{sup} & 0 & \cdots & 0 \\ G^2_{sup} & G^2_{sup} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ G^N_{sup} & G^N_{sup} & \cdots & G^N_{sup} \end{pmatrix} \tag{5.43}$$

Hence the $mN \times 1$ vector $\nu = (\nu^{(1)}, \dots, \nu^{(N)})^T$ required in (5.42) can be given by $\nu = c_{sup}^T \tilde{G}_{sup}^{-1}$.

Definition 11. *We say that the support control function $\{\tau_{sup}^k, u_k(t), k = 1, \dots, N\}$ is non-degenerate for the problem (5.31)-(5.35) if*

$$\frac{d\Delta_k(\tau_j)}{dt} \neq 0 \quad \forall\ \tau_j \in \tau_{sup}^k, \quad k = 1, \dots, N.$$

Remark 6. *Here non-degenerancy means that in a small neighborhood of the supporting points the admissible control can be replaced by constant functions whose values are less than those on the control constraint boundary and which satisfy (5.41), i.e. the support control function is non-singular if there exists numbers $\lambda_0 > 0, \quad \mu_0 > 0, \quad u_j^k(\lambda), \quad j = 1, \dots, m, \quad k = 1, \dots, N$ such that the following equalities*

$$\sum_{j=1}^{k}\sum_{i=1}^{m} u_j^i(\lambda) \int\limits_{\tau_{ij}-\lambda}^{\tau_{ij}+\lambda} g_{kj}(t)dt = \sum_{j=1}^{k}\sum_{i=1}^{m} \int\limits_{\tau_{ij}-\lambda}^{\tau_{ij}+\lambda} g_{kj}(t)u_j(t)dt,$$
$$|u_j^k| \leq 1 - \mu_0, \qquad j = 1, \dots, m, \quad k = 1, \dots, N. \tag{5.44}$$

hold for all $\lambda, \quad 0 < \lambda < \lambda_0$ and $k, \quad 1 \leq k \leq N$. Also (5.44) will be used below in the proof of the optimality conditions.

Associate with each supporting time instance τ_{kj} a small sub-interval T_{kj} from T such that the matrix $G_{gen}^k := \{ \int\limits_{T_{kj}} g_{kk}(\tau)d\tau, j = 1, \dots, m\}$ is non-singular, and, without loss of generality we can further assume that τ_{kj} is one of the ends of T_{kj} and the supporting control function $u_k(t) = u_j^k$ for $t \in T_{kj}, \quad j = 1, \dots, N$ are constant over the segments T_{kj}.

Now we have the following result.

Theorem 10. *A supporting control function $\{\tau_{sup}^k, \quad u_k^0(t), \quad k = 1, \dots, N\}$ is an optimal solution of the problem (5.31)-(5.35) if*

$$u_k^0(t) = \operatorname{sign}\big(\Delta_k(t)\big), \qquad k = 1, \dots, N. \tag{5.45}$$

Moreover, if this supporting control function is non-degenerate then the above condition is necessary and sufficient.

Proof. *Sufficiency.* Let $u_k(t) \neq u_k^0(t), \quad k = 1, \dots, N$ be an admissible control and $x_k(t)$ be the corresponding trajectory of the system (5.31)–(5.32). Since from (5.41) it follows that

$$G(t)[u^0(t) - u(t)] = 0$$

for the admissible control functions $u^0(t) = \{u_1^0(t), \dots, u_N^0(t)\}$ and $u(t) = \{u_1(t), \dots, u_N(t)\}$ then (5.40) and the definition of the co-control function $\Delta(t)$ yilds that the increment, $\Delta J(u) := J(u^0) - J(u)$, of the cost function can be expressed in the form

$$\Delta J(u) = \int\limits_0^{t^*} \sum_{j=1}^{N} c_j(t)\big[u_j^0(t) - u_j(t)\big]dt = -\sum_{j=1}^{N} \int\limits_0^{t*} \Delta_j(t)\big[u_j^0(t) - u_j(t)\big]dt.$$

Hence, (5.45) yields that $\Delta J(u) \geq 0$ for any admissible control u, i.e. $\{\tau_{sup}^k, u_k^0\}$ is an optimal (maximizing) supporting control function.

Necessity. Let $\{\tau_{sup}^k,\ u_k^0(t),\ k = 1, \dots, N\}$ be an optimal non-degenerate control but $\exists\ k_*, 1 \leq k_* \leq N$ and $\exists\ t_* \in T$, such that the theorem is not valid. If we suppose that $t_* \in [\tau_{k_*j} - \lambda, \tau_{k_*j} + \lambda]$ where $\lambda > 0$ is a small number, i.e. the instance t_* lies in the neighborhood of some supporting time instance τ_{k_*j}, then using the fact that the supporting control is non-degenerate yields that there exists a control variation $\Delta u_{k_*}^0(t)$, defined on the intervals $[\tau_{k_*j} - \lambda, \tau_{k_*j} + \lambda]$, such that $J(u^0) > 0$, which contradicts the optimality of $u_k^0(t)$. Therefore, we suppose that $t_* \notin [\tau_{k_*j} - \lambda, \tau_{k_*j} + \lambda] \quad \forall\, j = 1, \dots, m$ for some small $\lambda > 0$.

Next, without loss of generality, assume that $\Delta_{k_*}^0(t_*) > 0$ and $u_{k_*}(t_*) > 0$. Then by continuity of $\Delta_{k_*}(t)$ and piecewise-continuity of $u_{k_*}(t)$ there exists an neighborhood $T_{k_*}(t_*)$ of t_*, such that $\Delta k_*(t) > 0, \quad u_{k_*}(t) > -1$ for $t \in T_{k_*}(t_*)$. Now, we have to construct the admissible control variation such that the corresponding increment of the cost function satisfies $\Delta J(u) > 0$, which is impossible for the optimal controls $u_k^0(t)$.

Consider now the case of a small real number $\lambda_0 > 0$ (we see below that the existence of such a number λ_0 is guaranteed by the fact that the supporting control is non-degenerate) and for all $\lambda,\ 0 < \lambda < \lambda_0$ defines the control variation $\Delta u(t) = (\Delta u_1(t), ..., \Delta u_N(t)), \quad t \in T$ as

$$\Delta u_k(t) \quad = \quad 0, \quad k < k_*,\ t \in T;$$

$$\Delta u_{k_*}(t) \quad = \quad \begin{cases} \theta(-1 - u_{k_*}(t)), & \theta > 0, \quad t \in T_{k_*}(t); \\ 0, & t \in T \setminus \left(\bigcup\limits_{j=1}^{m} [\tau_{k_*j} - \lambda, \tau_{k_*j} + \lambda] \cup T_{k_*}(t) \right). \end{cases}$$

Hence the control variations on the intervals $[\tau_{k_*j} - \lambda, \tau_{k_*j} - \lambda],\ j = 1, \dots, m$ can be chosen as constant functions $\Delta u_{k_*}(t) \equiv \Delta \vartheta_j^k(\lambda)$. The control variations for the remaining passes $k > k_*$ are defined as

$$\Delta u_k(t) \equiv 0, \quad k = k_* + 1, \dots, N, \quad t \in T \setminus \bigcup_{j=1}^{m} [\tau_{kj} - \lambda, \tau_{kj} + \lambda];$$

$$\Delta u_k(t) \equiv \Delta \vartheta_j^k(\lambda),\ t \in [\tau_{kj} - \lambda, \tau_{kj} + \lambda], \quad j = 1, \dots, m, \quad k > k_*$$

where $\Delta \vartheta_j^k(\lambda)$ are unknown constants which are determined below.

Using (5.41), it follows that the conditions

$$\int_0^{t^*} \sum_{s=1}^{k} g_{ks}(\tau) \Delta u_s(\tau) d\tau = 0,\ k = 1, \dots, N, \tag{5.46}$$

hold for any admissible variation $\Delta u(t)$ and can be re-written in the form

$$\phi_{k_*}(\lambda) \triangleq \sum_{j=1}^{m} \int_{\tau_{k_*j}-\lambda}^{\tau_{k_*j}+\lambda} g_{k_*k_*}(\tau)\vartheta_j^{k_*}(\lambda)d\tau = -\theta \int_{T_{k_*}(t_*)} g_{k_*k_*}(\tau)(-1-u_{k_*}(\tau))d\tau,$$

$$\phi_{k_*+1}(\lambda) \triangleq \sum_{j=1}^{m} \int_{\tau_{k_*+1j}-\lambda}^{\tau_{k_*+1j}+\lambda} g_{k_*+1k_*+1}(\tau)\vartheta_j^{k_*+1}(\lambda)d\tau =$$

$$-\sum_{j=1}^{m} \int_{\tau_{k_*j}-\lambda}^{\tau_{k_*j}+\lambda} g_{k_*+1k_*}(\tau)\vartheta_j^{k_*}(\lambda)d\tau - \theta \int_{T_{k_*}(t_*)} g_{k_*+1k_*}(\tau)(-1-u_{k_*}(\tau))d\tau, \tag{5.47}$$

....

$$\phi_N(\lambda) \triangleq \sum_{j=1}^{m} \int_{\tau_{Nj}-\lambda}^{\tau_{Nj}+\lambda} g_{NN}(\tau)\vartheta_j^{N}(\lambda)d\tau = -\sum_{j=1}^{m} \int_{\tau_{k_*j}-\lambda}^{\tau_{k_*j}+\lambda} g_{Nk_*}(\tau)\vartheta_j^{k_*}(\lambda)d\tau -$$

$$-\theta \int_{T_{k_*}(t_*)} g_{Nk_*}(\tau)(-1-u_{k_*}(\tau))d\tau - \cdots - \sum_{j=1}^{m} \int_{\tau_{N-1j}-\lambda}^{\tau_{N-1j}+\lambda} g_{NN-1}(\tau)\vartheta_j^{N-1}(\lambda)d\tau$$

Expanding the function $\phi_{k_*}(\lambda)$ of (5.47) in a Taylor series truncated at the second order and setting $\Delta\vartheta_\lambda^{k_*} = \Delta\vartheta_1^{k_*}(\lambda), \ldots, \Delta\vartheta_m^{k_*}(\lambda)$ leads to

$$2\lambda G_{sup}^{k_*}\Delta\vartheta_\lambda^{k_*} + \frac{\lambda^3}{3}\{\frac{d^2 g_{k_*k_*}(\tau_{k_*j})}{d\tau},\ j=1,\ldots,m\}\Delta\vartheta_\lambda^{k_*} + o_{k_*}(\lambda^3) =$$

$$-\theta \int_{T_{k_*}(t_*)} g_{k_*k_*}(\tau)(-1-u_{k_*}(\tau))d\tau.$$

where $o_{k_*}(\lambda^3)$ denotes terms of 3 degrees and above which are neglected here. Hence the required vector $\Delta\vartheta_\lambda^{k_*}$ can be represented as

$$\Delta\vartheta_\lambda^{k_*} = \frac{1}{\lambda}\theta\hat{u}_{k_*} + \theta O_{k_*}(\lambda), \quad \text{where} \quad \hat{u}_{k_*} = -\frac{1}{2}G_{sup}^{k_*^{-1}} \int_{T_{k_*}(t_*)} g_{k_*k_*}(\tau)(-1-u_{k_*}(\tau))d\tau. \tag{5.48}$$

and $O_{k_*}(\lambda)$ denotes a residual first order term. Using (5.44) and (5.48), then for a small value of λ, $0 < \lambda < \lambda_0$, there exists a real number $\theta = \theta(\lambda)$, such that $\theta(\lambda) = \mu_{k_*}\lambda \le 1$, where $\mu_{k_*} > 0$ does not depend on λ, and the following inequalities

$$|u_j^{k_*}(\lambda) + \Delta\vartheta_j^{k_*}(\lambda)| \le 1, \quad j=1,\ldots,m$$

hold. Here we have exploited the fact that the admissible controls are constants $u_j^k(\lambda)$ over the intervals T_j^k, containing the supporting points τ_{kj}. Hence, the function

$$\bar{u}_{k_*}(t) = \begin{cases} u_j^{k_*}(\lambda) + \Delta\vartheta_j^{k_*}(\lambda), & t \in [\tau_{k_*j} - \lambda, \tau_{k_*j} + \lambda] \\ u_{k_*}(t) + \theta(\lambda)(-1 - u_{k_*}(t)), & t \in T_{k_*}(t_*) \end{cases}$$

is an admissible control function for $\theta(\lambda) = \mu_{k_*}\lambda \leq 1$ and a small μ_{k_*}.

In order to find $\Delta\vartheta_\lambda^{k_*+1}$ and $\theta(\lambda)$, expand $\phi_{k_*+1}(\lambda)$ as a Taylor series to yield

$$\sum_{j=1}^{m} \int_{\tau_{k_*j}-\lambda}^{\tau_{k_*j}+\lambda} g_{k_*+1k_*}(\tau)\Delta\vartheta_j^{k_*}(\lambda)d\tau = 2\lambda \sum g_{k_*+1k_*}(\xi_j)\Delta\vartheta_j^{k_*}(\lambda) = 2\lambda \tilde{G}_\xi^{k_*+1}\Delta\vartheta_\lambda^{k_*+1}$$

$$= 2\lambda\tilde{G}_\xi^{k_*+1}\left(\frac{1}{\lambda}\mu_{k_*}\lambda\hat{u}_{k_*} + \mu_{k_*}\lambda O_{k_*}(\lambda)\right) = 2\tilde{G}_\xi^{k_*+1}\mu_{k_*}\lambda\hat{u}_{k_*} + \mu_{k_*}\breve{o}_{k_*}(\lambda^3) \qquad (5.49)$$

Here the matrix $\tilde{G}_\xi^{k_*+1}$ is constructed from the rows $\{g_{k_*+1k_*}(\xi_j),\ j = 1, \ldots, m\}$, where ξ_j are points from the intervals $[\tau_{k_*j} - \lambda, \tau_{k_*j} + \lambda]$.

Next, set $\Delta\vartheta_\lambda^{k_*+1} = \left(\Delta\vartheta_1^{k_*+1}(\lambda), \ldots, \Delta\vartheta_m^{k_*+1}(\lambda)\right)$ to write

$$2\lambda G_{sup}^{k_*+1}\Delta\vartheta_\lambda^{k_*+1} + \frac{\lambda^3}{3}\left\{\frac{d^2 g_{k_*+1k_*+1}(\tau_{k_*j})}{d\tau}, j = 1, \ldots, m\right\}\Delta\vartheta_\lambda^{k_*+1} + o_{k^*+1}(\lambda^3) =$$

$$-\mu_{k_*}\lambda\left\{\tilde{G}_\xi^{k_*+1}\hat{u}_{k_*} + \int_{T_{k_*}(t_*)} g_{k_*+1k_*+1}(\tau)(-1 - u_{k_*}(\tau))d\tau\right\} + \mu_{k_*}\breve{o}_{k_*}(\lambda^3). \qquad (5.50)$$

which means that the required vector $\Delta\vartheta_\lambda^{k_*+1}$ can be expressed as

$$\begin{aligned} \Delta\vartheta_\lambda^{k_*+1} &= \frac{1}{\lambda}\mu_{k_*}\lambda\hat{u}_{k_*+1} + \mu_{k_*}\lambda O_{k^*+1}(\lambda), \\ \hat{u}_{k_*+1} &= -\frac{1}{2}(G_{sup}^{k_*+1})^{-1}\left\{\tilde{G}_\xi^{k_*+1}\hat{u}_{k_*} - \int_{T_{k_*}(t_*)} g_{k_*+1k_*+1}(\tau)(-1 - u_{k_*}(\tau))d\tau\right\}. \end{aligned} \qquad (5.51)$$

Now choose $\Delta\vartheta_\lambda^{k_*+1}$ such that the following inequalities hold

$$|u_j^{k_*+1}(\lambda) + \Delta\vartheta_j^{k_*+1}(\lambda)| \leq 1, \quad j = 1, \ldots, m$$

and hence the values of μ_{k_*} and λ_0 can be decreased as required. Continuing this expansion procedure for the remaining equations in (5.47), we obtain the desired admissible control function in the form

$$\bar{u}(t) = u^0(t) + \Delta u(t) = \left\{u_1^0(t) + \Delta u_1(t), \ldots, u_N^0(t) + \Delta u_N(t)\right\}, \quad t \in T.$$

and note here that $\Delta u_k(t) = 0 \quad \forall\, k < k_*$.

Now calculate the increment of the cost function generated by the designed control function $\bar{u}(t)$ as

$$\begin{aligned}
\Delta J(u) &= J(\bar{u}) - J(u^0) = \sum_{k=1}^{N} \int_0^{t^*} \Delta_k(t)\Delta u_k(t)dt = \sum_{k=k_*}^{N} \int_0^{t^*} \Delta_k(t)\Delta u_k(t)dt = \\
&- \theta \int_{T_{k_*}(t_*)} \Delta_{k_*}(t)\big(-1 - u_{k_*}(t)\big)dt - \sum_{j=1}^{m} \int_{\tau_{k_*j}-\lambda}^{\tau_{k_*j}+\lambda} \Delta_{k_*}(t)\big[u_j^{k_*}(\lambda) + \Delta\vartheta_j^{k_*}(\lambda) - u_j^{k_*}(t)\big]dt \\
&- \sum_{s=k_*+1}^{N} \sum_{j=1}^{m} \int_{\tau_{sj}-\lambda}^{\tau_{sj}+\lambda} \Delta_s(t)\big[u_j^s(\lambda) + \Delta\vartheta_j^s(\lambda) - u_j^s(t)\big]dt.
\end{aligned} \tag{5.52}$$

Since $\Delta_k(\tau_{kj}) = 0,\ k = k_*,\ \dots, N,\ j = 1, \dots, m$, then again from the Taylor series expansion in λ, we have the following estimate for the integral components

$$\begin{aligned}
\int_{\tau_{sj}-\lambda}^{\tau_{sj}+\lambda} \Delta_s(t)\big[u_j^s(\lambda) + \Delta\vartheta_j^s(\lambda) - u_j^s(t)\big]dt &= \int_{\tau_{sj}}^{\tau_{sj}} \Delta_s(t)\big[u_j^s(\lambda) + \Delta\vartheta_j^s(\lambda) - u_j^s(t)\big]dt \\
+2\lambda\Delta_s(\tau_{sj})\big[u_j^s(\lambda) + \Delta\vartheta_j^s(\lambda) - u_j^s(\tau_{sj})\big] &+ \\
\lambda^2 \frac{d\Delta_s(\tau_{sj})}{dt}\big[u_j^s(\lambda) &+ \Delta\vartheta_j^s(\lambda) - u_j^s(\tau_{sj})\big] + o_1(\lambda^2) \cong o(\lambda^2).
\end{aligned} \tag{5.53}$$

Hence, (5.52) and (5.53) yield

$$\Delta J(u) = -\mu_{k_*}\lambda \int_{T_{k_*}(t_*)} \Delta_{k_*}(t)\big(-1 - u_{k_*}(t)\big)dt + o(\lambda) > 0 \tag{5.54}$$

for a small $\lambda > 0$, which contradicts the optimality of control functions $u_k^0(t),\ k = 1, .., N$. ∎

The optimality conditions for the supporting control functions can also be expressed in maximum principle form. Let $\psi_N(t)$ be the solution of the following differential equations

$$\frac{d\psi_N(t)}{dt} = -A^T\psi_N(t), \quad \psi_N(t^*) = p_N - H_N^T\nu^N, \quad t \in T. \tag{5.55}$$

which can be represented as

$$\psi_N(t) = K_1^T(t^* - t)\psi(t^*), \quad t \in T. \tag{5.56}$$

Hence, the following equalities

$$\begin{aligned}
\psi_N^T(t)b &= \big(p_N^T - (\nu^N)^T H_N\big)K_1(t^* - t)b = p_N^T K_1(t^* - t)b \\
&- (\nu^N)^T H_N K_1(t^* - t)b = c_N(t) - (\nu^N)^T g_{NN}(t) = -\Delta_N(t)
\end{aligned} \tag{5.57}$$

hold. In order to verify the validity of the corresponding conditions for subsequent passes we apply (5.39) for the differential equations (5.37). Let $\psi_{N-1}(t),\ t \in T$ be a solution of the differential equation

$$\frac{d\psi_{N-1}(t)}{dt} = -A^T\psi_{N-1}(t) - D^T\psi_N(t), \quad \psi_{N-1}(t^*) = p_{N-1} - H_{N-1}^T\nu^{N-1},\ t \in T. \tag{5.58}$$

Then

$$\begin{aligned} \psi_k^T(t)b &= (p_{N-1}^T - (\nu^{N-1})^T H_{N-1})K_1(t^* - t)b - \\ &- (p_N^T - (\nu^N)^T H_N)\int_{t_*}^{t} K_1^T(t-\tau)D^T K_1^T(t^*-\tau)bd\tau \\ &= p_{N-1}^T K_1(t^*-t)b - (\nu^{N-1})^T H_{N-1}K_1(t^*-t)b - (p_N^T - (\nu^N)^T H_N)K_2(t^*-t)b \\ &= c_{N-1}(t) - (\nu^{N-1})^T g_{N-1N-1}(t) - (\nu^N)^T g_{NN-1}(t) = -\Delta_{N-1}(t) \end{aligned} \tag{5.59}$$

By analogy with the case considered above, we have

$$\psi_k^T(t)b = -\Delta_k(t), \quad k = 2, \ldots, N, \tag{5.60}$$

where $\psi_k(t),\quad t \in T$ are the solutions of the following differential equations

$$\frac{d\psi_k(t)}{dt} = -A^T\psi_k(t) - D^T\psi_{k+1}(t), \quad \psi_k(t^*) = p_k - H_k^T\nu^k, \quad t \in T. \tag{5.61}$$

For each $k = 1, \ldots, N$ introduce the associated Hamilton function as

$$H_k(x_{k-1}, x_k, \psi_k, u_k) = \psi_k^T\big(Ax_k + Dx_{k-1} + bu_k\big), \quad t \in T. \tag{5.62}$$

Then use (5.60) to yield the optimality conditions (5.45) can be re-formulated in the maximum principle form as follows

Corollary 1. *The admissible supporting control $\{\tau_{sup}^k,\ u_k^0(t),\ k = 1, \ldots, N\}$ is optimal if along the corresponding trajectories $x_k^0(t),\ \psi_k(t)$ of (5.31)-(5.32) and (5.61) the Hamiltonian function takes the maximum value, i. e.*

$$H_k(x_{k-1}^0(t), x_k^0(t), \psi_k, u_k^0(t)) = \max_{|v|\le 1} H_k(x_{k-1}^0(t), x_k^0(t), \psi_k, v), \quad t \in T \tag{5.63}$$

for $k = 1, ..., N$. If the admissible supporting control is non-degenerate then this condition is necessary and sufficient.

In the next section, the maximum principle for arbitrary admissible control functions of the form of (5.31)–(5.35) is established using the sub-optimality conditions.

5.2.2 ϵ- optimality conditions.

Usually, in the design of numerical implementation of optimal control algorithms we exploit approximate solutions with corresponding error estimation. Hence it is necessary to introduce the 'sub-optimality' concept, as it is often sufficient to stop the numerical computations when a satisfactory accuracy level has been achieved.

Assume that $\{u_k^0(t),\ k \in K\}$ is the optimal control for (5.31)–(5.35), and let $J(u^0)$ denote the corresponding optimal cost function value.

Definition 12. *We say that the admissible control function $\{u_k^\epsilon(t),\ k \in K\}$ is ϵ- optimal, if the corresponding solution $\{x_k^\epsilon(t),\ t \in T,\ k \in K\}$ of (5.31)-(5.33) satisfies $J(u^0) - J(u^\epsilon) \le \epsilon$.*

Now we proceed to calculate an estimate of a supporting control function $\{u_k^0,\ \tau_{sup}^k,\ k \in k\ t \in T\}$, i.e. a measure of non-optimality of the control. Note also that this estimate can be partitioned into two principal parts: one of which evaluates the degree of non-optimality of the chosen admissible control functions $u_k(t)$, and the second the error produced by non-optimality of the support τ_{sup}^k. This partition is a major advantage in the design of numerically applicable solution algorithms.

Introduce an estimate of optimality $\beta = \beta(\tau_{sup}, u)$ as the value of the maximum increment for the cost function of (5.31)–(5.35) calculated in the absence of the the principal constraints (5.33), this estimate is given by the solution of the following relaxed optimization problem

$$\Delta J(u) \to \max_{\Delta u_k}, \quad |u_k(t) + \Delta u_k(t)| \le 1,\ t \in T,\ k = 1, \dots, N. \tag{5.64}$$

It is easy to see that

$$\beta = \beta(\tau_{sup}, u) = \sum_{k=1}^{N} \int_0^{t_*} \Delta_k(t) \Delta u_k(t) dt = \sum_{k=1}^{N} \int_{T_k^+} \Delta_k(t)(u_k(t)+1)dt + \sum_{k=1}^{N} \int_{T_k^-} \Delta_k(t)(u_k(t)-1)dt, \tag{5.65}$$

where

$$T_k^+ = \{t \in T:\ \Delta_k(t) > 0\}, \quad T_k^+ = \{t \in T:\ \Delta_k(t) > 0\}.$$

and we have the following result.

Theorem 11. *(ϵ-maximum principle) Given any $\epsilon \ge 0$, the admissible control $\{u_k(t),\ t \in T,\ k \in K\}$ is ϵ-optimal for (5.31)-(5.35) if, and only if, $\exists$ the support $\{\tau_{sup}^k,\ k \in K\}$ such that along the solutions $x_k(t), \psi_k(t), t \in T,\ k \in K$ of (5.31)-(5.33) and (5.61) the Hamiltonian attains its ϵ-maximum value, i.e.*

$$H_k(x_{k-1}^0(t), x_k^0(t), \psi_k, u_k^0(t)) = \max_{|v| \le 1} H_k(x_{k-1}^0(t), x_k^0(t), \psi_k, v) - \epsilon_k(t), \quad t \in T, \tag{5.66}$$

where the functions $\epsilon_k(t),\ k \in K$ satisfy the following inequality

$$\sum_{k \in K} \int_T \epsilon_k(t) dt \le \epsilon. \tag{5.67}$$

Proof. *Sufficiency.* Assume that (5.66)–(5.67) holds for an admissible control $\{u_k(t),\ t \in T,\ k \in K\}$. Then by (5.60) the suboptimality estimate can be calculated as

$$\begin{aligned}
\beta &= \beta(\tau_{sup}, u) = \sum_{k=1}^{N} \int_{T_k^+} \psi_k^T(t) b\big(-u_k(t) - 1\big) dt + \sum_{k=1}^{N} \int_{T_k^-} \psi_k^T(t) b\big(1 - u_k(t)\big) dt \\
&= \sum_{k=1}^{N} \int_{T_k^+} \psi_k^T(t)\big(Ax_k(t) + Dx_{k-1}(t) - b\big) dt - \sum_{k=1}^{N} \int_{T_k^+} \psi_k^T(t)\big(Ax_k(t) + Dx_{k-1}(t) + bu_k(t)\big) dt \\
&= \sum_{k=1}^{N} \int_{T_k^-} \psi_k^T(t)\big(Ax_k(t) + Dx_{k-1}(t) + b\big) dt - \sum_{k=1}^{N} \int_{T_k^-} \psi_k^T(t)\big(Ax_k(t) + Dx_{k-1}(t) - bu_k(t)\big) dt \\
&+ \sum_{k=1}^{N} \int_{T} \Big[\max_{|v| \le 1} H_k\big(x_{k-1}(t), x_k(t), \psi_k(t), v\big) - H_k\big(x_{k-1}(t), x_k(t), \psi_k(t), u_k(t)\big)\Big] dt \\
&= \sum_{k=1}^{N} \int_{T} \epsilon_k(t) dt \le \epsilon.
\end{aligned}$$

Since the sub-optimal estimate (5.64) has been calculated in the absence of constraints (5.33), then it is obvious that the following inequalities hold

$$J(u^0) - J(u) \le \beta(\tau_{sup}, u) \le \epsilon.$$

This proves the ϵ- optimality property of the admissible control $\{u_k(t),\ t \in T,\ k \in K\}$.

Necessity. Let $\{u_k(t),\ t \in T,\ k \in K\}$ be an ϵ-optimal admissible control and let $\{\tau_{sup}^k,\ k \in K\}$ be an arbitrary support. Then the sub-optimal estimate of the control corresponding to the chosen support can be calculated as

$$\beta(\tau_{sup}, u) = \sum_{k=1}^{N} \int_{T} \Delta_k(t) u_k(t) dt + \sum_{k=1}^{N} \int_{T_k^+} \Delta_k(t) dt - \sum_{k=1}^{N} \int_{T_k^-} \Delta_k(t) dt. \tag{5.68}$$

Now introduce the following dual optimization problem

$$I(y, v, w) = \sum_{k \in K} \Big[h_k^T y_k + \int_T v_k(t) dt + \int_T w_k(t) dt\Big] \longrightarrow \min_{y,v,w} \tag{5.69}$$

subject to

$$\sum_{s=k}^{N} y_s^T g_{sk}(t) - v_k(t) + w_k(t) = c_k(t), \quad v_k(t) \ge 0,\ w_k(t) \ge 0,\ t \in T,\ k \in K. \tag{5.70}$$

It also can be shown that (5.69)–(5.70) has an optimal solution if there exists an optimal control for (5.31)–(5.35). The chosen support is denoted by $\{\tau_{sup}^k,\ k \in K\}$ and then use (5.42) to construct the vectors $z_k = \{y_k, v_k, w_k,\ k \in K\}$ as

$$y_k = \nu_k;\ v_k(t) = \Delta_k(t),\ w_k(t) = 0 \qquad if \quad \Delta_k(t) \geq 0;$$

$$v_k(t) = 0,\ w_k(t) = -\Delta_k(t) \qquad if \quad \Delta_k(t) < 0; \tag{5.71}$$

where, by the definition of $\Delta_k(t)$, these satisfy the constraint (5.70) of the dual problem.

Let $\{y_k^0, v_k^0(t), w_k^0(t), t \in T, k \in K\}$ denote an optimal solution of (5.69)–(5.70). Then (5.69) and (5.45) yield

$$\begin{aligned}
\beta(\tau_{sup}, u) &= \sum_{k=1}^{N}\sum_{s=k}^{N}\int_T \nu_s^T(t) g_{sk}(t) u_k(t) dt - \sum_{k=1}^{N}\int_T c_k^T(t) u_k(t) dt \\
&+ \sum_{k=1}^{N}\int_T v_k(t) dt - \sum_{k=1}^{N}\int_T w_k(t) dt \\
&= \left[\sum_{k=1}^{N}(\nu^k)^T \sum_{s=1}^{k}\int_T g_{ks}(t) u_s(t) dt + \sum_{k=1}^{N}\int_T v_k(t) dt - \sum_{k=1}^{N}\int_T w_k(t) dt\right]
\end{aligned}$$

$$- \left[\sum_{k=1}^{N}\sum_{s=k}^{N}\int_T (y_s^0)^T g_{sk}(t) u_k^0(t) dt + \sum_{k=1}^{N}\int_T v_k^0(t) dt - \sum_{k=1}^{N}\int_T w_k^0(t) dt\right]$$

$$+ \sum_{k=1}^{N}\int_T c_k(t) u_k^0(t) dt - \sum_{k=1}^{N}\int_T c_k(t) u_k(t) dt$$

$$= \left[\sum_{k=1}^{N}(\nu^k)^T h_k + \sum_{k=1}^{N}\int_T (v_k(t) - w_k(t)) dt\right] - \left[\sum_{k=1}^{N}(y_k^0)^T h_k + \sum_{k=1}^{N}\int_T (v_k^0(t) - w_k^0(t)) dt\right]$$

$$+ \sum_{k=1}^{N}\int_T c_k(t) u_k^0(t) dt - \sum_{k=1}^{N}\int_T c_k(t) u_k(t) dt.$$

Finally, the sub-optimal estimate can be written in the form

$$\beta = \beta(\tau_{sup}, u) = \beta_{sup} + \beta_u, \tag{5.72}$$

where

$$\beta_{sup} = \sum_{k=1}^{N} h_k^T(\nu_k - y^{0k}) + \sum_{k=1}^{N}\int_T \Big[(v_k(t) - v_k^0(t)) - (w_k(t) - w_k^0(t))\Big] dt \tag{5.73}$$

denotes the non-optimality measure of the chosen support $\{\tau_{sup}^k,\ k \in K\}$, and

$$\beta_u = \sum_{k=1}^{N}\int_T c_k(t)\big(u_k(t) - u_k^0(t)\big) dt \tag{5.74}$$

denotes the non-optimality measure of the given control function $\{u_k(t),\ t \in T,\ k \in K\}$.

Now choose the support $\tau^0_{sup} = \{\tilde{\tau}^k_{sup},\ k \in K\}$ such that the corresponding collection $z^0_k = \{y^0_k, v^0_k, w^0_k, k \in K\}$ of dual variables is an optimal solution of (5.69)–(5.70). First, we show that the chosen support $\tau^0_{sup} = \{\tilde{\tau}^k_{sup}(\epsilon),\ k \in K\}$ is the one required for the given ϵ- optimal control functions $\{u_k(t),\ k \in K\}$. In particular, since $\beta_{sup} = 0$ then $\beta = \beta(u, \tau^0_{sup}) = \beta_u \leq \epsilon$. Next set

$$
\begin{aligned}
\epsilon_k(t) &= \Delta_k(t)(u_k(t)+1), \quad t \in T^+_k, \\
\epsilon_k(t) &= \Delta_k(t)(u_k(t)-1), \quad t \in T^-_k, \\
\epsilon_k(t) &= 0 \qquad if \quad \Delta_k(t) = 0,\ t \in T.
\end{aligned}
$$

and note from the definition of $\Delta_k(t)$ that we have

$$
\begin{aligned}
\epsilon_k(t) &= -\psi^T_k(t)b(u_k(t)+1) = \psi^T_k(t)(Ax_k(t) + Dx_{k-1}(t) + b(-1)) \\
&- \psi^T_k(t)(Ax_k(t) + Dx_{k-1}(t) + bu_k(t)) \quad \text{if } \psi_k(t)b < 0; \\
\epsilon_k(t) &= \psi^T_k(t)(Ax_k(t) + Dx_{k-1}(t) + b(+1)) \\
&- \psi^T_k(t)(Ax_k(t) + Dx_{k-1}(t) + bu_k(t)) \quad \text{if } \psi_k(t)b > 0; \\
\epsilon_k(t) &= 0 \qquad \text{if} \qquad \psi_k(t)b = 0, \quad t \in T, \quad k \in K.
\end{aligned}
$$

The use of the Hamiltonian (5.62) now enables the last expressions to be written in the form

$$\epsilon_k(t) = \max_{|v|\leq 1} H_k\big(x^0_{k-1}(t), x^0_k(t), \psi_k, v\big) - H_k\big(x^0_{k-1}(t), x^0_k(t), \psi_k, u^0_k(t)\big), \quad t \in T,\ k \in K.$$

Adding these last expressions and noting that $\{u_k(t)\}$ is an suboptimal control, yields

$$
\begin{aligned}
\sum_{k=1}^{N} \int_T \epsilon_k(t)dt &= \sum_{k=1}^{N} \int_{T^+_k} \Delta_k(t)(u_k(t)+1)dt \\
&+ \sum_{k=1}^{N} \int_{T^-_k} \Delta_k(t)(u_k(t)-1)dt = \beta(u, \tau^0_{sup}) = \beta_u \leq \epsilon.
\end{aligned}
$$

which completes the proof. ∎

Note now that that maximum principle follows from the theorem above on setting $\epsilon = 0$.

Corollary 2. *The admissible control $\{u^0_k(t), k \in K,\ t \in T\}$ is optimal if, and only if, there exists a support $\{\tau^{0k}_{sup},\ k \in K\}$ such that the supporting control $\{u^0_k(t), \tau^{0k}_{sup},\ t \in T,\ k \in K\}$ satisfies the maximum conditions*

$$\max_{|v|\leq 1} H_k\big(x^0_{k-1}(t), x^0_k(t), \psi_k, v\big) = H_k\big(x^0_{k-1}(t), x^0_k(t), \psi_k, u^0_k(t)\big)$$

for all $k \in K$, $t \in T$, where $\psi_k(t)$ are the corresponding solutions of (5.61).

5.2.3 Differential properties of optimal solutions

An important aspect of optimization theory is sensitivity analysis of optimal control problems. In practice, control problems are often subject to disturbances or perturbations of the system data. In mathematical terms, perturbations can be described by some parameters in the initial data, boundary conditions, control and state constraints. It is clearly important to know how a problem solution depends on these parameters. The aim in this sub-section is determine the changes in the solutions developed here due to 'small' perturbations in the parameters, which should enable us to design a fast and reliable real-time algorithm for correcting the solutions for these effects. The major advantage of the proposed constructive approach is that the sensitivity analysis and some differential properties of the optimal controls under disturbances can be studied, which is very critical if they are to be applied to control synthesis problems.

Suppose that disturbances influence the initial data for (5.31)–(5.33). In particular, consider the system (5.31)–(5.33) on the interval $T_s = [s, t^*]$ with the initial data $x_k(s) = z_k, z_k \in G_k, k \in K$ where $G_k \subset \mathbb{R}^n$ is in some neighborhood of the point $x_k = \alpha_k$ and s belongs to the neighborhood G_0 of the instant $t = 0$. In addition, we assume that the following regularity condition holds: for the given disturbance domain $G_k, k \in K \cup \{0\}$, the structure of the optimal control functions for the non-disturbed data is preserved, i. e. the number of switching instances together with their order are constant.

Using Theorem 10, the optimal controls $\{u_k^0(t,s,z), k \in K\}$ are determined by the supporting time instances $\tau_{kj} = \tau_{kj}(s,z)$, $k \in K$, $j = 1, \dots, m$ which are dependent on the disturbances $(s, z_k), s \in G_0$, $z_k \in G_k$, $k \in K$. The aim of this section is to study the differential properties of the functions $\tau_{kj} = \tau_{kj}(s,z)$, $k \in K$, $j = 1, \dots, m$. For ease of notation we set $\tau \equiv \tau(s,z) = \{\tau_{kj}(s,z),\ k \in K,\ j = 1, \dots, m\}, z = \{z_k, k \in K\}$ in what follows.

Theorem 12. *If (5.31)-(5.33) is regular then for any $k \in K$ and $j = 1, ..., m$ the functions $\tau_{kj} = \tau_{kj}(s,z)$ are differentiable in the domain $G_0 \times G_k \subset \mathbb{R} \times \mathbb{R}^n$.*

Proof. Using (5.40)–(5.41) and Theorem 10 it follows immediately that the switching instances $\tau_{kj} = \tau_{kj}(s,z)$, $k \in K$, $j = 1, \dots, m$ of the optimal bang-bang control $\{u_k^0(t,s,z),\ k \in K\}$ for the disturbed problem (5.31)–(5.33) are the solutions of the following optimization problem

$$\max_{\tau_{kj}} \sum_{k \in K} sign\ R_k(s,z) \sum_{j=1}^{m+1} (-1)^j \int_{\tau kj-1}^{\tau_{kj}} c_k(t)dt \tag{5.75}$$

subject to

$$\sum_{l \in K} sign\ R_l(s,z) \sum_{j=1}^{m+1} (-1)^j \int_{\tau lj-1}^{\tau_{lj}} g_{kl}(t)dt = h_k(s,z), \quad k \in K \tag{5.76}$$

Here $sign\ R_k(s,z) = \pm 1$ denotes the value ($u = +1$ or $u = -1$) of the optimal control on pass k over the first control interval $t \in [s, \tau_{k1}]$, and

$$h_k(s,z) = g_k - \sum_{j=1}^{k} H_k K_j(t^*) z_{k+1-j} - \int_s^{t^*} H_k K_k(t^* - t) D f(t) dt. \tag{5.77}$$

It is obvious that the switching instances $\tau_{kj} = \tau_{kj}(s,z)$ satisfy the following inequalities

$$\tau_{k0} < \tau_{k1} < \tau_{k2} < \cdots < \tau_{km} < \tau_{km+1}, \quad \tau_{k0} = s, \quad \tau_{km+1} = t^*,$$

Since $\{u_k^0, \tau_{sup}^0, k \in K\}$ is the optimal supporting control for the non-disturbed problem (5.31)-(5.33) then the optimization problem (5.75)-(5.76) has the optimal solution $\tau_{kj}^0, k \in K, j = 1, \ldots, m\}$ at $s = 0, z_k = \alpha_k, k \in k, j = 1, \ldots, m$. Hence there exists Lagrange multipliers $\lambda_k^0 \in \mathbb{R}^m, k \in K$ which are not simultaneously equal to zero and such that the collection $\{\lambda_k^0, \tau_{kj}^0\}$ is a stationary point for the following Lagrange function associated with the optimization problem (5.75)-(5.76)

$$\begin{aligned} L(\lambda, \tau_{sup}) &= \sum_{k \in K} sign\ R_k(s,z) \sum_{j=1}^{m+1} (-1)^j \int_{\tau kj-1}^{\tau_{kj}} c_k(t) dt \\ &+ \sum_{k \in K} \lambda_k \Big[\sum_{l \in K} sign\ R_l(s,z) \sum_{j=1}^{m+1} (-1)^j \int_{\tau lj-1}^{\tau_{lj}} g_{kl}(t) dt - h_k(s,z) \Big]. \end{aligned} \tag{5.78}$$

The well known stationary conditions for the Lagrange function L lead to the following equalities

$$2\ sign\ R_k(s,z) \Big[c_k(\tau_{kj}) + \sum_{l=k}^{N} \lambda_l g_{lk}(\tau_{kj}) \Big] = 0, \quad j = 1, \ldots, m, \quad k \in K \tag{5.79}$$

$$\sum_{l=1}^{k} sign\ R_l(s,z) \sum_{j=1}^{m+1} (-1)^j \int_{\tau lj-1}^{\tau_{lj}} g_{kl}(t) dt - h_k(s,z) = 0, \quad k \in K \tag{5.80}$$

with respect to the unknown λ_k and $\tau_k(s,z)$, $k \in K$, $j = 1, \ldots, m$. The Jacobian matrix D of the mapping (5.79) with respect to variables (λ, τ_{sup}) calculated at $s = 0$ and $z_k = \alpha_k$ can be written in the form

$$D = \prod_{k \in K} 2\ sign R_k(0, \alpha) \begin{pmatrix} \hat{G}_{sup} & F \\ 0 & \hat{G}_{sup} \end{pmatrix} \tag{5.81}$$

where the matrix $\hat{G}_{sup}$ is defined as follows

$$\hat{G}_{sup} = \begin{pmatrix} g_{kj}(t), & t \in \tau_{sup}^k \\ j \geq k, & k = 1, \ldots, N \end{pmatrix} \tag{5.82}$$

and the matrix F is formed from the derivatives of the functions $c_k(t), g_{kl}(t)$ taken at the corresponding points. By the definition of the supporting time instances we have that $\det D \neq 0$ and by the implicit function theorem there exists a neighborhood of the point $(0, \alpha_k,\ k \in K)$ where (5.79) has a unique solution $\lambda = \lambda(s,z), \quad \tau_{kj} = \tau_{kj}(s,z)$ where these functions are also differentiable. This completes the proof. ∎

The above differential properties of the optimal controls can be used for sensitivity analysis and the solution of the synthesis problem for the repetitive processes considered here. In particular, the supporting control approach can be applied [30] to produce the differential equations for the switching time functions $\tau(s,z)$ necessary to design the optimal controllers. By analogy with [26] it follows that they satisfy the following differential equations

$$G\frac{\partial \tau}{\partial s} + Q = \frac{\partial h}{\partial s}, \quad P\frac{\partial \tau}{\partial z} = \frac{\partial h}{\partial z} \tag{5.83}$$

where $h(s,z) = (h_1(s,z), \ldots, h_m(t,s))$ is an $mN \times 1$-vector given by (79) and the matrices G, Q, P are defined (see [26]) by those defining the process dynamics and information associated with the non-disturbed optimal solution. For example

$$G = -\Big(g_{11}(s) sign\dot{\Delta}_1(\tau_{11}), g_{21}(s) sign\dot{\Delta}_1(\tau_{11}) + g_{22}(s) sign\dot{\Delta}_2(\tau_{21}), \ldots, \sum_{j=1}^{N} g_{Nj}(s) sign\dot{\Delta}_j(\tau_{j1})\Big)^T$$

where the functions $\Delta_j(t),\ j = 1, \ldots, N$ are designed with the help of the switching moments of the basic optimal control function. Note, that the analogous differential equations can be established for the optimal values of the cost function, treated as the function $J(s,z) \equiv J\big(u(\tau(s,z))\big)$.

Remark 7. *The equations (5.83) are (sometimes) termed Pfaff differential equations and model an essentially distinct class of continuous $n-D$ systems. The main characteristic feature of this model is that it is overdetermined (in the sense that the number of equations exceeds the unknown functions). It can also be shown that the non-degenerate assumption on the supporting control functions leads to the validity of the so-called Frobenious conditions that guarantee the existence and uniqueness of a solution of Pfaff differential equations.*

5.2.4 Examples

In order to demonstrate the advantages of the supporting control function approach, we now give the following examples.

Example 1. Consider the case of $N = 1$, where the superscript $(\cdot)$ is used to denote a particular element in the state vector on the pass, and the following optimal control problem

$$\max_{|u| \leq 1} J(u) \triangleq x^{(2)}(1) \tag{5.84}$$

for

$$\begin{aligned}
\frac{dx^{(1)}(t)}{dt} &= x^{(2)}, \qquad x^{(1)}(t),\ x^{(2)}(t) \in \mathbb{R} \quad t \in [s,1], \\
\frac{dx^{(2)}(t)}{dt} &= u(t), \quad x^{(1)}(s) = z_1, \quad x^{(2)}(s) = z_2
\end{aligned} \tag{5.85}$$

subject to the following constraints on control variables and a terminal state constraint

$$|u(t)| \leq 1, \qquad x^{(1)}(1) = 1/8, \tag{5.86}$$

respectively.

In this case it is easy to verify that for $s = 0$ and $x^{(1)}(0) = 0,\ x^{(2)}(0) = 0$ the optimal control signal is given by

$$u^0(t) = -1 \quad \text{for} \quad 0 \leq t \leq 1 - \sqrt{5/8}; \qquad \text{and} \quad u^0(t) = +1 \quad \text{for} \quad 1 - \sqrt{5/8} < t \leq 1.$$

Synthesis of the optimal control can be realized using the switching instance function $\tau = \tau(z_1, z_2, s)$, which has to satisfy the following differential equations

$$\begin{aligned}
\frac{\partial \tau}{\partial z_1} &= \frac{1}{2(1-\tau)} \\
\frac{\partial \tau}{\partial z_2} &= \frac{1-s}{2(1-\tau)} \\
\frac{\partial \tau}{\partial s} &= \frac{1-s-z_2}{2(1-\tau)}
\end{aligned} \tag{5.87}$$

with initial condition

$$\tau(0,0,0) = 1 - \sqrt{5/8},$$

which is a particular case of (5.83).

The solution of this Pffaf differential system is given by

$$\tau(z_1, z_2, s) = 1 - \sqrt{5/8 + (s-1)z_2 - z_1 - s + s^2/2}$$

Without loss of generality, assume $s = 0$ and then the optimal switching function is

$$\tau(z_1, z_2, 0) = 1 - \sqrt{5/8 - z_1 - z_2}.$$

Figures 1 and 2 illustrate the form of this solution. Figure 1 shows the state space variables together with additional variable t. The optimal trajectories (5.84)–(5.86) corresponding to the bang-bang control law lie on the parabolic cylinders $(Z_1): \ x^{(1)} = -\frac{1}{2}(x^{(2)})^2 + C_1 + C_2$ and $(Z_2):$ $x^{(1)} = +\frac{1}{2}(x^{(2)})^2 + \tilde{C}_1 + \tilde{C}_2$ where the constants $C_i, \tilde{C}_i,\ i = 1, 2$ are determined by the initial data

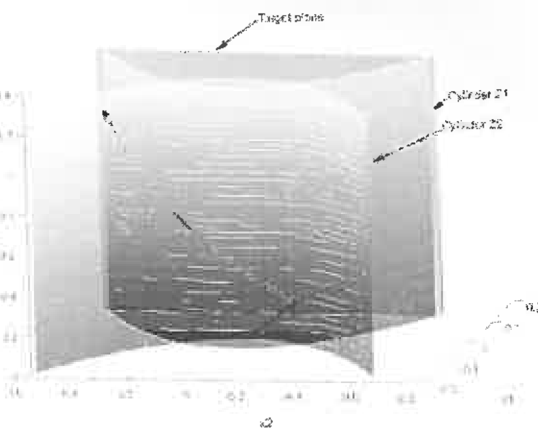


Figure 5.1: Optimal synthesis control

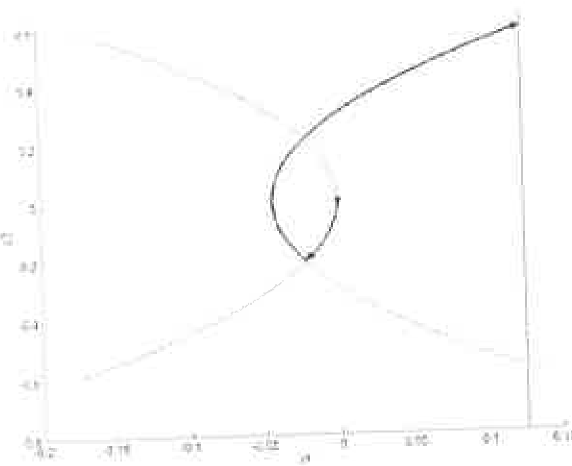


Figure 5.2: Projection on the Ox_1x_2 plane

$x^{(1)}(0) = z_1$, $x^{(2)}(0) = z_2$. These cylinders correspond to the solutions of the differential equations (5.85) with $u \equiv -1$ or $u \equiv +1$, respectively. It can also be shown that the admissible initial domain for which the problem can be solved is determined by the inequalities: $-\frac{3}{8} \leq z_1 + z_2 \leq \frac{5}{8}$. The switching manifold Z_h is described in parametric form by

$$\begin{cases} x^{(1)} = -\frac{\left(1-\sqrt{5/8-z_2-z_1}\right)^2}{2} + z_2\left(1-\sqrt{5/8-z_2-z_1}\right) + z_1, \\ x^{(2)} = -1 + \sqrt{5/8-z_2-z_1} + z_2, \\ T = 1 - \sqrt{5/8-z_2-z_1}, \\ -\frac{3}{8} \le z_1 + z_2 \le \frac{5}{8} \end{cases}$$

Finally, each optimal trajectory consists of two parts — first it evolves along the vertical parabolic cylinder Z_1 until $\tau = 1 - \sqrt{5/8 - z_2 - z_1}$ when it meets the switching manifold Z_h, and then immediately is switched to continue along the second vertical cylinder Z_2 to meet the target plane $x^{(1)} = 1/8$. Figure 1 shows the optimal trajectory in the space $\mathbb{R}^3$ for zero initial data, and Figure 2 shows the projection of this trajectory onto the $x^{(1)}, x^{(2)}$ plane.

Example 2. Consider the following optimization problem for $N = 2$, where again the superscript $(\cdot)$ is used to denote a particular element in the state or control vector on any pass:

$$\max_{u_1,u_2} J(u) = x_1^{(2)}(1) + x_2^{(2)}(1) \tag{5.88}$$

for the process

$$\begin{aligned} \frac{dx_1^{(1)}(t)}{dt} &= x_1^{(2)}(t), & \frac{dx_2^{(1)}(t)}{dt} &= x_2^{(2)}(t), \ t \in [s,1] \\ \frac{dx_1^{(2)}(t)}{dt} &= u_1(t), & \frac{dx_2^{(2)}(t)}{dt} &= x_1^{(1)}(t) + u_2(t), \end{aligned} \tag{5.89}$$

with boundary conditions of the form

$$x_1^{(1)}(s) = z_1^{(1)},\ x_1^{(2)}(s) = z_1^{(2)},\ x_2^{(1)}(s) = z_2^{(1)},\ x_2^{(2)}(s) = z_2^{(2)} \tag{5.90}$$

subject to

$$x_1^{(1)}(1) = 1/8, \quad x_2^{(1)}(1) = 1/384,\ |u_1(t)| \le 1,\ |u_2(t)| \le 1, \tag{5.91}$$

The dynamic here can be written as a stationary differential linear repetitive process of the form

$$\begin{bmatrix} \dot{x}_{k+1}^{(1)}(t) \\ \dot{x}_{k+1}^{(2)}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{k+1}^{(1)}(t) \\ x_{k+1}^{(2)}(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_k^{(1)}(t) \\ x_k^{(2)}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{k+1}(t),\ k = 0, 1. \tag{5.92}$$

Without loss of generality we set $x_0(t) = 0,\ t \in [s, 1]$.

To apply the results developed here to this example we first rewrite (5.89)–(5.91) in the following integral form:

$$\max_{u_1,u_2} \left\{ z_2^{(1)} + z_2^{(2)} + (1-s)z_1^{(1)} + \frac{(1-s)^2}{2} z_2^{(2)} + \int_s^1 \frac{(1-t)^2+2}{2} u_1(t)dt + \int_s^1 u_2(t)dt \right\} \tag{5.93}$$

subject to

$$\int_s^1 (1-t)u_1(t)dt = \frac{1}{8} - z_1^{(1)} + (1-s)z_1^{(2)},$$

$$\int_s^1 \left[\frac{(1-t)^3}{6}u_1(t) + (1-t)u_2(t)\right] dt = \frac{1}{384} - z_2^{(1)} - (1-s)z_2^{(2)} - \frac{(1-s)^2}{2}z_1^{(1)} - \frac{(1-s)^3}{6}z_1^{(2)}. \tag{5.94}$$

Hence

$$g_{11}(t) = 1-t, \quad g_{21}(t) = \frac{(1-t)^3}{6}, \quad g_{22}(t) = 1-t, \tag{5.95}$$

$$c_1(t) = \frac{(1-t)^2+2}{2}, \quad c_2(t) = 1 \tag{5.96}$$

and the multipliers required to design the co-control function $\Delta_i(t)$, $i = 1,2$ can, noting (5.42), be written as

$$\nu^{(2)} g_{22}(\tau_{2sup}) - c_2(\tau_{2sup}) = 0, \tag{5.97}$$

$$\nu^{(1)} g_{11}(\tau_{1sup}) + \nu^{(2)} g_{21}(\tau_{1sup}) - c_1(\tau_{1sup}) = 0$$

Then

$$\Delta_1(t) = (1-t)\left[\frac{1}{1-\tau_{1sup}} + \frac{1-\tau_{1sup}}{2} - \frac{(1-\tau_{1sup})^2}{6(1-\tau_{2sup})}\right] + \frac{(1-t)^3}{6(1-\tau_{2sup})} - \frac{(1-t)^2}{2} - 1, \tag{5.98}$$

$$\Delta_2(t) = \frac{1-t}{1-\tau_{2sup}} - 1$$

Now the problem is how to find the basic optimal trajectory when all variables in (5.90) are zero, i.e.

$$s = 0,\ x_1^{(1)}(0) = 0,\ x_1^{(2)}(0) = 0,\ x_2^{(1)}(0) = 0,\ x_2^{(2)}(0) = 0. \tag{5.99}$$

and take the supporting instances as

$$\tau_{1sup} = 1 - \sqrt{\frac{5}{8}}, \quad \tau_{2sup} = 1 - \sqrt{\frac{131}{256}} \tag{5.100}$$

Then it follows immediately from Theorem 10 that the optimal control functions for (5.88)–(5.91) with the initial data (5.99) are given by

$$u_1^0(t) = \begin{cases} -1, & 0 \le t < 1 - \sqrt{\frac{5}{8}}, \\ +1, & 1 - \sqrt{\frac{5}{8}} \le t \le 1 \end{cases}, \quad u_2^0(t) = \begin{cases} -1, & 0 \le t < 1 - \sqrt{\frac{131}{256}}, \\ +1, & 1 - \sqrt{\frac{131}{256}} \le t \le 1 \end{cases} \tag{5.101}$$

and the differential equations (5.83) give the switching functions

$$\tau_1 \equiv \tau_1(z_1^{(1)}, z_1^{(2)}, s), \quad \tau_2 \equiv \tau_2(z_1^{(1)}, z_2^{(1)}, z_1^{(2)}, z_2^{(2)}, s)$$

as

$$
\begin{aligned}
&-2\frac{\partial\tau_2}{\partial s}(1-\tau_2)-\frac{2(1-\tau_1)^3}{6}\frac{\partial\tau_1}{\partial s}=\frac{(1-s)^2}{2}z_1^{(2)}+(1-s)z_1^{(1)}+z_2^{(2)}-\frac{(1-s)^3}{6}-(1-s),\\
&-2\frac{\partial\tau_2}{\partial z_1^{(1)}}(1-\tau_2)-\frac{(1-\tau_1)^3}{3}\frac{\partial\tau_1}{\partial z_1^{(1)}}=-\frac{(1-s)^2}{2},\\
&-2\frac{\partial\tau_2}{\partial z_1^{(2)}}(1-\tau_2)-\frac{(1-\tau_1)^3}{3}\frac{\partial\tau_1}{\partial z_1^{(2)}}=-\frac{(1-s)^3}{6},\\
&-2\frac{\partial\tau_2}{\partial z_2^{(1)}}(1-\tau_2)=-1,\quad -2\frac{\partial\tau_2}{\partial z_2^{(2)}}(1-\tau_2)=-(1-s),
\end{aligned}
\tag{5.102}
$$

with initial conditions

$$
\tau_1(0,0,0)=1-\sqrt{\frac{5}{8}},\qquad \tau_2(0,0,0,0,0)=1-\sqrt{\frac{131}{16^2}}
\tag{5.103}
$$

The solutions of this differential system are

$$
\begin{aligned}
\tau_1(z_1^{(1)},z_1^{(2)},s)&=1-\sqrt{SR_1(z_1^{(1)},z_1^{(2)},s)}\\
\tau^{(2)}(z_1^{(1)},z_2^{(1)},z_1^{(2)},z_2^{(2)},s)&=1-\sqrt{SR_2(z_1^{(1)},z_2^{(1)},z_1^{(2)},z_2^{(2)},s)}
\end{aligned}
\tag{5.104}
$$

where

$$
\begin{aligned}
SR_1(z_1^{(1)},z_1^{(2)},s)&=\frac{5}{8}+(s-1)z_1^{(2)}-z_1^{(1)}-s+s^2/2,\\
SR_2(z_1^{(1)},z_2^{(1)},z_1^{(2)},z_2^{(2)},s)&=\frac{131}{256}+\frac{2s^4-8s^3+59s^2-102s}{96}+\\
\frac{-20s^2+40s-19}{48}z_1^{(1)}-\frac{1}{12}z_1^{(1)2}&+\frac{4s^3-12s^2+11s-3}{48}z_1^{(2)}+\frac{-s^2+2s-1}{12}z_1^{(2)2}+\\
&+\frac{sz_1^{(1)}z_1^{(2)}}{6}-\frac{z_1^{(1)}z_1^{(2)}}{6}-z_1^{(2)}+(s-1)z_2^{(2)}
\end{aligned}
\tag{5.105}
$$

It easy to see that the solution of the differential equations describing the process dynamics with $u_1=const,\ u_2=const$ are

$$
\begin{aligned}
x_1^{(1)}(t)=&u_1\frac{t^2}{2}+tC_1+C_2,\\
x_1^{(2)}(t)=&u_1t+C_1\\
x_2^{(1)}(t)=&u_1\frac{t^4}{24}+C_1\frac{t^3}{6}+C_2\frac{t^2}{2}+u_2\frac{t^2}{2}+tC_3+C_4,\\
x_2^{(2)}(t)=&u_1\frac{t^3}{6}+C_1\frac{t^2}{2}+tC_2+tu_2+C_3
\end{aligned}
\tag{5.106}
$$

and in this case that the optimal control for pass $k = 1$ coincides with that of Example 1.

Now consider disturbances Ω such that the optimal control is preserved for the case of zero initial conditions, i.e. $u_1 = -1$ for $t \leq \tau_1(z_1^{(1)}, z_1^{(2)}, s)$; $(u_2^0 = -1$, for $t \leq \tau_2(z_1^{(1)}, z_2^{(1)}, z_1^{(2)}, z_2^{(2)}, s)$ and the inequality $\tau_1(z_1^{(1)}, z_1^{(2)}, s) < \tau_2(z_1^{(1)}, z_2^{(1)}, z_1^{(2)}, z_2^{(2)}, s)$ holds. Using (5.104) we have that the domain Ω is described by

$$0 \leq \tau_1(z_1^{(1)}, z_1^{(2)}, s) < \tau_2(z_1^{(1)}, z_1^{(2)}, z_2^{(1)}, z_2^{(2)}, s) \leq 1$$

$$SR_1(z_1^{(1)}, z_1^{(2)}, s) \geq 0, \qquad SR_2(z_1^{(1)}, z_2^{(1)}, z_1^{(2)}, z_2^{(2)}, s) \geq 0$$

To construct the solution for pass $k = 2$, it is necessary to construct the switching surface $\mathfrak{F}$ which is defined by the vectors

$$x_2^{(1)}(t) \mid_{t=\tau_2} = x_2^{(1)}\big(\tau_2(z_1^{(1)}, z_2^{(1)}, z_1^{(2)}, z_2^{(2)}, s), \tau)\big),$$

$$x_2^{(2)}(t) \mid_{t=\tau_2} = x_2^{(2)}\big(\tau_2(z_1^{(1)}, z_2^{(1)}, z_1^{(2)}, z_2^{(2)}, s)\big)$$

when the parameters $z_1^{(1)}, z_2^{(1)}, z_1^{(2)}, z_2^{(2)}, s$ are members of the set Ω. The parametric description of the switching surface $\mathfrak{F}$ is given by

$$x_2^{(1)}(t) = -\frac{t^4}{24} + C_1\frac{t^3}{6} + C_2\frac{t^2}{2} - \frac{t^2}{2} + tC_3 + C_4, \tag{5.107}$$

$$x_2^{(2)}(t) = -\frac{t^3}{6} + C_1\frac{t^2}{2} + tC_2 - t + C_3$$

where the coefficients C_i are found from the parameters $z_1^{(1)}, z_2^{(1)}, z_1^{(2)}, z_2^{(2)}, s$.

5.3 Conclusions

In this thesis the supporting control functions setting is applied to study the optimization problems for continuous-discrete linear repetitive processes. The main goal achieved here is to develop the constructive necessary and sufficient optimality conditions in a form, which can be effectively used for the design of numerical algorithms. The iterative method proposed here is based on the principle of a decrease of the suboptimality estimate, i. e. the iteration $\{\tau_{sup}^k, \quad u_k(t), \quad k = 1, \ldots, N\} \rightarrow \{\hat{\tau}_{sup}^k, \quad \hat{u}_k(t), \quad k = 1, ..., N\}$ is performed in such a way to archive $\beta(\hat{\tau}_{sup}, \hat{u}) < \beta(\tau_{sup}, u)$. It turns out that this procedure can be separated into two stages: 1) transformation of the admissible control functions $\{u_k(t), \quad k = 1, ..., N\} \rightarrow \{\hat{u}_k(t), \quad k = 1, ..., N\}$ which decreases the non-optimality measure of the admissible controls $\beta(\hat{u}) < \beta(u)$; 2) variation of the support $\{\tau_{sup}^k, \quad k = 1, ..., N\} \rightarrow \{\hat{\tau}_{sup}^k, \quad k = 1, ..., N\}$ again to decrease the non-optimality measure of the support, i. e. $\beta(\hat{\tau}_{sup}) < \beta(\tau_{sup})$. These transformations involve essentially the duality theory for the problems (5.31) — (5.35) and (5.69) — (5.70) and exploit the ϵ-optimality conditions developed here. These results are the first in this general area and work is currently proceeding in a number of follow up areas. They can be used for sensitivity analysis of optimal control in the presence of disturbances. In the case of the ordinary linear control systems, some details of such analysis can be found in [39]. The developed methods can also be used to construct the differential equations for the switching functions of optimal control law that can be applied for the design of optimal controllers and synthesis of the optimal regimes in many control processes.

Chapter 6

Delay System Approach to Linear Differential Repetitive Processes: Controllability and Optimization

It is already known that repetitive processes can be represented in various dynamical system forms, which can, where appropriate, be used to great effect in control related analysis of these processes. In this chapter, we further investigate the already known links between some classes of linear repetitive processes and delay systems and apply this to analyze control theory problems arising in controllability and optimal control of these repetitive processes. In particular, the so-called characteristic mappings introduced in [29] are used to establish controllability properties criteria. Next, time optimal control problems are considered, where it is well known that the separation theorem for convex sets is a useful approach for studying a wide class of extremal problems. Here we adopt this method to establish optimality conditions in the classic form.

It has been conjectured that such a setting is appropriate for the development of numerical methods for optimal control problems and related studies and on which very little work has been reported to date. The results developed here provide (part of) the theoretical background for further work aimed at the efficient computation of optimal controllers for these processes. Some areas for further research are also briefly discussed.

6.1 Background

The differential linear repetitive processes [51] are defined over $0 \le t \le \hat{\alpha}$, $k \ge 0$, by the state space model

$$\begin{aligned}\dot{x}_{k+1}(t) &= \hat{A}x_{k+1}(t) + \hat{B}u_{k+1}(t) + \hat{B}_0 y_k(t) \\ y_{k+1}(t) &= \hat{C}x_{k+1}(t) + \hat{D}u_{k+1}(t) + \hat{D}_0 y_k(t)\end{aligned} \tag{6.1}$$

Here on pass k, $x_k(t)$ is the $n \times 1$ state vector, $y_k(t)$ is the $m \times 1$ pass profile vector, and $u_k(t)$ is the $r \times 1$ vector of control inputs. To complete the process description, it is necessary to specify the boundary conditions, i. e. the state initial vector on each pass and the initial pass profile. Here no loss of generality arises from assuming $x_{k+1}(0) = d_{k+1}$, $k \geq 0$, and $y_0(t) = \hat{g}(t)$, where d_{k+1} is an $n \times 1$ vector of known constant entries and $\hat{g}(t)$ is an $m \times 1$ vector whose entries are known functions of t over $0 \leq t \leq \hat{\alpha}$.

As mentioned before, the repetitive processes posses many other equivalent representations which can be better suited to the analysis of particular problems as, for example, $1D$ equivalent models enable much simpler characterization of the so-called pass controllability or observability [29, 32]. Revisit now a few such examples.

1) Singularly perturbed model with slow and fast modes

$$\begin{aligned} \dot{x}_{k+1}(t) =& \hat{A}x_{k+1}(t) + \hat{B}u_{k+1}(t) + \hat{B}_0 y_k(t) \\ \mu \dot{y}_{k+1}(t) =& \hat{C}_0 y_{k+1}(t) + \hat{C}x_{k+1}(t) + \hat{D}u_{k+1}(t) + \hat{D}_0 y_k(t), \ 0 \leq t \leq \hat{\alpha}, \ k \geq 0. \end{aligned} \tag{6.2}$$

Hence, the standard repetitive process is a limiting case of that of (6.2) for $\mu = 0$, $det\hat{C}_0 \neq 0$. This approach is the subject of ongoing work and the results will be reported in due course.

2) the Volterra type equation (with respect to the variable k)

$$\dot{x}_{k+1}(t) = \sum_{i-0}^{k} \left[A_i x_{k+1-i}(t) + B_i u_{k+1-i}(t) \right] + D_k g(t), x_{k+1}(0) = d_{k+1}, \ k \geq 0 \tag{6.3}$$

where

$$A_0 = \hat{A}, \ A_i = \hat{B}_0 \hat{D}_0^{i-1} \hat{C}, \ B_0 = \hat{B}, \ B_i = \hat{B}_0 \hat{D}_0^{i-1} \hat{D}, \ D_i = \hat{B}_0 \hat{D}_0^{i-1}, \ i \geq 1$$

Discrete Volterra equations and their applications to the discrete repetitive models are given in [25]. The Volterra approach can be also effectively used for the differential case that is outside the scope of this thesis.

To obtain another representation of processes described by (6.1) which is the subject of this paper (for the case $1 \leq k \leq N$ where N is a fixed positive integer), introduce the new variables $x : [0, \hat{\alpha}N] \to \mathbb{R}^n$, $y : [0, \hat{\alpha}N] \to \mathbb{R}^m$, $u[0, \hat{\alpha}N] \to \mathbb{R}^r$, where

$$x(t) = \begin{cases} x_1(t), & 0 < t < \hat{\alpha} \\ x_2(t - \hat{\alpha}), & \hat{\alpha} < t < 2\hat{\alpha}, \\ \ldots\ldots & \ldots\ldots\ldots \\ x_N(t - \hat{\alpha}(N-1)), & \hat{\alpha}(N-1) < t < \hat{\alpha}N \end{cases}$$

$$y(t) = \begin{cases} y_1(t), & 0 < t < \hat{\alpha} \\ y_2(t - \hat{\alpha}), & \hat{\alpha} < t < 2\hat{\alpha}, \\ \ldots\ldots & \ldots\ldots\ldots \\ y_N(t - \hat{\alpha}(N-1)), & \hat{\alpha}(N-1) < t < \hat{\alpha}N \end{cases},$$

$$u(t) = \begin{cases} u_1(t), & 0 < t < \hat{\alpha} \\ u_2(t-\hat{\alpha}), & \hat{\alpha} < t < 2\hat{\alpha}, \\ \ldots\ldots & \ldots\ldots\ldots \\ u_N(t-\hat{\alpha}(N-1)), & \hat{\alpha}(N-1) < t < \hat{\alpha}N \end{cases}$$

Then, (6.1) can be rewritten in the form of the following delay system

$$\begin{bmatrix} \frac{d}{dt} & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \hat{A} & 0 \\ \hat{C} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 & \hat{B}_0 \\ 0 & \hat{D}_0 \end{bmatrix} \begin{bmatrix} x(t-\hat{\alpha}) \\ y(t-\hat{\alpha}) \end{bmatrix} + \begin{bmatrix} \hat{B} & 0 \\ 0 & \hat{D} \end{bmatrix} \begin{bmatrix} u(t) \\ u(t) \end{bmatrix} \tag{6.4}$$

with initial condition

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{g}(t) \end{bmatrix}, \quad t \in [-\hat{\alpha}, 0]. \tag{6.5}$$

Here, I_m denotes the identity matrix in $\mathbb{R}^m$. In order to complete the correspondence between the delay system (6.4) and the repetitive process (6.1) we require additional constraints at $t = \hat{\alpha}k,\ k = 1, \ldots N-1$, which demand that the solution $x(t)$ is discontinuous and has "jumps/pushes". This leads to the so-called nonlocal conditions of the form

$$x(k\hat{\alpha} + 0) = d_k, \quad k \in K, \tag{6.6}$$

where $x(k\hat{\alpha}+0)$ denotes $x(t)$ as $t \to k\hat{\alpha}$ from the right. We also assume that the control functions $u(t)$ and pass profile vectors $y(t)$ are continuous from the right hand side at $t = \hat{\alpha}k,\ k = 1, \ldots, N-1$.

It is clear to see that this last representation is a special singular case of

$$\begin{bmatrix} \frac{d}{dt} & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x(t-\hat{\alpha}) \\ y(t-\hat{\alpha}) \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} u(t) \\ u(t) \end{bmatrix} \tag{6.7}$$

which is equivalent to

$$\begin{aligned} \dot{x}(t) =& A_{11}x(t) + A_{12}y(t) + D_{11}x(t-\hat{\alpha}) + D_{12}y(t-\hat{\alpha}) + Bu(t) \\ y(t) =& A_{21}x(t) + A_{22}y(t) + D_{21}x(t-\hat{\alpha}) + D_{22}y(t-\hat{\alpha}) + Du(t) \end{aligned} \tag{6.8}$$

Finally, if the matrix $(I_m - A_{22})$ is nonsingular, then the second equation can be re-arranged to the form

$$\begin{aligned} \dot{x}(t) =& A_{11}x(t) + A_{12}y(t) + D_{11}x(t-\hat{\alpha}) + D_{12}y(t-\hat{\alpha}) + Bu(t) \\ y(t) =& \tilde{A}_{21}x(t) + \tilde{D}_{21}x(t-\tilde{\alpha}) + \tilde{D}_{22}y(t-\tilde{\alpha}) + \tilde{D}u(t) \end{aligned} \tag{6.9}$$

where $\tilde{H} = (I_m - A_{22})^{-1}H$ for H belonging to the set $H \triangleq \{A_{21},\ D_{21},\ D_{22},\ D\}$.

If a linear repetitive process of the form of (6.1) contains time delays such that the resulting process model has the following form over $0 \le t \le \hat{\alpha}$, $1 \le k \le N$,

$$\begin{aligned} \dot{x}_{k+1}(t) =& \hat{A}x_{k+1}(t) + \hat{A}_{-1}x_{k+1}(t-\hat{h}) + \hat{B}u_{k+1}(t) + \hat{B}_0 y_k(t) + \hat{B}_{-1}y_k(t-\hat{h}) \\ y_{k+1}(t) =& \hat{C}x_{k+1}(t) + \hat{C}_{-1}x_{k+1}(t-\hat{h}) + \hat{D}u_{k+1}(t) + \hat{D}_0 y_k(t) + \hat{D}_{-1}y_k(t-\hat{h}) \end{aligned} \tag{6.10}$$

where $\hat{h}$ is a real number such that $0 < \hat{h} \le \hat{\alpha}$. Then such linear repetitive processes can be presented in the multiple delay differential system form of

$$\begin{aligned} \begin{bmatrix} \frac{d}{dt} & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} =& \begin{bmatrix} \hat{A} & 0 \\ \hat{C} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 & \hat{B}_0 \\ 0 & \hat{D}_0 \end{bmatrix} \begin{bmatrix} x(t-\hat{\alpha}) \\ y(t-\hat{\alpha} \end{bmatrix} + \begin{bmatrix} \hat{B} & 0 \\ 0 & \hat{D} \end{bmatrix} \begin{bmatrix} u(t) \\ u(t) \end{bmatrix} \\ &+ \begin{bmatrix} \hat{A}_{-1} & 0 \\ \hat{C}_{-1} & 0 \end{bmatrix} \begin{bmatrix} x(t-h) \\ y(t-h) \end{bmatrix} + \begin{bmatrix} 0 & \hat{B}_{-1} \\ 0 & \hat{D}_{-1} \end{bmatrix} \begin{bmatrix} x(t-\hat{h}-\hat{\alpha}) \\ y(t-\hat{h}-\hat{\alpha}) \end{bmatrix} \end{aligned} \tag{6.11}$$

with initial conditions

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{g}(t) \end{bmatrix}, \quad t \in [-\hat{\alpha}, 0], \quad \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \phi(t) \\ \psi(t) \end{bmatrix}, \quad t \in [-\hat{\alpha}-\hat{h}, -\hat{\alpha}) \tag{6.12}$$

and t nonlocal conditions

$$x(k\hat{\alpha}+0) = d_k, \quad k \in K. \tag{6.13}$$

where $\psi(t), \phi(t), \hat{g}(t)$ are the corresponding initial conditions in (6.11).

6.2 Hybrid delay model for differential repetitive processes

As a basis for further study consider first the case when the nonlocal conditions of (6.6) are absent. This can be realized under the assumption that, for example, the initial condition in (6.1) for the current pass coincides with the end point state of the previous pass, i. e. $x_{k+1}(0) = x_k(\alpha)$, that occur often in machining operations. Such an assumption is needed to avoid the presence of a nonlocal impulse initial conditions at the primary stage, which can be the source of significant difficulties.

The system under the consideration is now given by the following pair of differential and difference equations

$$\begin{aligned} \dot{x}(t) =& Ax(t) + A_{-1}x(t-h) + B_0 y(t) + B_{-1}y(t-h) + Bu(t) \\ y(t) =& Cx(t) + C_{-1}x(t-h) + D_{-1}y(t-h) + Du(t), \ t \in T \doteq [0, \alpha] \end{aligned} \tag{6.14}$$

with initial conditions

$$x(t) = f(t), \ t \in [-h, 0), \ x(0) = x_0, \ y(t) = g(t), \ t \in [-h, 0] \tag{6.15}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $u \in \mathbb{R}^r$, and α and h are given real numbers such that $h < \alpha$. We also assume that the control function $u(t)$ is piecewise continuous on the interval $[0, \alpha]$. The differential linear repetitive process (6.1) now follows immediately as a special case of this last model structure when choosing the matrices in (6.14) as

$$A = \hat{A},\ A_{-1} = 0,\ B_0 = 0,\ B_{-1} = \hat{B}_0,\ B = \hat{B},\ C = \hat{C},\ C_{-1} = 0,\ D_{-1} = \hat{D}_0,\ D = \hat{D}$$

and $\alpha = \hat{\alpha}N,\ h = \alpha$.

It is well known [35, 49] that a solution of the time delay differential equation can be found by the step method. In other words, by the application of the standard integration step-by-step method on each subinterval $[kh, (k+1)h)$ (with nonnegative integer k) we can construct the solution as the solution of an appropriate ODE. Let us focus on the smoothness property of the solutions as it follows from this procedure. Consider the first delay-interval, and more at the moment $t = 0$. Due to the form of the differential equation (6.14), and since the initial condition (6.15) is chosen arbitrarily, one can say that

$$\dot{x}(t)|_{t=0+} \neq \dot{f}(t)|_{t=0-} = \dot{x}(t)|_{t=0-} \tag{6.16}$$

i. e. there is a discontinuity in the first derivative of the solution $x(t)$ at the moment $t = 0$. Due to this fact we consider the differential equation (6.14) for $t > 0$ and use the separate function value $x(0) = x_0$ in the initial data (6.15). This remark can be extended to the next delay-intervals $[kh, (k+1)h)$, $k > 0$, but, note that the solution is getting smoother from one delay-interval to the next at the moments $t = kh$, $k > 1$. Next, from the difference equation (6.14) it follows that at the moment $t = 0$ we have

$$y(0) = Cx(0) + C_{-1}x(-h) + D_{-1}y(-h) + Du(0), \tag{6.17}$$

i. e. the value $y(0)$ of the pass profile $y(t)$ is determined by the initial data and the value of the control function $u(0)$. For this reason we consider the control functions $u(t)$ that are continuous from the right hand side, and, for the sake of brevity, let the left side limit value be $u(0-) = \lim\limits_{t \to 0-} u(t)$ which coincides with $g(0)$.

The pair of functions $(x(t), y(t))$ is termed as a solution of the system (6.14) — (6.15) for the given control function $u(t)$, if they satisfy the differential equation (6.14) almost everywhere on the interval $[0, \alpha]$ and the difference equation (6.14) for all $t \in [0, h]$. It is known that under the given assumptions the solution $x(t)$ is absolutely continuous and $y(t)$ is piecewise continuous on the interval $[0, \alpha]$.

Systems described by the equations of this form have been discussed in [1, 47], the first that results in the optimality conditions for the nonlinear version of the system (6.14) — (6.15) were obtained in [63], and some observability and controllability problems for a particular case of the system can be found in [3]. Here we present in an unified form some results on controllability and optimization that are relevant and necessary for a deep theoretical background for control of repetitive processes.

6.2.1 General response formula

The solution of the system (6.14) — (6.15) can be constructed by the step-by-step procedure for each subinterval of the form $[ih,(i+1)h),\ i=0,1,\dots,q_\alpha$, where $q_\alpha=\left[\frac{\alpha}{h}\right]$ denotes the integer part of the fraction $\frac{\alpha}{h}$. First, it is straightforward to show that the recurrent procedure based on the equation (6.14) leads to the following representation of $y(t)$ on the time interval $[0,t],\ t>h,\ t\in[q_th,(q_t+1)h)$ where $q_t=\left[\frac{t}{h}\right]$

$$
\begin{aligned}
y(t) =&Cx(t)+\sum_{j=0}^{q_t-1}M_{j+1}x(t-(j+1)h)+\sum_{j=0}^{q_t}G_ju(t-jh) \\
&+K_{q_t}g(t-(q_t+1)h)+W_{q_t}f(t-(q_t+1)h)
\end{aligned}
\tag{6.18}
$$

and for $t\in[0,h)$

$$
y(t)=Cx(t)+Du(t)+D_{-1}g(t-h)+C_{-1}f(t-h),
$$

where

$$
M_{j+1}=D_{-1}^j(C_{-1}+D_{-1}C),\ G_j=D_{-1}^jD,\ K_i=D_{-1}^iC_{-1},\ W_i=D_{-1}^{i+1},\ M_0=C,\ j=0,1,\dots \tag{6.19}
$$

Noting the formula (6.18)—(6.19) and using the recurrent procedure on the intervals $[0,h),[h,2h),\dots$ allows us to rewrite (6.14) as

$$
\begin{aligned}
\dot{x}(t) &= \sum_{j=1}^{q_t+1}H_jx(t-(j-1)h)+\sum_{j=1}^{q_t+1}V_ju(t-(j-1)h) \\
&+ Q_{q_t+1}g(t-(q_t+1)h)+P_{q_t+1}f(t-(q_t+1)h)
\end{aligned}
\tag{6.20}
$$

where

$$
\begin{aligned}
H_1 =&A+B_0C,\ H_2=A_{-1}+B_0(C_{-1}+D_{-1}C)+B_{-1}C,\\
H_j =&(B_0D_{-1}^{j-1}+B_{-1}D_{-1}^{j-2})(C_{-1}+D_{-1}C),\ j=2,\dots,\ q_t+1\\
V_1 =&B+B_0D,\ V_j=(B_0D_{-1}^{j-1}+B_{-1}D_{-1}^{j-2})D,\ j=2,\dots,\ q_t+1,\\
P_i =&(B_0D_{-1}+B_{-1}D_{-1}^{i-2})C_{-1},\ Q_i=(B_0D_{-1}+B_{-1})D_{-1}^{i-1},\ P_1=A_{-1}+B_0C_{-1}
\end{aligned}
\tag{6.21}
$$

Formula (6.20) says, in fact, that the hybrid system of (6.14) can be represented by retarded differential equations with varying number of delays. The amount of delays is increases with the growth of t.

Next, multiplying both sides of the equation (6.20) by the function $F(t,\tau)$, which is unknown at present, and then integrating yields on the left hand side

$$
\int_0^t F(t,\tau)\dot{x}(\tau)d\tau=x(t)-F(t,0)x_0-\int_0^t\frac{\partial F(t,\tau)}{\partial\tau}x(\tau)d\tau \tag{6.22}
$$

where we set $F(t,\tau) \equiv 0,\ \forall\ \tau > t$, and $F(t,t-0) = I_n$. Next, substituting $s = \tau - ih$ in each of the integrals on the right hand side, and noting that

$$F(t,\tau) \equiv 0,\ \forall\ \tau > t,\ x(t) = f(t),\ t \in [-h,0),\ x(t) \equiv 0,\ \forall t < -h$$

together with (6.22) leads to the following formula

$$x(t) = F(t,0)x_0 + \int_0^t \sum_{j=1}^{q_t+1} F(t,\tau)H_j x(\tau-(j-1)h)d\tau + \int_0^t \frac{\partial F(t,\tau)}{\partial \tau} x(\tau)d\tau + \tag{6.23}$$

$$\int_0^t \sum_{j=1}^{q_t+1} F(t,\tau)V_j u(\tau-(j-1)h)d\tau + \int_0^t Q_{q_t+1} g(\tau-(q_t+1)h)d\tau + \int_0^t P_{q_t+1} f(\tau-(q_t+1)h)d\tau$$

$$= \quad F(t,0)x_0 + \sum_{j=1}^{q_t+1} \int_0^t F(t,\tau+(j-1)h)H_j x(\tau)d\tau + \sum_{j=1}^{q_t+1} \int_0^t F(t,\tau+(j-1)h)V_j u(\tau)d\tau$$

$$+ \quad \sum_{j=1}^{q_t+1} \int_{-h}^0 F(t,\tau+(j-1)h)H_j f(\tau)d\tau + \int_{-h}^0 F(t,\tau+(q_t+1)h)\Big[P_{q_t+1}f(\tau) + Q_{q_t+1}g(\tau)\Big]d\tau$$

Now define the required function $F(t,\tau)$ as a solution of the following differential equation

$$\frac{\partial F(t,\tau)}{\partial \tau} = -\sum_{j=1}^{q_t+1} F(t,\tau+(j-1)h)H_j,\ F(t,\tau) \equiv 0,\ \forall\ \tau > t,\ F(t,t-0) = I_n, \tag{6.24}$$

(where $F(t,t-0)$ denotes $F(t,\tau)$ evaluated as $t \to \tau$ from the left) whose properties can be found, for example, in [29]. Finally, noting (6.18), we have the following formula for the solutions of the system (6.14)—(6.15)

$$x(t) = F(t,0)x_0 + \sum_{j=1}^{q_t+1} \int_{-h}^0 F(t,\tau+(j-1)h)H_j f(\tau)d\tau + \sum_{j=1}^{q_t+1} \int_0^t F(t,\tau+(j-1)h)V_j u(\tau)d\tau +$$

$$\int_{-h}^0 F(t,\tau+(q_t+1)h)\big[P_{q_t+1}f(\tau) + Q_{q_t+1}g(\tau)\big]d\tau,\ t \geq 0; \tag{6.25}$$

$$y(t) = CF(t,0)x_0 + \int_{-h}^0 CF(t,\tau)H_1 f(\tau)d\tau + \int_{-h}^0 CF(t,\tau+h)\Big[P_1 f(\tau) + Q_1 g(\tau)\Big]d\tau +$$

$$+ \int_0^t CF(t,\tau)V_1 u(\tau)d\tau + C_{-1}f(t-h) + D_{-1}g(t-h) + Du(t),\ t \in [0,h)$$

$$
\begin{aligned}
y(t) =& CF(t,0)x_0 + \sum_{j=0}^{q_t-1} M_{j+1}F(t-(j+1)h,0)x_0 + \\
& \sum_{l=0}^{q_t-1}\sum_{j=0}^{q_t-l}\int_{-h}^{0} M_l F(t-lh,\tau+jh)H_{j+1}f(\tau)d\tau + \\
& + \sum_{l=0}^{q_t}\int_{-h}^{0} M_l F(t-lh,\tau+(q_t+1-l)h)\left[P_{q_t+1-l}f(\tau)+Q_{q_t+1-l}g(\tau)\right]d\tau + \\
& + \sum_{l=0}^{q_t-1}\sum_{j=0}^{q_t-l}\int_{0}^{t} M_l F(t-lh,\tau+jh)V_{j+1}u(\tau)d\tau + \\
& + \sum_{j=0}^{q_t} G_j u(t-jh) + K_{q_t-1}g(t-q_t h) + W_{q_t-1}f(t-q_t h), \quad q_t = \left[\frac{t}{h}\right],\ t \geq h
\end{aligned}
\tag{6.26}
$$

which clearly is the general response formula for (6.14).

6.3 Controllability

In this section we consider a controllability of hybrid system of (6.14) which clearly must be a fundamental element of a mature systems theory for linear repetitive processes and play a significant role for application areas. The formula (6.25)–(6.26) is a required starting point for this study. Here it should also be noted that there exists more than one distinct controllability notion, see e. g. [41], and that this area is far from being complete for the repetitive processes and delay systems considered here.

6.3.1 Pointwise completness and controllability with respect to initial data

In general, for differential systems with retarded arguments and, in particular, for hybrid differential-difference systems, the so-called pointwise completness [65, 66] pays a key role. In order to formulate this notion we introduce the following notations. Let $C^n[-h,0],\ h>0$ denote the vector space of the continuous n-vector function $f:[-h,0]\to\mathbb{R}^n$. The solution of system (6.14)—(6.14) (in the absence of input actions, i. e. with $B=0,\ D=0$) corresponding to the initial data (6.15) where $f\in C^n[-h,0],\ g\in C^m[-h,0],\ x_0\in\mathbb{R}^n$ is denoted by $x(t)=x(t,f,g,x_0),\ y(t)=y(t,f,g,x_0)$. The reachability set for the state variable $x(t)$ of the system (6.14)—(6.15) at the given moment $t^*\in[0,T]$ is defined as follows

$$
\mathcal{R}_x(t^*) = \{x\in\mathbb{R}^n : x = x(t^*,g,f,x_0), \text{ for all } f\in C^n[-h,0],\ g\in C^m[-h,0],\ x_0\in\mathbb{R}^n\} \tag{6.27}
$$

By analogy, the reachability set for the pass profile $y(t)$ of the system (6.14)—(6.15) at the given moment $t^* \in [0,T]$ is defined as

$$\mathcal{R}_y(t^*) = \{y \in \mathbb{R}^m : y = y(t^*, g, f, x_0), \text{ for all } f \in C^n[-h,0],\ g \in C^m[-h,0],\ x_0 \in \mathbb{R}^n\} \quad (6.28)$$

For many cases an essential question is: is reaching the desired state and/or pass profile position dependent on the choice of the initial data? The following definition is a formal description of this problem.

Definition 13. *It is said that the system (6.14)—(6.15) is pointwise complete on the interval* $[0,T]$ *if*

$$\mathcal{R}_x(t) = \mathbb{R}^n \ \textit{and}\ \mathcal{R}_y(t) = \mathbb{R}^m \ \textit{for all}\ \ t \in [0,T]. \quad (6.29)$$

If for some $t^* \in [0,T]$ *the conditions (6.29) are not true then the system is called pointwise degenerate at the moment* t^*.

The notion of pointwise completness was introduced first in [65] for the study of the controllability of linear differential time delay systems. Some details and an overview of existing results can also be found in the survey [42]. It is obvious that the ordinary linear differential system of the form $\dot{x}(t) = Ax(t)$ is pointwise complete since for any t^* and $x^* \in \mathbb{R}^n$ there exists $x(0) = x_0 \in \mathbb{R}^n$ such that the corresponding solution satisfies the condition $x(t^*, x_0) = x^*$. Also, it is proved that each stationary linear differential system with constant time delay is pointwise complete in the case $n = 2$. The following example shows that the presence of a "difference" equation in the hybrid system destroys the pointwise completness of differential time delay system with $n = 2$.

Example. Consider the hybrid system of (6.14)—(6.15) on the interval $t \in [0,T]$ where $h \leq T \leq 2h$, $n = 2$, $m = 2, h = \ln 2$ and the following choice of the matrices

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix},\ A_{-1} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix},\ B_{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix},\ B_0 = 0, \quad (6.30)$$

$$C = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix},\ C_{-1} = \begin{pmatrix} 0 & 0 \\ -4 & 0 \end{pmatrix},\ D_{-1} = 0,\ B = 0,\ D = 0$$

Substituting the function $y(t)$ from second equation into the first of the system (6.14)—(6.15) corresponding to the given choice of matrices leads to the following time delay system

$$\dot{x}(t) = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -2 & 0 \\ -1 & 2 \end{bmatrix} x(t-h) + \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} x(t-2h) \quad (6.31)$$

Thus the state variable of the considered hybrid system (6.30) is described by the retarded differential system (6.31) with multiple delays. For simplicity, next the matrices involved in (6.31) are denoted by A, A_1, A_2, respectively. It is known (see, [42]) that a linear stationary differential system with multiple delays is pointwise complete if, and only if, the following conditions

$$rank M^0 = n + n_1, \qquad \text{where } n_1 = \sum_{i=1}^{N} rank M_i(\lambda_i) \tag{6.32}$$

hold. Here the matrices M^0 and $M_i(\lambda_i)$ are defined by spectral parameters of the operator

$$W(\lambda, e^{-\lambda h}) = (\lambda I - A - e^{-\lambda h} A_1 - e^{-2\lambda h} A_2), \qquad \lambda \in \mathbb{C} \tag{6.33}$$

associated with the system (6.31). In the considered case we have

$$W(\lambda, e^{-\lambda h}) = \begin{bmatrix} \lambda + 2e^{-\lambda h} & -2 \\ e^{-\lambda h} + 2e^{-2\lambda h} & -\lambda - 1 - 2e^{-\lambda h} \end{bmatrix}, \quad \det W(\lambda, e^{-\lambda h}) = \lambda^2 - \lambda. \tag{6.34}$$

Hence, the eigenvalues are $\lambda_1 = 0$ and $\lambda = 1$. Further, noting $h = \ln 2$, we have

$$M_1(\lambda_1) = W(\lambda, e^{-\lambda h})|_{\lambda=0} = \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}, \quad M_2(\lambda_2) = W(\lambda, e^{-\lambda h})|_{\lambda=1} = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \tag{6.35}$$

and the constant $(n+1)n \times n^2$ (in this case 6×4) matrix M^0 is defined as

$$M^0 = \begin{bmatrix} M_1(\lambda_1) & O \\ O & M_2(\lambda_2) \\ I & I \end{bmatrix}, \quad \text{where } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6.36}$$

It is easy to verify that

$$rank M_1 = rank \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} = 1, \quad rank M_2 = rank \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} = 1$$

and

$$rank M^0 = rank \begin{bmatrix} 2 & -2 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = 3.$$

Hence

$$3 = rank\ M^0 < n + n_1 = 4,$$

which immediately shows that the considered system is not pointwise complete for the delay value $h = \ln 2$.

Note that the eigenfuncions corresponding to the given eigenvalues $\lambda_1 = 0,\ \lambda_2 = 1$ are $\phi_1(t) = (1,1)^T,\ \phi_2(t) = (e^t, e^t)^T$. It is obvious that the rank of the fundamental matrix, the entries of which are the given eigenfunctions, is equal 1. Hence the linear space formed by these basic functions is isomorphic to the space $\mathbb{R}$. This means [66] again that the system under consideration is degenerate.

For the hybrid differential-difference systems links between the pointwise completness and controllability notions exist. We start here with a particular case of the state controllability with respect initial conditions.

Definition 14. *The system (6.14)—(6.15) (with $B = 0,\ D = 0$) is said to be state controllable with respect to initial data at the given moment $t = T$ if for any n-vector $c_T \in \mathbb{R}^n$ there exists the initial functions $g(t),\ f(t),\ t \in [-h, 0]$ such that the corresponding solution $x(t, g, f, x_0)$ of the system (6.14)—(6.15) satisfies the following condition*

$$x(T, g, f, x_0) = c_T \tag{6.37}$$

Now, the following theorem can be stated.

Theorem 13. *[42] The system (6.14)—(6.15) (with $B = 0,\ D = 0$) is state controllable with respect to the initial data at the given moment $t = T$ if, and only if,*

i) system (6.14)—(6.15) is pointwise complete;

ii)

$$rank\{H_0^i[H_1, H_2, \ldots, H_{q_T}, G_{q_T}, P_{q_T}],\ i = 0, \ldots, n\} = n \tag{6.38}$$

where the matrices $H_i, i = 0, \ldots, q_T,\ G_{q_T},\ P_{q_T}$ are defined in (6.21).

The proof of the theorem and other results can be found in [42] and, hence the details are omitted here.

By analogy to Definition 14, profile controllability with respect to initial data can be introduced and studied.

6.3.2 Point pass profile controllability

For nD systems as well as for repetitive processes there many possibilities for introducing various controllability notions. In this subsection we introduce and study the following point pass profile controllability, which plays a significant role in further analysis.

Definition 15. *The system (6.14)-(6.15) is said to be pass profile controllable at the given points $\beta_0, \beta_1, \ldots, \beta_\nu$, such that $0 = \beta_0 < \beta_1 < \ldots \beta_\nu \leq \alpha$, if for any $c_i \in \mathbb{R}^m,\ i = 0, \ldots, \nu$ there exists a control vector $u(t),\ t \in [0, \alpha]$ such that the solution $y(t, g, f, x_0, u)$ of the system (6.14)—(6.15) corresponding to the zero initial data $g(t) = 0,\ t \in [-h, 0),\ f(t) = 0,\ t \in [-h, 0),\ x_0 = 0$ satisfies the following conditions*

$$y(\alpha - \beta_j, 0, 0, 0, u) = c_j,\ j = 0, 1, \ldots, \nu. \tag{6.39}$$

We suppose that the admissible control functions $u(t)$ belong to the class of all piecewise continuous functions on the interval $t \in [0, \alpha]$ with values in the space $\mathbb{R}^r$. This class is denoted by $U(\cdot)$.

Physical motivation for this form of controllability is the requirement that the pass profile vector takes pre-assigned values at particular points along the pass. Note also that some first results concerning observability and controllability problems for particular cases of the system model structure considered here can be found in the earlier paper [3].

From (6.25) we have

$$y(t) = \sum_{j=0}^{i} G_j u(t - jh) + \int_0^t R(t, \tau) u(\tau) d\tau, \qquad t \geq h, \tag{6.40}$$

where

$$R(t, \tau) = \sum_{l=0}^{q_t - 1} \sum_{j=0}^{q_t - l} \int_0^t M_l F(t - lh, \tau + jh) V_{j+1}, \qquad q_t = \left[\frac{t}{h}\right] \tag{6.41}$$

Note that in (6.25) $r(t, g, f, x_0) = 0$ for the zero initial data $g(t) = 0,\ t \in [-h, 0),\ f(t) = 0,\ t \in [-h, 0),\ x_0 = 0$.

Theorem 14. *The system (6.14)—(6.15) is pass profile controllable at the given points $\beta_0, \beta_1, \ldots, \beta_\nu$ if, and only if, the following equalities*

$$g_i^T G_0 = 0,\ \ldots,\ g_i^T G_{q_i} = 0, \qquad g_i^T R(\alpha - \beta_i, \tau) \equiv 0,\ \tau \in [0, \alpha - \beta_i],\ i = 0, 1, \ldots, \nu \tag{6.42}$$

hold only when $g_i = 0$, where $g_i \in \mathbb{R}^m$, $i = 0, 1, \ldots \nu$ and $q_i = \left[\frac{\alpha - \beta_i}{h}\right]$.

Proof. The property to be established here requires that the following set of equations

$$c_i = G_0 u(\alpha - \beta_i) + \ldots + G_{q_i} u(\alpha - \beta_i - q_i h) + \int_0^{\alpha - \beta_i} R(\alpha - \beta_i, \tau) u(\tau) d\tau,\ i = 0, \ldots, \nu \tag{6.43}$$

can be solved with respect to the unknown vector $u(t),\ t \in [0, \alpha]$ with piecewise continuous entries and r-vectors $u(\alpha - \beta_i - q_i h),\ i = 0, \ldots, \nu$. Consider therefore the following set

$$\begin{aligned} Y = \Big\{ y =(y_0, \ldots, y_\nu) \in \mathbb{R}^{m(\nu+1)} :\ y_s = \sum_{j=0}^{q_s} G U_j v_{js} + \\ + \int_0^{\alpha - \beta_s} R(\alpha - \beta_s, \tau) u(\tau) d\tau,\ \forall\ v_{js} \in \mathbb{R}^{m(\nu+1)},\ \forall u(\cdot) \in U(\cdot) \Big\} \end{aligned} \tag{6.44}$$

where $U(\cdot)$ denotes the set of all admissible control vectors. Then it is easy to see that the set $Y \subset \mathbb{R}^{m(\nu+1)}$ is a linear subspace of $\mathbb{R}^{m(\nu+1)}$.

Now suppose that conditions (6.42) hold but the system is not pass profile controllable. Then this means that $Y \neq \mathbb{R}^{m(\nu+1)}$. Since the set Y is a linear subspace of $\mathbb{R}^{m(\nu+1)}$, there exists a nontrivial vector $\bar{g} = (\bar{g}_1, \ldots, \bar{g}_\nu) \in \mathbb{R}^{m(\nu+1)}$, $\bar{g} \neq 0$, such that $\bar{g} \perp Y$. This, in turn, means that there exists a nontrivial vector $\bar{g} \neq 0$ which satisfies the conditions of (6.42) and a contradiction has been established.

Suppose now the system is controllable but condition (6.42) holds for some nontrivial vector $g^* \in \mathbb{R}^{m(\nu+1)}$. This means that $g^* \perp Y$. Hence $Y \neq \mathbb{R}^{m(\nu+1)}$ which is a contradiction and the proof is complete. ∎

Theorem 14, however, is hard to apply to checking controllability. Another approach would be to apply the so-called characteristic equations approach introduced in [29] to obtain the effective criteria to check the controllability properties of the considered model. To obtain the characteristic equations that follow, apply the Laplace transform to the system (6.14)—(6.15) with zero initial data

$$\begin{aligned} pX(p) &= AX(p) + A_{-1}e^{-ph}X(p) + B_0Y(p) + B_{-1}e^{-ph}Y(p) + BU(p), \\ Y(p) &= CX(p) + C_{-1}e^{-ph}X(p) + D_{-1}e^{-ph}Y(p) + DU(p). \end{aligned} \tag{6.45}$$

Next the following substitutions are to be done: replace $X(p), Y(p), U(p)$ by the $(n \times r)$, $(m \times r)$ and $(r \times r)$ - matrices $X_{k-1}(t)$, $Y_{k-1}(t)$, $U_{k-1}(t)$, $k = 1, 2, \ldots$, $t \in [0, \alpha]$; the differention operator p is replaced by the shift operator with respect to the discrete variable k, the operator e^{-ph} is replaced by the time delay operator such that the following relations

$$\begin{aligned} &X(p) \longrightarrow X_{k-1}(t),\ e^{-ph}X(p) \longrightarrow X_{k-1}(t-h),\ pX(p) \longrightarrow X_k(t) \\ &Y(p) \longrightarrow Y_{k-1}(t),\ e^{-ph}Y(p) \longrightarrow Y_{k-1}(t-h) \end{aligned} \tag{6.46}$$

hold. This enables rewriting (6.14)—(6.15) in the following form

$$\begin{aligned} X_k(t) =& AX_{k-1}(t) + A_{-1}X_{k-1}(t-h) + B_0Y_{k-1}(t) + B_{-1}Y_{k-1}(t-h) + BU_{k-1}(t) \\ Y_{k-1}(t) =& CX_{k-1}(t) + C_{-1}X_{k-1}(t-h) + D_{-1}Y_{k-1}(t-h) + DU_{k-1}(t),\ t \in [0, \alpha] \end{aligned} \tag{6.47}$$

In order to complete this setting it is necessary to determine the initial conditions

$$\begin{aligned} &X_0(0) = 0,\ X_i(t) \equiv 0,\ \forall\, i \leq 0,\ t \leq 0;\ Y_0(0) = 0,\ Y_i(t) \equiv 0,\ \forall\, i \leq 0,\ t \leq 0. \\ &U_0(0) = I_r,\ U_i(t) \equiv 0,\ \forall\, i \neq 0,\ t \neq 0. \end{aligned} \tag{6.48}$$

Now, the following theorem can be stated, the proof of which is very strongly motivated by the results of the earlier paper [3].

Theorem 15. *The system (6.14)—(6.15) is pass profile controllable at the given points* $\beta_0, \beta_1, \ldots, \beta_\nu$ *if, and only if, the following rank condition holds*

$$rank \begin{bmatrix} Y_i(t-\beta_0) & \\ Y_i(t-\beta_1) & i=0,\ldots,n(q_\alpha+1) \\ \cdots & \\ Y_i(t-\beta_\nu) & t\in[0,\beta_\nu+(q_\alpha+1)h] \end{bmatrix} = (\nu+1)m. \tag{6.49}$$

Proof. The conditions (6.42) of Theorem 14 yield that the process (6.14)—(6.15) is pointwise profile pass controllable if, and only if, the following conditions hold

$$\text{rank}\{\mathcal{G}_0, G_1, \ldots, \mathcal{G}_{q_\alpha}\} = m(\nu+1) \tag{6.50}$$

and

$$g^T\mathfrak{F}(\alpha,\tau) \not\equiv 0 \quad \tau\in[0,\alpha] \qquad \text{for all} \quad g\in\mathbb{R}^{m(\nu+1)},\ g\neq 0 \tag{6.51}$$

where $\big(m(\nu+1)\times r(\nu+1)\big)-$ matrices $\mathcal{G}_i$ are given by

$$\mathcal{G}_i = diag\{G_i,\ldots,G_i\}, \qquad \text{where } G_i = D_{-1}^iD,\ i=0,\ldots,q_\alpha, \tag{6.52}$$

and $[n(\nu+1)\times m(\nu+1)]$ matrix function $\mathfrak{F}(\alpha,\tau)$ is defined as

$$\mathfrak{F}(\alpha,\tau) = \begin{cases} \mathcal{M}_0\mathcal{F}(\alpha,\tau)\mathcal{V}, & \tau\in[\alpha-\beta_1,\alpha) \\ (\mathcal{M}_0\mathcal{F}(\alpha,\tau)+\mathcal{M}_1\mathcal{F}(\alpha,\tau+\beta_1))\mathcal{V}, & \tau\in[\alpha-\beta_1,\alpha) \\ \cdots\cdots\cdots & \cdots\cdots \\ \sum_{i=0}^{\nu}\mathcal{M}_i\mathcal{F}(\alpha,\tau+\beta_i)\mathcal{V}, & \tau\in(\alpha-\beta_\nu,\alpha-\beta_{\nu-1}], \end{cases} \tag{6.53}$$

where

$$\begin{aligned} &\mathcal{V} = [V_1, V_2, \cdots, V_{q_\alpha+1}]^T, \\ &\mathcal{M}_k = \Big[O_{n(q+1)\times mk}, [M_0,\ldots,M_q]^T, O_{n(q+1)\times m(\nu-k)}\Big], \quad k=0,\ldots,\nu \end{aligned} \tag{6.54}$$

and

$$\mathcal{F}(\alpha,\tau) = \begin{bmatrix} F(\alpha,\tau) & F(\alpha,\tau+h) & F(\alpha,\tau+2h) & \cdots & F(\alpha,\tau+(\nu-1)h) & F(\alpha,\tau+\nu h) \\ F(\alpha,\tau+h) & F(\alpha,\tau+2h) & F(\alpha,\tau+3h) & \cdots & F(\alpha,\tau+\nu h) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ F(\alpha,\tau+\nu h) & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Note that to design the function (6.53) the well-known properties $F(t,\tau)\equiv 0,\ t<\tau,\quad F(t-s,\tau)=F(t,s+\tau),\ t\le s\le \tau$ of the fundamental matrix $F(t,\tau)$ are used. From the characteristic equation (6.47)—(6.48) it follows immediately that $Y_0(ih)=D_{-1}^{i}D=G_i$, and, hence,

$$\mathcal{G}_i = diag\{Y_0(ih),\dots,Y_0(ih)\}$$

which contains a part of the required matrices in (6.49).

To select the remaining matrices in (6.49), consider the $n(q_\alpha+1)$ vector-valued function $\psi(g,\tau)=g^T\mathfrak{F}(\alpha,\tau)$. Then it can be shown that the function $\mathfrak{F}(\alpha,\tau)$ satisfies the following matrix differential equation

$$\frac{\partial\mathfrak{F}(\alpha,\tau)}{\partial\tau}=-\mathfrak{F}(\alpha,\tau)\mathcal{H},\quad \text{where } \mathcal{H}=\begin{bmatrix} H_1 & 0 & 0 & \dots & 0 & 0\\ H_2 & H_1 & 0 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ H_{q_\alpha} & H_{q_\alpha-1} & H_{q_\alpha-2} & \dots & H_2 & H_1\end{bmatrix} \tag{6.55}$$

Hence the function $\psi(g,\tau)$ satisfies an ordinary homogeneous differential equation of order $n(q_\alpha+1)$, and it is known that such a differential equation has a trivial solution if, and only if, its initial conditions are zero. From [29] it now follows that the function considered here is analytic on each sub-interval $(\alpha-kh,\alpha-(k+1)h)$, and dis-continuous at $\tau=\alpha-\beta_i-jh,\ i=0,\dots,\nu,\ j=1,\dots,q$ where the jumps are given by

$$\begin{aligned}\Delta\psi^{(s)}(g,\alpha-\beta_i-jh) \doteq&\psi^{(s)}(g,\alpha-\beta_i-jh+0)-\psi^{(s)}(g,\alpha-\beta_i-jh-0)=\\ =&(-1)^{i+j}g^TY_s(\alpha-\beta_i-jh),\ i=0,\dots,\nu,\ j=1,\dots,q\end{aligned} \tag{6.56}$$

Thus, $\Delta\psi^{(s)}(g,\alpha-\beta_i-jh)=0$, and, therefore $\psi(g,\tau)\equiv 0$, if, and only if, the matrix (6.49) has the maximal rank, which completes the proof. ■

6.4 Optimization

In this section the following time optimal control problem for the process (6.14)—(6.15) is considered.

Optimization Problem. *For the given initial data $x(t)=f(t),\ y(t)=g(t),\ t\in[-h,0),\ x(0)=x_0$ find the minimal time T and the control function $u(t),\ t\in[0,T]$ such that the corresponding trajectory*

$$x(t)\equiv 0,\ t\in[T-h,T] \tag{6.57}$$

is in the equilibrium state.

In effect, the solution to this problem will drive the system dynamics to the zero equilibrium state as quickly as possible.

Note that in this case, subject to some additional assumptions, the control function on the last interval $[T-h,T]$ can be represented in the feedback form. In particular, suppose that the matrix B from (6.14) is invertible. Then from (6.14) we have

$$u(t) = -B^{-1}\left[A_{-1}x(t-h) + B_0 y(t) + B_{-1}y(t-h)\right] \tag{6.58}$$

Substituting this into (6.14), and assuming that there exists $\left[E + DB^{-1}B_0\right]^{-1}$ yields finally

$$u(t) = Nx(t-h) + My(t-h), \ t \in [T-h,T] \tag{6.59}$$

where

$$N = \left[E + DB^{-1}B_0\right]^{-1}\left[C_{-1} - DB^{-1}A_{-1}\right], \ M = \left[E + DB^{-1}B_0\right]^{-1}\left[D_{-1} - DB^2_{-1}\right] \tag{6.60}$$

The representation (6.59) shows also that the complete controllability can be solved for the particular given case on the base of relative (pointwise) controllability formulated above. Indeed, if there exists a control function $u(t), \ t \in [0, T-h]$ such that $x(T-h) = 0$ (for the considered case it is sufficient to choose the single point for $\nu = 0, \ \beta_\nu = h$ and $c_\nu = 0$) then the following setting

$$\bar{u}(t) = \begin{cases} u(t), \ t \in [0, T-h) \\ Nx(t-h) + My(t-h), \ t \in [T-h, T] \end{cases}$$

solves the problem of the complete controllability. Note however that this approach is however of a limited significance as in the majority cases the state dimension considerably exceeds the input dimension. Then, the semi inverse approach can be applied, which is the subject of ongoing work.

Now let T be a fixed time moment. The class of the admissible controls $u(t), \ t \in [0,T]$ is the set of all piecewise continuous functions such that $u(t) \in U, \ t \in [0,T]$, where U is a compact convex set from $\mathbb{R}^r$. By analogy to (6.40,) the solution of the process (6.14)—(6.15) can be rewritten in the following form

$$\begin{aligned} x(t) =& s(t,f,g,x_0) + \int_0^t S(t,\tau)u(\tau)d\tau, \\ y(t) =& r(t,f,g,x_0) + \sum_{j=0}^{i} G_j u(t-jh) + \int_0^t R(t,\tau)u(\tau)d\tau, \end{aligned} \tag{6.61}$$

where $r(t,f,g,x_0)$ and $R(t,\tau)$ were defined by (6.41) and

$$\begin{aligned} s(t,f,g,x_0) =& F(t,0)x_0 + \sum_{j=1}^{i+1} \int_{-h}^{0} F(t,\tau+(j-1)h)H_j f(\tau)d\tau + \\ &+ \int_{-h}^{0} F(t,\tau+(i+1)h)\left[P_{i+1}f(\tau) + Q_{i+1}g(\tau)\right]d\tau, \\ S(t,\tau) =& \sum_{j=1}^{i+1} F(t,\tau+(j-1)h)V_j, \ i = \left[\frac{T}{h}\right] \end{aligned} \tag{6.62}$$

Definition 16. *We say that the control function* $u(t),\ t \in [0,T]$ *is* T*-admissible for the system (6.14)—(6.15), if the corresponding trajectory satisfies the following condition*

$$x(t) \equiv 0,\ t \in [T-h, T] \tag{6.63}$$

Introduce

$$Z = \left\{ x \in \mathbb{R}^n \mid x = s(T-h, f, g, x_0) \text{ for all } (f, g, x_0) \in C_{[-h,0]} \times C_{[-h,0]} \times \mathbb{R}^n \right\}$$

and

$$\mathcal{R} = \left\{ s \in Z \mid \text{ such that for } x = s \ \exists \text{ T-admissible control } u(\cdot) \right\}. \tag{6.64}$$

In fact, the set $\mathcal{R}$ is the reachability set for the system (6.14)—(6.15) with (6.63) in place. We assume that $\mathcal{R} \neq O$, which is true if the system is controllable. Equivalently, we suppose that there exists at least one collection of the initial data

$$x(t) = f(t),\ t \in [-h, 0),\ x(0) = x_0,\ y(t) = g(t),\ t \in [-h, 0]$$

for which there exists $T-$ admissible control functions. Let $U_T(\cdot)$ denote the set of the all T-admissible control vectors for the system (6.14)—(6.15) corresponding to the set $\mathcal{R}$. Then it is easy to show that $\mathcal{R}$ is closed and convex.

Theorem 16. *For the given initial data* $f(t),\ g(t),\ t \in [-h, 0), x(0) = x_0$ *there exists* T*-admissible control if, and only if, the following inequality*

$$\max_{\|g\|=1} \left\{ g^T s(T-h, f, g, x_0) + \inf_{u \in U_T(\cdot)} \int_0^{T-h} g^T S(T-h, \tau) u(\tau) d\tau \right\} \leq 0 \tag{6.65}$$

holds.

Proof. *Necessity.* Let Optimization Problem be solvable for the moment T and $u(t),\ t \in [0,T]$ is a T-admissible control function. This means that

$$0 = x(T-h) = s(T-h, f, g, x_0) + \int_0^{T-h} S(T-h, \tau) u(\tau) d\tau.$$

Multiplying both sides of the last equality by the vector $g \in \mathbb{R}^n$ yields

$$g^T s(T-h, f, g, x_0) + \int_0^{T-h} g^T S(T-h, \tau) u(\tau) d\tau = 0.$$

Hence

$$g^T s(T-h, f, g, x_0) + \inf_{u \in U_T(\cdot)} \int_0^{T-h} g^T S(T-h, \tau) u(\tau) d\tau \leq 0$$

and (6.65) holds.

Sufficiency. Let the inequality (6.65) hold for the given initial data $(f,\ g, x_0)$. On the contrary, assume that for this data there is no any T-admissible control $u(\cdot)$ which solves the problem. This means that the corresponding vector $s^* = s^*(T-h, f,\ g, x_0) \notin \mathcal{R}$ does not belong to the set $\mathcal{R}$, i. e. $s^*(T-h, f,\ g, x_0) \notin \mathcal{R}$. Since $\mathcal{R}$ is a convex set then there exists a supporting hyperplane with the nontrivial normal vector $g* \in \mathbb{R}^n$, $\|g^*\| = 1$ such that the following inequality

$$g^{*T} s^*(t, f, g, x_0) > g^{*T} s, \qquad \forall\ s \in \mathcal{R} \tag{6.66}$$

holds. Since $s \in \mathcal{R}$ then there exists a $T-$ admissible control function $u(t),\ t \in [0, T-h]$ such that

$$s + \int_0^{T-h} S(T-h, \tau) u(\tau) d\tau = 0.$$

Hence, (6.66) yields that

$$g^{*T} s^*(T-h, f, g, x_0) + \int_0^{T-h} g^{*T} S(T-h, \tau) u(\tau) d\tau > 0$$

and since s is an arbitrary vector from the set $\mathcal{R}$ then the last inequality is true for all $u(\cdot) \in U_T(\cdot)$. Therefore

$$g^{*T} s^*(T-h, f, g, x_0) + \inf_{u \in U_T(\cdot)} \int_0^{T-h} g^{*T} S(T-h, \tau) u(\tau) d\tau > 0$$

■

which contradicts (6.65).

Next, consider

$$\Lambda(T) = \max_{\|g\|=1} \left\{ g^T s(T-h, f, g, x_0) + \inf_{u \in U_T(\cdot)} \int_0^{T-h} g^T S(T-h, \tau) u(\tau) d\tau \right\}. \tag{6.67}$$

It can be shown that $\Lambda(T)$ is a non decreasing, continuous function, and hence we have the result below for which we also require the following definition.

Definition 17. *Let $p : \mathbb{R} \to \mathbb{R}$ be a function. Then we say that $Z^0 \in \mathbb{R}$ is the minimal root of equation $p(z) = 0$ if $p(z^0) = 0$ and there is no $z^* \in \mathbb{R}$ such that $z^* < z^0$ and $p(z^*) = 0$.*

Theorem 17. *Given the initial data $f(t), g(t),\ t \in [-h, 0), x(0) = x_0$, the moment T^0 is optimal if, and only if, T^0 is a minimal root of the equation*

$$\Lambda(T) = 0. \tag{6.68}$$

Proof. *Necessity.* Let $u^0(\cdot)$ be the optimal control for the optimization Problem. Then Theorem 16 gives $\Lambda(T^0) \le 0$. At first, suppose that $\Lambda(T^0) < 0$. Since $\Lambda(T)$ is a non decreasing and continuous function then $\exists \bar{T},\ \bar{T} < T^0$ such that $\Lambda(T^0) \le \Lambda(\bar{T}) \le 0$. In accordance with Theorem

16,the optimization Problem is solvable with $\bar{T} < T^0$ which is impossible. Thus, T^0 is a root for the equation (6.68). The minimality of T^0 can be shown analogously.

Sufficiency. Let T^0 be the minimal root of $\Lambda(T) = 0$ for the control function $u^0(t),\ t \in [0, T^0 - h]$. Suppose now that this control function is not optimal for the given initial data. Hence, there is the $\bar{T}$-admissible control function $\bar{u}(t),\ t \in [0, \bar{T} - h]$ where $\bar{T} < T^0$. Then Theorem 16 yields $\Lambda(\bar{T}) \leq 0$. However, noting that the function $\Lambda(T)$ is non-decreasing, we have $\Lambda(\bar{T}) \geq \Lambda(T^0) = 0$, which contradicts the minimality of the root T^0, which completes the proof. ∎

Finally, the optimal time T^0 is given by the equality (6.68) and the optimal control function $u^0(t)$ is determined as

$$\min_{u \in U_T(\cdot)} \int_0^{T-h} g^{0T} S(T-h, \tau) u(\tau) d\tau = \int_0^{T-h} g^{0T} S(T-h, \tau) u^0(\tau) d\tau \tag{6.69}$$

where g^0 is the vector which maximizes (6.67).

These optimality conditions can be presented in a more practically usable form for some particular sets of admissible controls $U(\cdot)$. In the next section the time optimal control problem subject to integral control constraints is solved.

6.4.1 Time optimal problem subject to integral control constraints

Consider the following optimization problem: Minimize

$$T \longrightarrow \min_{u \in U(\cdot)} \tag{6.70}$$

over the solutions of the process

$$\begin{aligned} \dot{x}(t) =& Ax(t) + A_{-1}x(t-h) + B_0 y(t) + B_{-1} y(t-h) + Bu(t) \\ y(t) =& Cx(t) + C_{-1}x(t-h) + D_{-1}y(t-h) + Du(t),\ t \in [0, T] \end{aligned} \tag{6.71}$$

with initial conditions

$$x(t) = f(t),\ t \in [-h, 0),\ x(0) = x_0,\ y(t) = g(t),\ t \in [-h, 0] \tag{6.72}$$

subject to state constraints

$$x(t) \equiv 0,\ t \in [T-h, T] \tag{6.73}$$

and integral control constraints

$$U(\cdot) \triangleq \left\{ u(\cdot) : \int_0^T u^T(\tau) u(\tau) d\tau \leq 1 \right\} \tag{6.74}$$

In accordance with Theorem 17 an optimal time control function has to satisfy the equation $\Lambda(T) = 0$ where the function $\Lambda(T)$ is defined in (6.67). Noting (6.62) reduces our task to calculate minimum in (6.67) as

$$\mathcal{M}(g) \triangleq \int_0^{T-h} g^T \sum_{j=1}^{i+1} F(T-h, \tau + (j-1)h)V_j u(\tau)d\tau \longrightarrow \inf_{u \in U_T(\cdot)} \tag{6.75}$$

subject to

$$\int_0^T u^T(\tau)u(\tau)dt \leq 1 \tag{6.76}$$

Using (6.59) and (6.61) allows rewriting (6.76) as

$$\begin{aligned}
&\int_0^{T-h} u^T(\tau)u(\tau)d\tau + \int_{T-h}^{T} (Nx(\tau-h) + My(\tau-h))^T (Nx(\tau-h) + My(\tau-h))d\tau = \\
&= \Upsilon + \int_0^{T-h} u^T(\tau)\left[I_m + \mathcal{G}(\tau)\right]^T u(\tau)d\tau + \int_0^{T-h} \left[\psi(\tau) + \varphi(\tau)\right]^T u(\tau)d\tau + \\
&+ \int_0^{T-h}\int_0^{T-h} u^T(\tau)\left[\Psi(\tau,\theta) + \Phi(\tau,\theta)\right]u(\theta)d\tau \leq 1
\end{aligned} \tag{6.77}$$

where I_m is the identity $(m \times m)$-matrix, and

$$\Upsilon = \int_{T-h}^{T} \left[Ns(\tau-h) + Mr(\tau-h)\right]^T \left[Ns(\tau-h) + Mr(\tau-h)\right]dt$$

$$\psi(\tau) = \int_{T-2h}^{T-h} \left\{S^T(\theta,\tau)N^T\left[Ns(\theta) + Mr(\theta)\right] + \left[s^T(\theta)N^T + r^T(\theta)M^T\right]MR(\theta,\tau)\right\}d\theta$$

$$\Phi(\theta,\tau) = \int_{T-2h}^{T-h} \left\{S^T(t,\tau)N^T\left[NS(t,\theta) + MR((t,\theta)\right] + R^T(t,\tau)M^TMR((t,\theta)\right]d\tau,$$

$$\Psi(\tau,\theta) = \begin{cases}
\left[S^T(\theta,\tau)N^T + R^T(\theta,\tau)M^T\right]MG_0, & \tau \in (T-2h, T-h] \\
\left[S^T(\theta+h,\tau)N^T + R^T(\theta+h,\tau)M^T\right]MG_1, & \tau \in (T-3h, T-2h] \\
\cdots\cdots & \cdots \\
\left[S^T(\theta+(q_T-1)h,\tau)N^T + R^T(\theta+(q_T-1)h,\tau)M^T\right]MG_{q_T-1}, & \tau \in [0,h],
\end{cases}$$

$$\varphi(\tau)=\begin{cases}\left(s^T(\tau)N^T+r^T(\tau)M^T\right)MG_0, & \tau\in(T-2h,T-h]\\ \left(s^T(\tau+h)N^T+r^T(\tau+h)M^T\right)MG_1, & \tau\in(T-3h,T-2h]\\ \dots\dots\dots\dots & \dots\\ \left(s^T(\tau+(q_T-1)h)N^T+r^T(\tau+(q_T-1)h)M^T\right)MG_{q_T-1}, & \tau\in[0,h]\end{cases} \tag{6.78}$$

and

$$\mathcal{G}(\tau)=\begin{cases}G_0^TM^TMG_0+G_0M^TMG_1e^{-ph}+\dots & \\ +G_0^TM^TMG_{g_T-1}e^{-(q_T-1)ph}, & \tau\in(T-2h,T-h]\\ G_1^TM^TMG_1+G_1M^TMG_2e^{-ph}+\dots & \\ +G_1^TM^TMG_{g_T-2}e^{-(q_T-2)ph}, & \tau\in(T-3h,T-2h]\\ \dots\dots\dots\dots & \dots\\ G_{g_T-2}^TM^TMG_{g_T-2}+G_{g_T-2}^TM^TMG_{g_T-1}e^{-ph}, & \tau\in(h,2h]\\ G_{g_T-1}^TM^TMG_{g_T-1} & \tau\in[0,h]\end{cases}$$

Here e^{-kph} denotes the shift operator such that $(e^{-kph}u)(\tau)=u(\tau-kh)$. Using the Lagrange multiplier method leads to the functional

$$\Pi(u)=\int_0^{T-h}g^TS(T-h,\tau)u(\tau)d\tau+\lambda\Bigg\{\Upsilon+\int_0^{T-h}u^T(\tau)\left[I_m+\mathcal{G}(\tau)\right]^Tu(\tau)d\tau+$$
$$+\int_0^{T-h}\left[\psi(\tau)+\varphi(\tau)\right]^Tu(\tau)d\tau+\int_0^{T-h}\int_0^{T-h}u^T(\tau)\left[\Psi(\tau,\theta)+\Phi(\tau,\theta)\right]u(\theta)d\tau\Bigg\} \tag{6.79}$$

which is subject to minimization with respect to the unknown λ and $u(t)$. Now, it is to find the first variation $\delta\Pi$ for $\Pi(u)$, which can be represented as

$$\delta\Pi(u)=\frac{\partial\Pi(u+\alpha v)}{\partial\alpha}|_{\alpha=0}=\int_0^{T-h}v^T(\tau)S^T(T-h,\tau)d\tau+$$
$$+\int_0^{T-h}\lambda\Bigg\{u^T(\tau)\left[I_m+\mathcal{G}(\tau)\right]^Td\tau+\int_0^{T-h}\left[\psi(\tau)+\varphi(\tau)\right]^T)d\tau+ \tag{6.80}$$
$$+\int_0^{T-h}\int_0^{T-h}u^T(\tau)K(\theta,\tau)u(\theta)d\theta\Bigg\}v(\tau)d\tau$$

where

$$K(\theta,\tau) = (\Psi(\tau,\theta) + \Phi(\tau,\theta)) + (\Psi^T(\tau,\theta) + \Phi^T(\tau,\theta))$$

Since $\delta\Pi(u) = 0 \quad \forall\, v(\tau)$ for the optimal solution then (6.80) yields

$$S^T(T-h,\tau)g + \lambda\left\{2u(\tau)\left[I_m + \mathcal{G}(\tau)\right] + \psi(\tau) + \varphi(\tau) + \int\limits_0^{T-h} u^T(\tau)K(\theta,\tau)u(\theta)d\theta\right\} = 0 \tag{6.81}$$

The solution of (6.81) can be represented as the following equality

$$u_g(t) = u_1(t) + u_2(t), \qquad \text{where} \quad u_2(t) = \frac{1}{\lambda}L(t)g. \tag{6.82}$$

Here the vector $u_1(t)$ and the $(n\times r)$- matrix $L(t)$ satisfy the following integral equations

$$2u_1(t)(I + \mathcal{G}(t)) + \psi(t) + \varphi(t) + \int\limits_0^{T-h} K(\theta,t)u_1(\theta)d\theta = 0 \tag{6.83}$$

and

$$2L(t) + S(T-h,t) + \int\limits_0^{T-h} K(\theta,t)L(\theta)d\theta = 0 \tag{6.84}$$

To show this it is sufficient to substitute (6.82) into (6.81), which gives

$$S^T(T-h,\tau)g + \lambda\left\{2(u_1(t) + \frac{1}{\lambda}L(t)g))\left[I_m + \mathcal{G}(\tau)\right] + \psi(\tau) + \varphi(\tau) + \right.$$

$$\left. + \int\limits_0^{T-h} u^T(\tau)K(\theta,\tau)(u_1(\theta) + \frac{1}{\lambda}L(\theta)g)d\theta\right\} = \left(S^T(T-h,\tau) + L(\tau) + \right.$$

$$\left. + \int\limits_0^{T-h} K(\theta,\tau)L(\theta)d\theta\right)g + \lambda\left[2u_1(\tau)(I_m + \mathcal{G}(\tau)) + \psi(\tau) + \varphi(\tau) + \int\limits_0^{T-h} u^T(\tau)K(\theta,\tau)u_1(\theta)\right] = 0$$

The unknown multiplier λ can be determined by the fact that the required control function belongs to the admissible set $U(\cdot)$, i. e. $\int\limits_0^T u^T(\tau)u(\tau)d\tau = 1$. Hence

$$\Upsilon + \int\limits_0^{T-h} \left[u_1(\tau) + \frac{1}{\lambda}L(\tau)g)\right]^T \left(I_m + \mathcal{G}(\tau)\right)\left[u_1(\tau) + \frac{1}{\lambda}L(\tau)g)\right]d\tau =$$

$$+ \int\limits_0^{T-h} \left(\psi(\tau) + \varphi(\tau)\right)^T \left[u_1(\tau) + \frac{1}{\lambda}L(\tau)g)\right]d\tau + \tag{6.85}$$

$$+ \int\limits_0^{T-h}\int\limits_0^{T-h} \left[u_1(\tau) + \frac{1}{\lambda}L(\tau)g)\right]^T \left(\Psi(\theta,\tau) + \Phi(\theta,\tau)\right)\left[u_1(\tau) + \frac{1}{\lambda}L(\tau)g)\right]d\theta d\tau = 1$$

This leads to the following equation for λ

$$a\frac{1}{\lambda^2}+2b\frac{1}{\lambda}+c=0 \tag{6.86}$$

where the required coefficients are

$$a=\int_0^{T-h} g^t L(\tau)^T L(\tau) g d\tau+\int_0^{T-h}\int_0^{T-h} g^T L^T(\tau) K(\theta,\tau) L(\theta) d\theta d\tau=-\tfrac{1}{2}\int_0^{T-h} S^T(T-h,\tau)L(\tau)d\tau,$$

$$b=\int_0^{T-h} u_1^T(\tau)L(\tau)gd\tau+\int_0^{T-h}[\psi(\tau)+\varphi(\tau)]L(\tau)gd\tau+\int_0^{T-h}\int_0^{T-h} g^t L^T(\tau)K(\theta,\tau)u_1(\theta)d\tau d\theta=0,$$

$$c=\quad \Upsilon-1+\int_0^{T-h} u_1^T(\tau)(I+\mathcal{G}(\tau))u_1(\tau)d\tau+2\int_0^{T-h}(\psi(\tau)+\varphi(\tau))u_1(\tau)d\tau+$$

$$+\int_0^{T-h}\int_0^{T-h} u_1^T(\tau)\left[\Phi(\theta,\tau)+\Psi(\theta,\tau)\right]u_1(\theta)d\tau d\theta=\Upsilon-1+\int_0^{T-h}(\psi(\tau)+\varphi(\tau))u_1(\tau)d\tau.$$

Thus the required λ is the positive root of equation (6.86), and the optimal control for the given T then is defined by (6.82). Substituting the obtained control function $u_g(t)$ of (6.82) into the basic condition (6.67) and noting Theorem 17, reduces the time optimisation problem to the following:

find the minimal root T^0 for the equation

$$\max_{\|g\|=1}\mathcal{L}(g,T)=0 \tag{6.87}$$

where

$$\mathcal{L}(g,T)=g^T s(T-h,f,g,x_0)+\int_0^{T-h} g^T S(T-h,\tau)u_g(\tau)d\tau$$

and the function $u_g(t)$ is given by (6.82).

Hence, the following theorem has been proved

Theorem 18. *Optimal time T^0 in optimisation problem (6.70)—(6.74) is the minimal root of the equation (6.87) and the corresponding optimal control is*

$$u^0(t)=\begin{cases}u_{g^0}(t), & t\in[0,T^0-h)\\ Nx^0(t-h)+My^0(t-h), & t\in[T^0-h,T^0]\end{cases} \tag{6.88}$$

where the vector g^0 realizes the maximum in (6.87), the function $u_g(t)$ is given by (6.82) and the matrices M, N are defined by (6.60).

6.5 Conclusions and Further work

In this chapter differential repetitive processes have been studied from the perspective of differential delayed systems. The new mathematical models for this class of systems have been introduced and primary analysis is provided. First of all, the controllability and time optimal control have been outlined. It is necessary to add that this work covers only first attempts to investigate the differential repetitive processes from that point of view, and hence a rich field remains to be the subject of further work. For example, new controllability and observability notions are of significant interest for further investigations. In particular, the controllability notion which includes so-called functional controllability (see [40]) when it is required to drive the state variables at the final interval $[\alpha, \alpha+h]$ to the pre-assigned functions $x(t)=\varphi(t), y(t)=\psi(t), t \in [\alpha, \alpha+h]$. This notion can be given as follows

Definition 18. *The process (6.14)—(6.15) is said to be completely controllable if for any initial data $g(t),\ t \in [-h,0],\ f(t),\ t \in [-h,0),\ x_0 = 0$ of (6.15) there exist the moment $t_1 < +\infty$ and the control function $u(t),\ t \geq 0,\ u(t) \equiv 0,\ t \geq t_1$ such that the corresponding solutions $x(t,g,f,x_0,u),\ y(t,g,f,x_0,u)$ of the system (6.14)—(6.15) satisfy the following conditions*

$$x(t) \equiv 0, \ y(t) \equiv 0, \ t \geq t_1 \tag{6.89}$$

In fact, this is required to drive the system at the interval $[t_1, t_1+h]$ to the zero position and to maintain it during the time $t > t_1 + h$. Related analysis for ordinary time delay system can be found in [41, 52] and some results on the controllability of the multiconnected system have been also given in [29, 43].

The results for the linear process (6.14)—(6.15) developed in this chapter can also be extended to obtain the necessary conditions for optimal control of nonlinear models. As known, the cost functional increment method [29] is based on the estimate for trajectory variation generated by the corresponding control function variation, and in fact uses the linear part of the model. For this purpose, we can consider the following nonlinear optimisation problem

$$\dot{x}(t) = F\big(x(t), x(t-h), y(t), y(t-h), u(t)\big) \tag{6.90}$$

$$y(t) = G\big(x(t), x(t-h), y(t-h), u(t)\big), \ t \in T \doteq [0,\alpha] \tag{6.91}$$

with the initial conditions

$$x(t) = f(t), \ t \in [-h,0), \ x(0) = x_0, \ y(t) = g(t), \ t \in [-h,0] \tag{6.92}$$

and $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $u \in \mathbb{R}^r$. Here it is necessary to assume that the control functions are piecewise continuous on the interval $[0,\alpha]$ and $u(t) \in U$ for all $t \in T$, where $U \subset \mathbb{R}^r$ is some prescribed set. The pair of functions $\big(x(t), y(t)\big)$ is a solution of the system (6.90)—(6.92) for the given control function $u(t)$, if they satisfy the differential equation (6.90) almost everywhere on the interval $[0,\alpha]$ and the difference equation (6.91) for all $t \in [0,h]$.

Let $\beta_0, \beta_1, \ldots, \beta_\nu$, be given time moments such that $0 = \beta_0 < \beta_1 < \ldots \beta_\nu \leq \alpha$, and, $M_i \subset \mathbb{R}^m$, $i = 0, \ldots, \nu$ be given convex closed sets from $\mathbb{R}^m$. Hence, the optimal control problem is to minimize the cost functional of the form

$$J(u) = \varphi(y(\alpha - \beta_0), y(\alpha - \beta_1), \ldots, y(\alpha - \beta_\nu)) \tag{6.93}$$

subject to the constraints

$$y(\alpha - \beta_i) \in M_i, \ i = 0, \ldots, \nu \tag{6.94}$$

over the solutions of the system (6.90)—(6.92)). Here $\varphi(x_1, \ldots, x_{\nu+1})$ is a continuously differentiable function. The introduction of this kind optimisation problem corresponds to the notion of the pass controllability for the given points when it is necessary to optimize the final pass profile running through the pre-assigned value set at the specified time moments. However, the major task is to solve the general problem with the nonlocal initial conditions omitted here. These problems are the subject of ongoing work and will be reported in due course.

Summary

Thus, the principal results of the Research work are:

• In Chapter 1, Theorem 1 where the estimate for approximating trajectories was established

$$\rho(x(t), G) \leq C \cdot \sqrt{1/i} \quad \forall t \in [0,T],\ i \geq i_0;$$

• In Chapter 2, Lemma 5 where the correct discrete approximation for the optimal control problems with "minmax" constraints was stated :

"...Then there is a sequence of perturbations $\epsilon_N \to 0$ as $N \to \infty$ such that one has a value convergence $\lim_{N\to\infty} J_N^0 = J^0$";

• In Chapter 3, the correct dicretization and test preparation for a robot model was discussed.

• In Chapter 4, Theorem 4 (and in Chapter 4, Theorem 9) where the Pontryagin maximum principle was proved for the control systems (ordinary and repetitive, respectively) in presence of intermediate constraints;

• In Chapter 5, for the optimal control in the linear repetitive processes , Theorem 10 an 11 where the constructive optimality and suboptimality conditions were obtained and their correlation with classic results were established;

• In Chapter 6, we investigated further the structural links between linear repetitive processes and a special class of time delay systems. This lead to a new controllability, Theorem 15, and the optimal control, Theorem 18, is the results for these processes.

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